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Representation of odd integers as the sum of one prime, two squares of primes and powers of 2

by

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1. Introduction. Let

$$\mathcal{A} = \{n : n \in \mathbb{N},\ n \not\equiv 0 \pmod 2,\ n \not\equiv 2 \pmod 3\}.$$

In 1938 Hua [3] proved that almost all $n \in \mathcal{A}$ are representable as sums of two squares of primes and a kth power of a prime for odd k,

$$n = p_1^2 + p_2^2 + p_3^k. \tag{1.1}$$

In 1999, Liu, Liu and Zhan [6] proved that every large odd integer N can be written as a sum of one prime, two squares of primes and k powers of 2,

$$N = p_1^2 + p_2^2 + p_3 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.2}$$

In 2004, Liu [8] proved that $k = 22000$ is acceptable in (1.2).

In this paper we shall prove the following result.

Theorem. *Every sufficiently large odd integer can be written as a sum of one prime, two squares of primes and* 106 *powers of* 2.

The substantial improvement is due to two facts: firstly, we use the method of [1] and [7] to enlarge the major arcs; secondly, Heath-Brown and Puchta's estimation for the measure of exponential sums of powers of 2 (Lemma 3) gives a good control for the minor arcs.

2. Outline and preliminary results. To prove the Theorem, it suffices to estimate the number of solutions of the equation

$$n = p_1^2 + p_2^2 + p_3 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{2.1}$$

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Suppose N is our main parameter, which we assume to be "sufficiently large". We write

$$P = N^{1/6-\varepsilon}, \quad Q = NP^{-1}L^{-10}, \quad M = NL^{-9}, \quad L = \log_2 N. \tag{2.2}$$

We use c and ε to denote an absolute constant and a sufficiently small positive number, not necessarily the same at each occurrence.

The circle method, in the form we require, begins with the observation that

$$R(N) := \sum_{\substack{p_1^2+p_2^2+p_3+2^{\nu_1}+\cdots+2^{\nu_k}=N \\ M<p_1^2,p_2^2,p_3\le N}} (\log p_1)(\log p_2)(\log p_3) = \int_0^1 f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)\,d\alpha, \tag{2.3}$$

where we write $e(x) = \exp(2\pi i x)$ and

$$\begin{aligned} f(\alpha) &= \sum_{M<p^2\le N} (\log p)e(\alpha p^2), \qquad g(\alpha) = \sum_{M<p\le N} (\log p)e(\alpha p), \\ h(\alpha) &= \sum_{2^\nu\le N} e(\alpha 2^\nu) := \sum_{\nu\le L} e(\alpha 2^\nu). \end{aligned} \tag{2.4}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q]$ can be written as

$$\alpha = \frac{a}{q} + \beta, \qquad |\beta| \le \frac{1}{qQ}, \tag{2.5}$$

for some integers a, q with $1 \le a \le q \le Q,\ (a,q) = 1$. Let

$$\mathfrak{M} = \bigcup_{1\le q\le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right]. \tag{2.6}$$

These are the *major arcs*, and the *minor arcs* $\mathfrak{m}$ are given by

$$\mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M}. \tag{2.7}$$

LEMMA 1 (Theorem 3 of [4] for $k = 2$). *Suppose that α is a real number and there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \le q \le Y, \qquad (a,q) = 1, \qquad |q\alpha - a| < Y^{-1},$$

with $Y = X^{3/2}$. Then for any fixed $\varepsilon > 0$ one has

$$\sum_{X<p\le 2X} (\log p)e(\alpha p^2) \ll X^{7/8+\varepsilon} + \frac{q^\varepsilon X(\log X)^c}{(q + X^2|q\alpha - a|)^{1/2}}.$$

For χ mod q, define

$$C_2(\chi,a)=\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{ah^2}{q}\right),\qquad C(\chi,a)=\sum_{h=1}^{q}\overline{\chi}(h)e\left(\frac{ah}{q}\right),\tag{2.8}$$

$$C_2(q,a)=C_2(\chi_0,a),\qquad C(q,a)=C(\chi_0,a).\tag{2.9}$$

Here χ_0 is the principal character modulo q.

If χ_1,χ_2,χ_3 are characters mod q, then let

$$B(n,q;\chi_1,\chi_2,\chi_3)=\frac{1}{\phi^3(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q}C(\chi_1,a)C_2(\chi_2,a)C_2(\chi_3,a)e\left(-\frac{an}{q}\right),\tag{2.10}$$

$$A(n,q)=B(n,q;\chi_0,\chi_0,\chi_0),\qquad \mathfrak{S}(n,X)=\sum_{q\le X}A(n,q).\tag{2.11}$$

Lemma 2 (Lemma 2.1 of [8]). *Let χ_j mod r_j with $j=1,2,3$ be primitive characters, $r_0=[r_1,r_2,r_3]$, and χ_0 the principal character mod q. Then*

$$\sum_{\substack{q\le x\\ r_0\mid q}}|B(n,q;\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)|\ll r_0^{-1/2+\varepsilon}(\log x)^c.$$

On the minor arcs, we need estimates for the measure of the set

$$\mathcal{E}_\lambda:=\{\alpha\in(0,1]:|h(\alpha)|\ge\lambda L\}.\tag{2.12}$$

The following lemma is due to Heath-Brown and Puchta [2].

Lemma 3. *We have*

$$\operatorname{meas}(\mathcal{E}_\lambda)\ll N^{-E(\lambda)}\quad\textit{with}\quad E(0.9108)>\frac{19}{24}+10^{-10}.$$

Proof. Let

$$T_h(\alpha)=\sum_{0\le n\le h-1}e(\alpha 2^n),$$

$$F(\xi,h)=\frac{1}{2^h}\sum_{r=0}^{2^h-1}\exp\{\xi\operatorname{Re}(T_h(r/2^h))\},$$

$$E(\lambda)=\frac{\xi\lambda}{\log 2}-\frac{\log F(\xi,h)}{h\log 2}-\frac{\varepsilon}{\log 2}.$$

Then for any $\xi,\varepsilon>0$, and any $h\in\mathbb{N}$, we have

$$\operatorname{meas}(\mathcal{E}_\lambda)\ll N^{-E(\lambda)}.$$

This was proved in Section 7 of [2]. Taking $\xi=1.31$, $h=18$, we get

$$E(0.9108)>\frac{19}{24}+10^{-10}.$$

This completes the proof of the lemma.

3. The major arcs. Let

$$(3.1)\qquad \begin{aligned} f^*(\alpha) &= \frac{C_2(q,a)}{\phi(q)} \sum_{M<m^2\le N} e(\beta m^2),\\ g^*(\alpha) &= \frac{C(q,a)}{\phi(q)} \sum_{M<m\le N} e(\beta m). \end{aligned}$$

We now proceed to estimate the quantity

$$(3.2)\qquad \int_{\mathfrak{M}} f^2(\alpha)g(\alpha)e(-\alpha n)\,d\alpha - \int_{\mathfrak{M}} f^{*2}(\alpha)g^*(\alpha)e(-\alpha n)\,d\alpha,$$

which we think of as the error of approximation of the integral over $\mathfrak{M}$ by the expected term.

By the standard major arcs techniques we have

$$(3.3)\qquad \int_{\mathfrak{M}} f^{*2}(\alpha)g^*(\alpha)e(-\alpha n)\,d\alpha = P_0\mathfrak{S}(n,P)(1+o(1)),$$

where

$$(3.4)\qquad P_0 = \pi n/4,$$

and $\mathfrak{S}(n,P)$ is defined by (2.11). Define

$$\begin{aligned} W(\chi,\beta) &= \sum_{M<p^2\le N} (\log p)\chi(p)e(\beta p^2) - D(\chi)\sum_{M<m^2\le N} e(\beta m^2),\\ W^\sharp(\chi,\beta) &= \sum_{M<p\le N} (\log p)\chi(p)e(\beta p) - D(\chi)\sum_{M<m\le N} e(\beta m), \end{aligned}$$

where $D(\chi)$ is 1 or 0 according as χ is principal or not.

Just as in [1, (4.1)] we can rewrite $f(\alpha)$ and $g(\alpha)$ as

$$(3.5)\qquad f\left(\frac{a}{q}+\beta\right) = \frac{C_2(q,a)}{\phi(q)} \sum_{M<m^2\le N} e(\beta m^2) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C_2(\chi,a)W(\chi,\beta),$$

$$(3.6)\qquad g\left(\frac{a}{q}+\beta\right) = \frac{C(q,a)}{\phi(q)} \sum_{M<m\le N} e(\beta m) + \frac{1}{\phi(q)} \sum_{\chi \bmod q} C(\chi,a)W^\sharp(\chi,\beta).$$

So we can use (3.5) and (3.6) to express the difference in (3.2) as a linear combination of error terms involving $f^*(\alpha)$ and $g^*(\alpha)$, and $W(\chi,\beta)$ and $W^\sharp(\chi,\beta)$.

We shall focus on the most troublesome among the error terms that arise, namely the multiple sum

$$(3.7)\qquad \sum_{q\le P}\sum_{\chi_1 \bmod q}\sum_{\chi_2 \bmod q}\sum_{\chi_3 \bmod q} B(n,q;\chi_1,\chi_2,\chi_3)J(n,q,\chi_1,\chi_2,\chi_3).$$

Here $B(n,q;\chi_1,\chi_2,\chi_3)$ is defined in (2.10), and

$$J(n,q,\chi_1,\chi_2,\chi_3)=\int_{-1/qQ}^{1/qQ} W^\sharp(\chi_1,\beta)W(\chi_2,\beta)W(\chi_3,\beta)e(-\beta n)\,d\beta.$$

We first reduce (3.7) to a sum over primitive characters. Suppose χ_j^* mod r_j with $r_j\,|\,q$ is the primitive character inducing χ_j. In general, if χ mod q, $q\le P$, is induced by a primitive character χ^* mod r with $r\,|\,q$, we have

$$W^\sharp(\chi,\beta)=W^\sharp(\chi^*,\beta),\qquad W(\chi,\beta)=W(\chi^*,\beta). \tag{3.8}$$

By Cauchy's inequality

$$J(n,q,\chi_1,\chi_2,\chi_3)\ll W^\sharp(\chi_1^*)W(\chi_2^*)W(\chi_3^*), \tag{3.9}$$

where for a character χ mod r,

$$W^\sharp(\chi)=\max_{|\beta|\le 1/rQ}|W^\sharp(\chi,\beta)|,\qquad W(\chi)=\Big(\int_{-1/rQ}^{1/rQ}|W(\chi,\beta)|^2\,d\beta\Big)^{1/2}. \tag{3.10}$$

Using (3.9) we can bound (3.7) by

$$\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1}\sum_{r_2\le P}\sideset{}{^*}\sum_{\chi_2}\sum_{r_3\le P}\sideset{}{^*}\sum_{\chi_3}W^\sharp(\chi_1)W(\chi_2)W(\chi_3)B(n,\chi_1,\chi_2,\chi_3). \tag{3.11}$$

Here $\sum_{r_j}\sum_{\chi}^*$ denotes a summation over the primitive characters mod r_j $\le P$, and

$$B(n,\chi_1,\chi_2,\chi_3)=\sum_{\substack{q\le P\\ r_0|q}}|B(n,q;\chi_1\chi_0,\chi_2\chi_0,\chi_3\chi_0)|,$$

where $r_0=[r_1,r_2,r_3]$ and χ_0 is the principal character mod q.

By Lemma 2 we have

$$B(n,\chi_1,\chi_2,\chi_3)\ll r_0^{-1/2+\varepsilon}L^c,$$

and by [7, Lemma 2.4] we have

$$\sum_{r\le R}\sideset{}{^*}\sum_{\chi}[r,d]^{-1/2+\varepsilon}W(\chi)\ll d^{-1/2+\varepsilon}L^c$$

whenever $R\le N^{1/6-\varepsilon}$. Thus the sixfold sum in (3.11) does not exceed

$$L^c\sum_{r_1\le P}\sideset{}{^*}\sum_{\chi_1}r_1^{-1/2+\varepsilon}W^\sharp(\chi_1).$$

To estimate $\sum_{r_1\le P}\sum_{\chi_1}^* r_1^{-1/2+\varepsilon}W^\sharp(\chi_1)$, we can modify the proof of Lemma 2.3 of [7] for $k=1$. For $L^B<R\le P$, where B is a constant depending on A, the right-hand sides of (5.1) and (5.2) of [7] should be replaced by

$N^{1/2}(T_1+1)^{1/2}L^{-A}$ and $N^{1/2}T_2L^{-A}$, by using Theorem 4.1 of [7]; moreover, since $R \le P = N^{1/6-\varepsilon}$, we get

$$\sum_{L^B<r_1\le P} \sum_{\chi_1}{}^{*} r_1^{-1/2+\varepsilon} W^{\sharp}(\chi_1) \ll NL^{-A} \quad \text{for any } A>0.$$

For the case $R \le L^B$, in the same way as in [7] we deduce that

$$\sum_{r_1\le L^B} \sum_{\chi_1}{}^{*} r_1^{-1/2+\varepsilon} W^{\sharp}(\chi_1) \ll NL^{-A} \quad \text{for any } A>0.$$

We have shown that the sum in (3.7) is $O(NL^{-A})$ for any fixed $A>0$. Recall that (3.7) was one of several error terms in a representation of (3.2). Since the other error terms in that representation can be estimated similarly, we conclude that the difference in (3.2) is $O(NL^{-A})$.

Together with (3.3) we obtain the following result:

LEMMA 4. *For all integers $n \in \mathcal{A}$, we have*

$$\int_{\mathfrak{M}} f^2(\alpha)g(\alpha)e(-\alpha n)\,d\alpha = (\pi/4+o(1))\mathfrak{S}(n,P)n + O(N/\log N). \tag{3.12}$$

LEMMA 5. *For $n \in \mathcal{A}$, we have*

$$\mathfrak{S}(n,P) \ge 2.27473966.$$

Otherwise, we have $\mathfrak{S}(n,P) = O(P^{-1+\varepsilon})$.

Proof. By [8, p. 114], we have

$$\mathfrak{S}(n,P) = \sum_{q=1}^{\infty} A(n,q) + O(P^{-1+\varepsilon}). \tag{3.13}$$

By [10, (3.14)], when $(a,q)=1$, we have $C(q,a)=\mu(q)$. Hence

$$A(n,q) = \frac{\mu(q)}{\phi^3(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} C_2^2(q,a)e\left(-\frac{an}{q}\right),$$

and for $k\ge 2$, $A(n,p^k)=0$. Since $A(n,q)$ is multiplicative, we have

$$\mathfrak{S}(n,P) = \prod_{p=2}^{\infty}(1+A(n,p)) + O(P^{-1+\varepsilon}). \tag{3.14}$$

By direct computation, for $n \in \mathcal{A}$ we have

$$1+A(n,2)=2, \qquad 1+A(n,3)=3/2. \tag{3.15}$$

If $n \equiv 0 \pmod 2$, we have $1+A(n,2)=0$. When $n \equiv 2 \pmod 3$, we have

$1+A(n,3)=0$. By [8, p. 114], for $p\ge 5$, we have

$$1+A(n,p)\ge\begin{cases}1-\dfrac{p+1}{(p-1)^3}, & p\equiv 1 \pmod 4,\\[2mm] 1-\dfrac{3p-1}{(p-1)^3}, & p\equiv -1 \pmod 4.\end{cases}\tag{3.16}$$

Hence

$$\prod_{p\ge 5}(1+A(n,p))\ge\prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ p\ge 5}}\left(1-\frac{p+1}{(p-1)^3}\right)\prod_{\substack{p\equiv -1 \,(\mathrm{mod}\, 4)\\ p\ge 5}}\left(1-\frac{3p-1}{(p-1)^3}\right).$$

By the elementary inequality

$$(1+x)^a<1+ax+\frac{a(a-1)}{2}\,x^2 \quad \text{if } a>2,\ -1<x<0,$$

for $p>82$ and $p\equiv 1 \pmod 4$ we have

$$1-\frac{p+1}{(p-1)^3}\ge\left(1-\frac{1}{(p-1)^2}\right)^{3.025},$$

and for $p>82$ and $p\equiv -1 \pmod 4$,

$$1-\frac{3p-1}{(p-1)^3}\ge\left(1-\frac{1}{(p-1)^2}\right)^{3.025}.$$

Thus

$$\begin{aligned}
&\prod_{p\ge 5}(1+A(n,p))\\
&\ge\prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}}\left(1-\frac{p+1}{(p-1)^3}\right)\prod_{\substack{p\equiv -1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}}\left(1-\frac{3p-1}{(p-1)^3}\right)\prod_{p>82}\left(1-\frac{1}{(p-1)^2}\right)^{3.025}\\
&=\prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}}\left(1-\frac{p+1}{(p-1)^3}\right)\prod_{\substack{p\equiv -1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}}\left(1-\frac{3p-1}{(p-1)^3}\right)\\
&\quad\times\prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ 5\le p<82}}\left(1-\frac{1}{(p-1)^2}\right)^{-3.025}\prod_{\substack{p\equiv -1 \,(\mathrm{mod}\, 4)\\ 3\le p<82}}\left(1-\frac{1}{(p-1)^2}\right)^{-3.025}\\
&\quad\times\prod_{p\ge 3}\left(1-\frac{1}{(p-1)^2}\right)^{3.025}\ge 1.11571\cdot\left(0.6601\cdot\frac{4}{3}\right)^{3.025}>0.7582465536,
\end{aligned}$$

where we have used the well known result $\prod_{p\ge 3}(1-1/(p-1)^2)=0.6601\ldots$. By (3.14), (3.15) and the above estimate, we get the lemma.

4. Proof of Theorem. We need the following lemmas.

LEMMA 6. *Let* $\mathcal{A}(N,k)=\{n\ge 2: n=N-2^{\nu_1}-\cdots-2^{\nu_k}\}$ *with* $k\ge 100$. *Then for odd* N, *we have*

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\not\equiv 2\,(\mathrm{mod}\,3)}} n\ge (2/3-2^{-90})NL^k.$$

Proof. Let $((\nu))$ mean that $\nu_1,\dots,\nu_k$ satisfies

$$1\le \nu_1,\dots,\nu_k\le \log_2(N/kL),\qquad N-2^{\nu_1}-\cdots-2^{\nu_k}\equiv 0\ (\mathrm{mod}\,3). \tag{4.1}$$

Then $n\ge N-N/L$, and

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\equiv 0\,(\mathrm{mod}\,3)}} n\ge \sum_{((\nu))}(N-2^{\nu_1}-\cdots-2^{\nu_k})\ge \left(N-\frac{N}{L}\right)\sum_{((\nu))}1. \tag{4.2}$$

For odd q, let $\varepsilon(q)$ be the order of 2 in the multiplicative group of integers modulo q. Let

$$H(d,N,K)=\sharp\Big\{(\nu_1,\dots,\nu_K):1\le\nu_i\le\varepsilon(d),\, d\,|\,N-\sum 2^{\nu_i}\Big\}.$$

When $d=3$, $\varepsilon(3)=2$, and it is an easy exercise to check that

$$H(3,N,K)=\begin{cases}\frac13(2^K-(-1)^K), & 3\nmid N,\\ \frac13(2^K+(-1)^K), & 3\,|\,N.\end{cases}$$

Thus if $K>100$ we have

$$H(3,N,K)\varepsilon(3)^{-K}\ge\frac13\,(1-2^{-98}),$$

and

$$\sum_{((\nu))}1\ge H(3,N,k)([\log_2(N/kL)/\varepsilon(3)]-2)^k\ge\frac13\,(1-2^{-96})L^k.$$

Hence

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\equiv 0\,(\mathrm{mod}\,3)}} n\ge (1/3-2^{-95})NL^k. \tag{4.3}$$

Similarly,

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\equiv 1\,(\mathrm{mod}\,3)}} n\ge (1/3-2^{-95})NL^k. \tag{4.4}$$

From this and (4.3) we get the lemma.

LEMMA 7 (Lemma 3 of [5]). *Let $f(\alpha)$ and $h(\alpha)$ be as in* (2.4). *Then*

$$\int_0^1 |f(\alpha)h(\alpha)|^4 \, d\alpha \le c_1 \frac{\pi^2}{16} NL^4,$$

where

$$c_1 \le \left(\frac{32^4 \cdot 101 \cdot 1.620767}{3} + \frac{8 \cdot \log^2 2}{\pi^2}\right)(1+\varepsilon)^9.$$

LEMMA 8. *Let $g(\alpha)$ and $h(\alpha)$ be as in* (2.4). *Then*

$$\int_{\mathfrak{m}} |g(\alpha)h(\alpha)|^2 \, d\alpha \le 12.3685 c_0 NL^2,$$

where

$$c_0 = \prod_{p \ge 3}\left(1 - \frac{1}{(p-1)^2}\right) = 0.6601\ldots.$$

Proof. This is actually Lemma 10 of [2]. By (8.14) of [9], we can replace (41) of [2] by $C_2 \le 1.94$, and then by the proof of Lemma 9 of [2] the assertion follows.

Now we prove the Theorem. Let $\mathcal{E}_\lambda$ be as defined in (2.12), and $\mathfrak{M}$ and $\mathfrak{m}$ as in (2.6) and (2.7) with P, Q determined in (2.2). Then (2.3) becomes

$$R(N) = \int_0^1 f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)\, d\alpha = \int_{\mathfrak{M}} + \int_{\mathfrak{m}\cap\mathcal{E}_\lambda} + \int_{\mathfrak{m}\setminus\mathcal{E}_\lambda}. \tag{4.5}$$

For the major arcs, by Lemma 4 we have

$$\begin{aligned}\int_{\mathfrak{M}} & f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)\, d\alpha \\ &= \sum_{n\in\mathcal{A}(N,k)} \int_{\mathfrak{M}} f^2(\alpha)g(\alpha)e(-\alpha n)\, d\alpha \\ &= \left(\frac{\pi}{4} + o(1)\right) \sum_{n\in\mathcal{A}(N,k)} \mathfrak{S}(n,P)n + O(NL^{k-1}) \\ &\ge 2.27473966\left(\frac{\pi}{4} + o(1)\right)\Big\{ \sum_{\substack{n\in\mathcal{A}(N,k)\\ n\not\equiv 2 \,(\mathrm{mod}\, 3)}} n\Big\} + O(NL^{k-1}) \\ &\ge 1.516492 \frac{\pi}{4} NL^k,\end{aligned} \tag{4.6}$$

where we have used Lemmas 5 and 6.

For the second integral in (4.5), by Dirichlet's lemma on rational approximation, any $\alpha \in \mathfrak{m}$ can be written as

$$\alpha = \frac{a}{q} + \beta, \qquad |\beta| \le \frac{1}{qN^{3/4}}, \tag{4.7}$$

for some integers a, q with $1 \le a \le q \le N^{3/4}$, $(a, q) = 1$. If $q \le P = N^{1/6-\varepsilon}$, since $\alpha \in \mathfrak{m}$, we have $PL^{10} < N|q\alpha - a|$; otherwise we have $q > P$; hence $q + N|q\alpha - a| > P$ for any $\alpha \in \mathfrak{m}$. By Lemma 1,

$$\max_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll N^{1/2-1/16+\varepsilon}.$$

By Theorem 3.1 of Vaughan [10],

$$\max_{\alpha \in \mathfrak{m}} |g(\alpha)| \ll N^{1-1/12+\varepsilon}.$$

Therefore

$$\int_{\mathfrak{m} \cap \mathcal{E}_\lambda} \ll N^{-E(0.9108)} N^{2-5/24+\varepsilon} L^k \ll N^{1-\varepsilon}, \tag{4.8}$$

where we have used Lemma 3 for $\lambda = 0.9108$.

For the last integral in (4.5), with the definition of $\mathcal{E}_\lambda$, and Lemmas 7 and 8, by Cauchy's inequality we have

$$\begin{aligned} \int_{\mathfrak{m} \setminus \mathcal{E}_\lambda} &\le (\lambda L)^{k-3} \Big(\int_0^1 |f(\alpha) h(\alpha)|^4 \, d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{m}} |g(\alpha) h(\alpha)|^2 \, d\alpha \Big)^{1/2} \\ &\le 21616 \lambda^{k-3} \frac{\pi}{4} N L^k. \end{aligned} \tag{4.9}$$

Combining this with (4.6) and (4.8), we get

$$R(N) \ge \frac{\pi}{4} N L^k (1.516492 - 21616 \lambda^{k-3}). \tag{4.10}$$

When $k \ge 106$, for $\lambda = 0.9108$, by the above estimate we have

$$R(N) > 0.$$

This means that every large odd integer N can be written in the form of (1.2) for $k \ge 106$.

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