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Functoriality and number of solutions of congruences

by

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In this note we use Langlands functoriality to prove certain results on the number of solutions of congruences, complementing results in [F1–3]. We would like to thank the referee for pointing out a mistake.

1. Number of solutions of congruences. Let $f(x) = x^d + a_1x^{d-1} + \cdots + a_d$, $a_1, \ldots, a_d \in \mathbb{Z}$ be an irreducible polynomial. Let $N_f(n)$ be the number of solutions of $f(x) \equiv 0 \pmod n$. It is an important problem to study $N_f(n)$. Let L be the splitting field of f with the Galois group G. Let $E = \mathbb{Q}[\alpha]$, where α is a root of f. Then $[E : \mathbb{Q}] = d$. Let $\mathrm{Gal}(L/E) = H$. Let $S(L/\mathbb{Q}) = \{p : N_f(p) = d\}$. Then it is known that $S(L/\mathbb{Q})$ determines L completely. It is a goal of the class field theory to determine the set $S(L/\mathbb{Q})$. Except for finitely many primes, $p \in S(L/\mathbb{Q})$ if and only if p splits completely in L: this comes from the fact that p splits completely in L if and only if p splits completely in E. It is clear that if p splits completely in L, then p splits completely in E. Conversely, let $\mathfrak{P}, \mathfrak{p}$ be primes in L, E, respectively, such that $\mathfrak{P} \mid \mathfrak{p}$, $\mathfrak{p} \mid p$. Then p splits completely in L if and only if $L_{\mathfrak{P}} = \mathbb{Q}_p$. We fix an embedding $\mathbb{Q}_p \hookrightarrow \overline{\mathbb{Q}}_p$. Suppose p splits completely in E. Then for any $\mathfrak{p} \mid p$, $E \subset E_{\mathfrak{p}} = \mathbb{Q}_p \hookrightarrow \overline{\mathbb{Q}}_p$. So every conjugate of E is contained in $\mathbb{Q}_p$. Hence L is contained in $\mathbb{Q}_p$. So p splits completely in L. By the well-known theorem of Dedekind (e.g. [N, Theorem 4.33]), except for finitely many primes (in fact, if p does not divide the discriminant of $f(x)$, or $(\mathcal{O}_E : \mathbb{Z}[\alpha])$), p splits completely in E if and only if $N_f(p) = d$.

Consider $\mathrm{Ind}_H^G 1 = 1 + \varrho$, where $\varrho : G \to \mathrm{GL}_{d-1}(\mathbb{C})$ is a direct sum of non-trivial irreducible representations of G, i.e.,

$$\varrho = n_1\varrho_1 + \cdots + n_k\varrho_k \tag{1}$$

where $\varrho_1, \ldots, \varrho_k$ are non-trivial irreducible representations of G. By Frobenius reciprocity, we see that if ϱ_i is a 1-dimensional character, then $n_i = 1$.

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(See [FH], for example.) We include the case $H = \{1\}$. In particular, if G is abelian, then $\mathbb{Q}[\alpha]/\mathbb{Q}$ is Galois, and hence $E = L$, and $H = \{1\}$.

The Artin conjecture asserts that $\zeta_E(s)/\zeta(s)$ is entire. Langlands functoriality (the strong Artin conjecture) predicts that there exists an automorphic representation $\pi = \bigotimes \pi_p$ of $\mathrm{GL}_{d-1}(\mathbb{A})$ which corresponds to ϱ. If ϱ is irreducible, then π is cuspidal [R2]. More precisely, let π_i be a cuspidal representation corresponding to ϱ_i. Then π is the isobaric sum

$$\pi = \underbrace{\pi_1 \boxplus \cdots \boxplus \pi_1}_{n_1} \boxplus \cdots \boxplus \underbrace{\pi_k \boxplus \cdots \boxplus \pi_k}_{n_k}.$$

In particular, the Langlands–Tunnell theorem says that if ϱ is a 2-dimensional representation with solvable image, then the strong Artin conjecture is true.

If p is unramified, then $\varrho(\mathrm{Frob}_p)$ is the semisimple conjugacy class of π_p. Let $\mathrm{diag}(\alpha_{1p}, \ldots, \alpha_{d-1,p})$ give rise to the semisimple conjugacy class of π_p, and let $a_p = \alpha_{1p} + \cdots + \alpha_{d-1,p}$. In particular, we have the L-function (without the Γ-factors)

$$L(s,\pi) = \prod_p L(s,\pi_p) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

such that $\zeta_E(s) = \zeta(s)L(s,\pi)$.

We prove that if $\sigma = \mathrm{Ind}_H^G 1$, then $\chi_\sigma(\mathrm{Frob}_p) = N_f(p)$, so that $N_f(p) = 1 + a_p$. We can see this in two ways. First, by the property of the Artin L-function, $L(s, \mathrm{Ind}_H^G 1, L/\mathbb{Q}) = \zeta_E(s)$. Let $\zeta_E(s) = \prod_p L_p(s)$. If $N_f(p) = a$, then $L_p(s)$ has the form $(1-p^{-s})^{-a} \prod_{i=1}^r (1-p^{-k_i s})^{-1}$, where $k_i \geq 2$. Hence $L_p(s) = (1 - ap^{-s} + \cdots \pm p^{-ds})^{-1}$.

Second, $\sigma = \mathrm{Ind}_H^G 1$ is the permutation representation of G on the left cosets of H in G. Let $\{g_i H : i = 1, \ldots, d\}$ be the left cosets. Then $\chi_\sigma(g)$ is the trace of the permutation matrix given by $g_i H \mapsto g g_i H$. It is the number of the left cosets such that $g_i^{-1} g g_i \in H$. Suppose p decomposes as $p\mathcal{O}_E = \mathfrak{p}_1 \cdots \mathfrak{p}_k$ such that each $\mathfrak{p}_i$ is unramified, and has the residual degree f_i. Then $d = f_1 + \cdots + f_k$. If $\mathfrak{P} \mid \mathfrak{p}_i$, then $\left(\frac{L/E}{\mathfrak{P}}\right) = \left(\frac{L/\mathbb{Q}}{\mathfrak{P}}\right)^{f_i} \in H$. Hence $\left(\frac{L/\mathbb{Q}}{\mathfrak{P}}\right) \in H$ if and only if $f_i = 1$. Pick $\mathfrak{P}_i \mid \mathfrak{p}_i$ for each i. Pick elements τ_i which send $\mathfrak{p}_1$ to $\mathfrak{p}_i$ for $i = 1, \ldots, k$. Then $\tau_i \left(\frac{L/\mathbb{Q}}{\mathfrak{P}_i}\right)^{k_i}$, $i = 1, \ldots, k$, $0 \leq k_i \leq f_i - 1$, are coset representatives [N, Theorem 7.29]. Hence $\chi_\sigma(\mathrm{Frob}_p)$ is the number of i's such that $f_i = 1$. It is exactly $N_f(p)$. So $\chi_\sigma(\mathrm{Frob}_p) = N_f(p)$.

Since π is an automorphic representation of $\mathrm{GL}_{d-1}(\mathbb{A})$, $L(s,\pi)$ has an analytic continuation to all of $\mathbb{C}$, and satisfies an appropriate functional equation. Hence we have

Proposition 1 ([FI]). *Let a_n be as above. Then*

$$\sum_{n\le x} a_n = O(x^{(d-2)/d+\varepsilon}).$$

Hence the series $\sum_{n=1}^\infty a_n/n^s$ converges for $\mathrm{Re}(s) > (d-2)/d+\varepsilon$. In particular,

$$\sum_{p\le x} \frac{a_p}{p} = O(1).$$

2. Distribution of values of $r_2(n)$. Let $r_2(n) = \sum_{x_1^2+x_2^2=n} 1$. We are interested in

$$\sum_{n\le x} r_2(f(n)).$$

If $f(x) = ax^2+bx+c$, then (see [F2] for the details)

$$\sum_{n\le x} r_2(f(n)) = \begin{cases} A(f)x\log x + O(x\log\log x) & \text{if } b^2-4ac=-\mu^2, \\ B(f)x + O(x^{8/9}(\log x)^3) & \text{if } b^2-4ac\ne-\mu^2. \end{cases} \tag{2}$$

We use (see [F2] for the precise reference)

Lemma 2. *Let $t(n)$ be a multiplicative function such that $t(n)\ge 0$ and $t(p^k)\ll k^c$, $k\in\mathbb{N}$ (p prime, and c constant). Let $f(x)=\sum_{i=0}^l a_i x^i \in \mathbb{Z}[x]$ be irreducible such that $(a_0,\ldots,a_l)=1$. Then*

$$\sum_{n\le x} t(f(n)) \ll x\exp\Bigl(\sum_{p\le x}\frac{N_f(p)(t(p)-1)}{p}\Bigr),$$

where the implied constant depends on $t(n)$ and $f(n)$.

Hence we need to compute

$$\sum_{p\le x}\frac{N_f(p)(r_2(p)-1)}{p}.$$

Here we have removed finitely many primes p where π_p is not spherical, or $p=2$. However, if $p\ne 2$, then $r_2(p) = 1+\bigl(\frac{-1}{p}\bigr)$. Let χ_4 be the non-trivial character of $(\mathbb{Z}/4\mathbb{Z})^\times$. Then $\chi_4(p)=\bigl(\frac{-1}{p}\bigr)$. Since $\sum_{n=1}^\infty \chi_4(n)/n^s$ is holomorphic at $s=1$, $\sum_{p\le x}\chi_4(p)/p = O(1)$. Also

$$\sum_{n=1}^\infty \frac{a_n\chi_4(n)}{n^s} = L(s,\pi\otimes\chi_4).$$

If π is not cuspidal, and χ_4 occurs in the decomposition (1), then it occurs with multiplicity one, and hence $L(s,\pi\otimes\chi_4)$ has a simple pole at $s=1$. So $\sum_{p\le x} a_p\chi_4(p)/p = \log\log x + O(1)$. Otherwise, $L(s,\pi\otimes\chi_4)$ is holomorphic at $s=1$, and $\sum_{p\le x} a_p\chi_4(p)/p = O(1)$. Here we note that χ_4 occurs in the decomposition (1) if and only if χ_4 is an irreducible character of G and $\chi_4|_H = 1$. Hence it is the case if and only if $\mathbb{Q}[\sqrt{-1}] \subset E$. Hence

$$\sum_{p\le x} \frac{N_f(p)(r_2(p)-1)}{p} = \begin{cases} \log\log x + O(1) & \text{if } \mathbb{Q}[\sqrt{-1}] \subset E, \\ O(1) & \text{otherwise.} \end{cases}$$

Therefore, we obtain

THEOREM 3. *Suppose we have the strong Artin conjecture for $L(s,\varrho)$. Then*

$$\sum_{n\le x} r_2(f(n)) \ll \begin{cases} x\log x & \textit{if } \mathbb{Q}[\sqrt{-1}] \subset E, \\ x & \textit{otherwise.} \end{cases}$$

If $f(x) = ax^2+bx+c$, $b^2-4ac = -\mu^2$, then $E = \mathbb{Q}[\sqrt{-1}]$, and hence $\zeta_E(s) = \zeta(s)L(s,\chi_4)$. So the estimate (2) is the best possible. If $f(x)$ is the mth cyclotomic polynomial, and $4\,|\,m$, then $\mathbb{Q}[\sqrt{-1}] \subset E = \mathbb{Q}[e^{2\pi i/m}]$. Hence $\sum_{n\le x} r_2(f(n)) \ll x\log x$.

We give five examples which satisfy the condition in Theorem 3.

EXAMPLE 1. Suppose $f(x) = x^3+ax^2+bx+c$, and its Galois group is S_3 with the discriminant D. Then $\varrho : S_3 \to \mathrm{GL}_2(\mathbb{C})$ is the irreducible 2-dimensional representation. Hence ϱ gives rise to a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Let $L(s,\pi) = \sum_{n=1}^\infty a_n n^{-s}$. Then $N_f(p) = 1+a_p$. In particular, if ϱ is odd, i.e., $D<0$, it comes from a holomorphic cusp form F of weight 1 and level $|D|$. Then $F(z) = \sum_{n=1}^\infty a_n q^n$, $q = e^{2\pi i z}$. In this case, $S(L/\mathbb{Q}) = \{p : a_p = 2,\ (\frac{D}{p}) = 1\}$ and $\sum_{n\le x} r_2(f(n)) \ll x$.

EXAMPLE 2. Let $f(x) = x^4+ax^3+bx^2+cx+d$, and assume its Galois group is S_4 with discriminant D. Here $\varrho : S_4 \to \mathrm{GL}_3(\mathbb{C})$ is one of the two irreducible 3-dimensional representations. There exists a Galois extension $\widetilde{L}/\mathbb{Q}$ such that $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \simeq \mathrm{GL}_2(\mathbb{F}_3)$, and $[\widetilde{L} : L] = 2$. Then $\varrho = \mathrm{Sym}^2(\sigma)$, where σ is the 2-dimensional representation $\sigma : \mathrm{GL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathbb{C})$ (see [Ki2] for the details). Since $\mathrm{GL}_2(\mathbb{F}_3)$ is solvable, by the Langlands–Tunnell theorem, σ gives rise to a cuspidal representation π (if $D<0$, it is odd and it comes from a holomorphic cusp form of weight 1). Let $L(s,\pi) = \sum_{n=1}^\infty b_n n^{-s}$. Then the central character is $\omega_\pi(p) = (\frac{p}{D})$. Then ϱ gives rise to the Gelbart–Jacquet lift $\mathrm{Sym}^2(\pi)$ and $a_p = b_p^2 - \omega_\pi(p)$. Hence $N_f(p) = 1 + b_p^2 - (\frac{p}{D})$. Since σ is not of dihedral type, $\mathrm{Sym}^2(\pi)$ is cuspidal.

In this case, $S(L/\mathbb{Q}) = \{p : a_p = \pm 2,\ (\frac{p}{D}) = 1\}$, and $\sum_{n\le x} r_2(f(n)) \ll x$.

EXAMPLE 3. Let $f(x) = x^4+ax^3+bx^2+cx+d$ with Galois group A_4 and discriminant D. Here $\varrho : S_4 \to \mathrm{GL}_3(\mathbb{C})$ is the irreducible 3-dimensional representation. In this case, there exists a Galois extension $\widetilde{L}/\mathbb{Q}$ such that $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_3)$, and $[\widetilde{L} : L] = 2$. Then $\varrho = \mathrm{Sym}^2(\sigma)$, where σ is the

2-dimensional representation $\sigma : \mathrm{SL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathbb{C})$. This is similar to the S_4 case.

Example 4. Let $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ with Galois group A_5 and discriminant D. Here $\varrho : A_5 \to \mathrm{GL}_4(\mathbb{C})$ is the irreducible 4-dimensional representation. There exists a Galois extension $\widetilde{L}/\mathbb{Q}$ such that $\mathrm{Gal}(\widetilde{L}/\mathbb{Q}) \simeq \mathrm{SL}_2(\mathbb{F}_5)$, and $[\widetilde{L} : L] = 2$. Then $\varrho = \sigma \otimes \sigma^\tau$, where σ is one of the 2-dimensional representations $\sigma : \mathrm{SL}_2(\mathbb{F}_5) \to \mathrm{GL}_2(\mathbb{C})$, and τ is the automorphism $\sqrt{5} \mapsto -\sqrt{5}$ (see [Ki1] for the details). Suppose σ is odd, and it gives rise to a cuspidal representation π which is attached to a holomorphic cusp form of weight 1, $F(z) = \sum_{n=1}^\infty b_n q^n$. Then ϱ gives rise to the functorial product $\pi \boxtimes \pi^\tau$ (see [R1]), and $a_p = b_p b_p^\tau$. Hence $N_f(p) = 1 + b_p b_p^\tau$.

R. Taylor [T] proved infinitely many cases of modularity of odd icosahedral Galois representations. In particular, the following quintic polynomials give rise to holomorphic cusp forms of weight 1:

$$\begin{aligned}
&x^5 + 2x^4 + 6x^3 + 8x^2 + 10x + 8,\\
&x^5 + 6x^4 + x^3 + 4x^2 - 24x + 32,\\
&x^5 - 2x^3 + 2x^2 + 5x + 6,\\
&x^5 + 5x^4 + 8x^3 - 20x^2 - 21x - 5.
\end{aligned}$$

In this case, $S(L/\mathbb{Q}) = \{p : b_p = \pm 2\}$, and $\sum_{n \le x} r_2(f(n)) \ll x$.

Example 5 ([F2]). Let $f(x) = x^4 - m$, where m is a positive integer which is not a square. Then $L = \mathbb{Q}[\sqrt{-1}, m^{1/4}]$. Let $E = \mathbb{Q}[m^{1/4}]$. Then we can show that $\mathrm{Ind}_H^G 1$ is the direct sum of the trivial character, one non-trivial 1-dimensional character χ and the unique 2-dimensional irreducible representation ϱ. Then ϱ gives rise to a holomorphic cusp form F of weight 1. Let $F(z) = \sum_{n=1}^\infty a_n q^n$, $q = e^{2\pi i z}$, and assume χ gives rise to a Dirichlet character $\chi(p) = \left(\frac{m_0}{p}\right)$, where m_0 is the square-free part of m. Then $N_f(p) = 1 + \left(\frac{m_0}{p}\right) + a_p$. In this case, $S(L/\mathbb{Q}) = \left\{p : a_p = 2, \left(\frac{m_0}{p}\right) = 1\right\}$, and $\sum_{n \le x} r_2(f(n)) \ll x$.

3. The sum $\sum_{n\le x} |b(f(n))|^2$. Let $\pi' = \bigotimes_p \pi'_p$ be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, and let $L(s, \pi') = \prod_p L(s, \pi'_p) = \sum_{n=1}^\infty b(n) n^{-s}$ (without the Γ-factor). Let f be as in Section 1. We assume that $L(s, \pi \times \mathrm{Ad}(\pi'))$ is holomorphic at $s = 1$, where π corresponds to ϱ. In particular, it is the case if π' is attached to a holomorphic cusp form of weight $k \ge 2$. Note that $L(s, \pi \times \mathrm{Ad}(\pi'))$ has a pole at $s = 1$ if and only if $\pi \simeq \mathrm{Ad}(\pi')$. So it is very rare.

We are interested in the sum

$$\sum_{n\le x}|b(f(n))|^2.$$

If the Ramanujan conjecture holds, then $|b(n)|\le d(n)$, where $d(n)$ is the number of divisors of n, and we know that

$$\sum_{n\le x}d(f(n))^2\ll x(\log x)^3.$$

(We can obtain this from Lemma 2 by observing that $\zeta(s)^2=\sum_{n=1}^{\infty}d(n)/n^s$. Namely, $d(n)$ is the Fourier coefficients for the automorphic representation $\pi'=1\boxplus 1$; $L(s,\pi')=\zeta(s)^2$. Then $L(s,\pi'\times\pi')$ has a pole of order 4 at $s=1$. Hence $\sum_{p\le x}d(p)^2/p=4\log\log x+O(1)$ and $\sum_{p\le x}a_p d(p)^2/p=O(1)$.)

This gives a trivial estimate (see [F3] for the details)

$$\sum_{n\le x}|b(f(n))|^2\ll x(\log x)^3.$$

We would like to obtain a better estimate. Furthermore, we do not assume the Ramanujan conjecture for π'.

THEOREM 4. *Let f,ϱ be as in Section 1. Suppose we have the strong Artin conjecture for $L(s,\varrho)$, and that $L(s,\pi\times\mathrm{Ad}(\pi'))$ is holomorphic at $s=1$, where π corresponds to ϱ. Then*

$$\sum_{n\le x}|b(f(n))|^2\ll x.$$

Proof. We follow [F3]. By Lemma 2, we need to compute

$$\sum_{p\le x}\frac{N_f(p)(|b(p)|^2-1)}{p}.$$

Here we have removed finitely many primes p where π_p or π'_p is not spherical. Since $N_f(p)=1+a_p$, we need to consider $\sum_{p\le x}|b(p)|^2/p$ and $\sum_{p\le x}a_p|b(p)|^2/p$.

Since $L(s,\pi'\times\widetilde{\pi}')$ has a simple pole at $s=1$, we have $\sum_{p\le x}|b(p)|^2/p=\log\log x+O(1)$. Since the triple product L-function $L(s,\pi\times\pi'\times\widetilde{\pi}')=L(s,\pi\times\mathrm{Ad}(\pi'))L(s,\pi)$ is holomorphic at $s=1$, we have $\sum_{p\le x}a_p|b(p)|^2/p=O(1)$. Since $\sum_{p\le x}1/p=\log\log x+O(1)$ we get

$$\sum_{p\le x}\frac{N_f(p)(|b(p)|^2-1)}{p}=O(1). \blacksquare$$

We have the result unconditionally for the polynomials in the examples in Section 2.

4. Distribution of values of $N_f(n)$. Let f, ϱ be as in Section 1. In this section we do not need to assume the strong Artin conjecture for $L(s,\varrho)$. We are interested in the quantities

$$\sum_{n\le x} N_f(n), \quad \sum_{p\le x} N_f(p).$$

Erdős proved (see [F1] for the precise reference)

$$\sum_{p\le x} N_f(p) = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \qquad \sum_{p\le x} \frac{N_f(p)}{p} = \log\log x + c(f) + o(1).$$

One can also show (see [F1] for the details)

$$\sum_{n\le x} N_f(n) = C(f)x + O\left(\frac{x}{(\log x)^{1/2-\varepsilon}}\right),$$

where $C(f) = e^{-\gamma+c(f)}P$. Here γ is the Euler constant and

$$P = \prod_p e^{-N_f(p)/p}\left(1 + \frac{N_f(p)}{p} + \frac{N_f(p^2)}{p^2} + \cdots\right).$$

We would like to obtain a better error term, following [F1]. Consider

$$L(s) = \sum_{n=1}^{\infty} \frac{N_f(n)}{n^s}.$$

Since $N_f(n)$ is multiplicative, we can write

$$L(s) = \prod_p \left(1 + \frac{N_f(p)}{p^s} + \frac{N_f(p^2)}{p^{2s}} + \cdots\right)$$

for $\mathrm{Re}(s) > 1$. Here

$$\zeta_E(s) = \prod_p \left(1 - \frac{N_f(p)}{p^s} + \cdots \pm \frac{1}{p^{ds}}\right)^{-1}.$$

Hence $L(s)/\zeta_E(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1/2$. Hence it is holomorphic and non-vanishing for $\mathrm{Re}(s) > 1/2$. Let $L(s) = \zeta_E(s)A(s)$ for $\mathrm{Re}(s) > 1$, where $A(s)$ is holomorphic and non-vanishing for $\mathrm{Re}(s) > 1/2$. This provides the meromorphic continuation of $L(s)$ to $\mathrm{Re}(s) > 1/2$. Since $\zeta_E(s)$ has a simple pole at $s = 1$, $L(s)$ has a simple pole at $s = 1$. We use Perron's formula:

$$\sum_{n\le x} N_f(n) = \frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \frac{x^s}{s}\, L(s)\, ds + O\left(\frac{x^{1+2\varepsilon}}{T}\right),$$

for any $1 \le T \le x$, where $\alpha = 1+\varepsilon$. We move the integration to the parallel segment with $\mathrm{Re}(s) = 1/2+\varepsilon$. Then

$$\sum_{n\le x} N_f(n) = x\,\mathrm{Res}_{s=1} L(s) + \frac{1}{2\pi i}\int_{1/2+\varepsilon-iT}^{1/2+\varepsilon+iT} \frac{x^s}{s} L(s)\,ds + O\biggl(\frac{x^{1+2\varepsilon}}{T}\biggr).$$

We have the convexity bound for $\zeta_E(s)$ at $\mathrm{Re}(s) = 1/2+\varepsilon$ (see [CN]):

$$|\zeta_E(1/2+\varepsilon+it)| \ll (1+|t|)^{d/4}.$$

Hence

$$\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT} \frac{x^s}{s} L(s)\,ds \ll x^{1/2+\varepsilon}\int_0^T t^{d/4-1}\,dt = O(x^{1/2+\varepsilon}T^{d/4}).$$

Take $T = x^{2/(d+4)}$. Then

$$\sum_{n\le x} N_f(n) = x\,\mathrm{Res}_{s=1} L(s) + O(x^{(d+2)/(d+4)+\varepsilon}).$$

We have proved

THEOREM 5. *Let $f, L(s)$ be as above. Then $L(s)$ has a simple pole at $s=1$, and*

$$\sum_{n\le x} N_f(n) = x\,\mathrm{Res}_{s=1} L(s) + O(x^{(d+2)/(d+4)+\varepsilon}).$$

REMARK. Note that the above error estimate holds even when G is abelian, or $G = S_3$, improving the result in [F1]. We have the convexity bounds for Dirichlet L-functions, namely, $|L(1/2+\varepsilon+it,\chi)| \ll (1+|t|)^{1/6+\varepsilon}$. So if G is abelian, $|\zeta_E(1/2+\varepsilon+it)| \ll (1+|t|)^{d/6+\varepsilon}$. Then the error bound is improved to $O(x^{(d+3)/(d+6)+\varepsilon})$.

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