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Mean square value of exponential sums related to the representation of integers as sums of squares

by

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1. Introduction. The ergodic theory of unipotent flows has proved to be a very useful tool in understanding the distribution of values of quadratic forms at integer argument (see [6]–[9] and references therein). In the present paper we use the approach developed in [8], [9] to calculate the mean square value of the exponential sums

$$r_{\boldsymbol{\alpha}}(\mu) = \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^k \\ \|\boldsymbol{m}\|^2=\mu}} e(\boldsymbol{m}\cdot\boldsymbol{\alpha}), \tag{1.1}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^k$ is fixed, $\mu \in \mathbb{Z}_+$, $e(t) := \exp(2\pi \mathrm{i} t)$, and $\|\cdot\|$ denotes the usual euclidean norm

$$\|\boldsymbol{m}\|^2 = m_1^2 + \cdots + m_k^2, \qquad \boldsymbol{m} = (m_1, \ldots, m_k). \tag{1.2}$$

The above sums were studied by Bleher, Cheng, Dyson and Lebowitz [2], Bleher and Dyson [3]–[5] and Bleher and Bourgain [1] in connection with the fluctuations of the number of lattice points inside a large sphere centered at $\boldsymbol{\alpha}$.

For $\boldsymbol{\alpha} = \mathbf{0}$ the sum (1.1) represents the number of ways of writing the integer μ as a sum of k squares. We are here interested in the behaviour of $r_{\boldsymbol{\alpha}}(\mu)$ for generic choices of $\boldsymbol{\alpha}$, which satisfy the following diophantine condition: a vector $\boldsymbol{\alpha} \in \mathbb{R}^k$ is called *diophantine* if there exist constants $\kappa, C > 0$ such that

$$\left|\boldsymbol{\alpha} + \frac{\boldsymbol{m}}{q}\right| > \frac{C}{q^{\kappa}} \tag{1.3}$$

for all $\boldsymbol{m} \in \mathbb{Z}^k$, $q \in \mathbb{Z}$, $q > 0$. Here $|\cdot|$ denotes the maximum norm on $\mathbb{R}^k$. The constant κ is called the *type* of $\boldsymbol{\alpha}$. The smallest possible value for κ is

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$\kappa = 1 + 1/k$; in this case $\boldsymbol{\alpha}$ is called *badly approximable* [13]. The set of all diophantine vectors is of full Lebesgue measure [13, Th. 6G].

We assume throughout this paper that $k \geq 2$.

THEOREM 1.1. *Assume $\boldsymbol{\alpha} \in \mathbb{R}^k$ is such that the components of $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Then*

$$\liminf_{M \to \infty} \frac{1}{M^{k/2}} \sum_{\mu=0}^{M} |r_{\boldsymbol{\alpha}}(\mu)|^2 \geq B_k, \tag{1.4}$$

where B_k is the volume of the k-dimensional unit ball. If, in addition, $\boldsymbol{\alpha}$ is diophantine of type $\kappa < (k-1)/(k-2)$, then

$$\lim_{M \to \infty} \frac{1}{M^{k/2}} \sum_{\mu=0}^{M} |r_{\boldsymbol{\alpha}}(\mu)|^2 = B_k. \tag{1.5}$$

The above statement also holds for $k = 1$, in fact without the diophantine condition, since

$$\begin{aligned} \frac{1}{\sqrt{M}} \sum_{\mu=1}^{M} |r_{\boldsymbol{\alpha}}(\mu)|^2 &= \frac{4}{\sqrt{M}} \sum_{0 < m \leq \sqrt{M}} \cos^2(2\pi m \boldsymbol{\alpha}) \\ &\to 4 \int_0^1 \cos^2(2\pi x)\, dx = 2 = B_1 \end{aligned} \tag{1.6}$$

holds for every irrational $\boldsymbol{\alpha} \in \mathbb{R}$ in the limit $M \to \infty$. (This follows directly from the equidistribution of the sequence $m\boldsymbol{\alpha}$ modulo one.) For $k \geq 2$ the diophantine conditions are indeed necessary, since the mean square value diverges for every rational $\boldsymbol{\alpha} \in \mathbb{Q}^k$ (unlike in the case $k = 1$); compare the discussion in [1]. Hence if $\boldsymbol{\alpha}$ is sufficiently well approximable by rationals, (1.5) fails.

Theorem 1.1 is proved by Bleher and Dyson [3] for $k = 2$. In the case $k > 2$, Bleher and Bourgain [1] obtain the bound

$$1 \ll \frac{1}{M^{k/2}} \sum_{\mu=0}^{M} |r_{\boldsymbol{\alpha}}(\mu)|^2 \ll M^{\varepsilon} \tag{1.7}$$

for any $\varepsilon \geq 0$, provided $\boldsymbol{\alpha}$ satisfies the diophantine condition

$$\Big(\prod_{j=1}^{k} |m_j|_+ \Big)^{1+\varepsilon} |\boldsymbol{m} \cdot \boldsymbol{\alpha} + p| > C \tag{1.8}$$

for some constant $C > 0$, and all $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, $p \in \mathbb{Z}$, where $|x|_+ := \max(1, |x|)$. Vectors $\boldsymbol{\alpha}$ satisfying such a diophantine condition are called *multiplicatively diophantine*. An equivalent characterization of the set of multiplicatively diophantine vectors $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is (cf. [15, p. 69]):

there exist constants $\varepsilon \geq 0$, $C > 0$ such that

$$q^{1+\varepsilon} \prod_{j=1}^{k} |q\alpha_j + m_j| > C \tag{1.9}$$

for all $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, $q \in \mathbb{Z}$, $q > 0$. Comparing this with (1.3) (set $\varepsilon = k(\kappa - 1) - 1$), it is evident that the set of multiplicatively diophantine vectors is contained in the set of diophantine vectors, and hence Theorem 1.1 tightens estimate (1.7). According to Littlewood's conjecture [10], it is expected that for $k \geq 2$ there are no multiplicatively badly approximable numbers, i.e., there are no $\boldsymbol{\alpha} \in \mathbb{R}^k$ which satisfy (1.8) or (1.9) for $\varepsilon = 0$ and some $C > 0$.

Our method is in principle also capable of evaluating the mean square value when the components of $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are *not* linearly independent over $\mathbb{Q}$, provided $\boldsymbol{\alpha}$ is still diophantine of type $\kappa < (k-1)/(k-2)$; compare the discussion in [8, App. A]. Note, however, that the limit is not necessarily equal to B_k.

Theorem 1.2 below is concerned with correlations between exponential sums $r_{\boldsymbol{\alpha}}(\mu)$ at different values of the argument. For technical reasons we average with smoothed cutoff functions $\psi \in \mathcal{S}(\mathbb{R}_+)$, i.e., infinitely differentiable functions $\mathbb{R}_+ := [0, \infty) \to \mathbb{C}$ which, together with their derivatives, decay rapidly at ∞. An example for a function in $\mathcal{S}(\mathbb{R}_+)$ is $\psi(t) = \exp(-t)$.

THEOREM 1.2. *Assume the components of $(\boldsymbol{\alpha}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$ and $\boldsymbol{\alpha}$ is diophantine of type $\kappa < (k-1)/(k-2)$. Let $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}_+)$, and $\Delta(\mu)$ be the Fourier coefficients of a piecewise continuous function $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$. Then*

$$\lim_{M \to \infty} \frac{1}{M^{k/2}} \sum_{\mu_1, \mu_2 = 0}^{\infty} \psi_1\left(\frac{\mu_1}{M}\right) \psi_2\left(\frac{\mu_2}{M}\right) r_{\boldsymbol{\alpha}}(\mu_1)\, \overline{r_{\boldsymbol{\alpha}}(\mu_2)} \Delta(\mu_1 - \mu_2) = \frac{k}{2}\, B_k\, \Delta(0) \int_0^\infty \psi_1(r)\psi_2(r) r^{k/2-1}\, dr. \tag{1.10}$$

Theorems 1.1 and 1.2 are proved in Section 6.

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2. Theta sums. The *Jacobi theta sum* Θ_f is defined for a given Schwartz function $f \in \mathcal{S}(\mathbb{R}^k)$ by

$$\Theta_f(\tau, \phi; \boldsymbol{\xi}) = v^{k/4} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f_\phi((\boldsymbol{m} - \boldsymbol{y})v^{1/2}) e\big(\tfrac{1}{2}\|\boldsymbol{m} - \boldsymbol{y}\|^2 u + \boldsymbol{m} \cdot \boldsymbol{x}\big), \tag{2.1}$$

where

$$\tau = u + \mathrm{i}v \in \mathfrak{H}, \qquad \phi \in [0, 2\pi), \qquad \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{2k}, \tag{2.2}$$

and $\mathfrak{H}$ denotes the upper half-plane $\mathfrak{H} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$. Furthermore, the family of functions f_ϕ is defined by

$$f_\phi(\boldsymbol{w}) = \int_{\mathbb{R}^k} G_\phi(\boldsymbol{w}, \boldsymbol{w}') f(\boldsymbol{w}')\, dw', \tag{2.3}$$

with the integral kernel

$$G_\phi(\boldsymbol{w}, \boldsymbol{w}') = e(-k\sigma_\phi/8) |\sin \phi|^{-k/2} e\left[\frac{\frac{1}{2}(\|\boldsymbol{w}\|^2 + \|\boldsymbol{w}'\|^2) \cos \phi - \boldsymbol{w} \cdot \boldsymbol{w}'}{\sin \phi} \right], \tag{2.4}$$

where $\sigma_\phi = 2\nu + 1$ if $\nu\pi < \phi < (\nu + 1)\pi$, $\nu \in \mathbb{Z}$. The operators $U^\phi : f \mapsto f_\phi$ are unitary. Note in particular $U^0 = \mathrm{id}$. The functions f_ϕ are decaying rapidly for large argument, uniformly in ϕ, that is, for any $R > 1$, there is a constant c_R such that for all $\boldsymbol{w} \in \mathbb{R}^k$, $\phi \in \mathbb{R}$, we have

$$|f_\phi(\boldsymbol{w})| \le c_R (1 + \|\boldsymbol{w}\|)^{-R}; \tag{2.5}$$

see [8, Lem. 4.3].

If $f, g \in \mathcal{S}(\mathbb{R}_+)$, the function $\Theta_f \overline{\Theta_g}$ can be realized as a smooth function on a homogeneous space $\Gamma^k \backslash G^k$ of finite measure; see Sections 3 and 4 in [8] for more details. Here G^k is the semi-direct product group $G^k = \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^{2k}$ with multiplication law

$$(M; \boldsymbol{\xi})(M'; \boldsymbol{\xi}') = (MM'; \boldsymbol{\xi} + M\boldsymbol{\xi}'), \tag{2.6}$$

where $M, M' \in \mathrm{SL}(2, \mathbb{R})$ and $\boldsymbol{\xi}, \boldsymbol{\xi}' \in \mathbb{R}^{2k}$; the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^{2k}$ is defined canonically as

$$M\boldsymbol{\xi} = \begin{pmatrix} a\boldsymbol{x} + b\boldsymbol{y} \\ c\boldsymbol{x} + d\boldsymbol{y} \end{pmatrix}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}, \tag{2.7}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^k$. The parametrization of $\mathrm{SL}(2, \mathbb{R})$ in terms of the variable (τ, ϕ) used in the definition of Θ_f is obtained by means of the Iwasawa decomposition

$$M = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \tag{2.8}$$

which is unique for $\tau = u + \mathrm{i}v \in \mathfrak{H}$, $\phi \in [0, 2\pi)$.

The relevant discrete subgroup is defined as

$$\Gamma^k = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} ab\boldsymbol{s} \\ cd\boldsymbol{s} \end{pmatrix} + \boldsymbol{m} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}),\ \boldsymbol{m} \in \mathbb{Z}^{2k} \right\} \subset G^k, \tag{2.9}$$

with $\boldsymbol{s} = (1/2,\dots,1/2) \in \mathbb{R}^k$. We shall later make use of the fact that Γ^k is of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes \left(\frac{1}{2}\mathbb{Z}\right)^{2k}$. The left action of the group Γ^k on G^k is properly discontinuous. A fundamental domain of Γ^k in G^k is given by

$$\mathcal{F}_{\Gamma^k} = \mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})} \times \{\phi \in [0,\pi)\} \times \{\boldsymbol{\xi} \in [-1/2,1/2)^{2k}\}, \tag{2.10}$$

where $\mathcal{F}_{\mathrm{SL}(2,\mathbb{Z})}$ is the fundamental domain in $\mathfrak{H}$ of the modular group $\mathrm{SL}(2,\mathbb{Z})$, given by $\{\tau \in \mathfrak{H} : u \in [-1/2,1/2),\ |\tau| > 1\}$. The space $\Gamma^k\backslash G^k$ is noncompact, and $\Theta_f\overline{\Theta_g}$ is in fact unbounded. The following proposition controls the behaviour in the cusps [8, Prop. 4.10].

PROPOSITION 2.1. *Let $f, g \in \mathcal{S}(\mathbb{R}^k)$. For any $R > 1$, we have*

$$\Theta_f\left(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right)\overline{\Theta_g\left(\tau,\phi;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right)} = v^{k/2} f_\phi(-\boldsymbol{y}v^{1/2})\overline{g_\phi(-\boldsymbol{y}v^{1/2})} + O_R(v^{-R}), \tag{2.11}$$

uniformly for all $(\tau,\phi;\boldsymbol{\xi}) \in \mathcal{F}_{\Gamma^k}$.

3. Equidistribution of closed orbits. Let Γ be a lattice in G^k. The unipotent flow Ψ^t on the homogeneous space $\Gamma\backslash G^k$ is defined as right translation by

$$\Psi_0^t = \left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix};\boldsymbol{0}\right), \tag{3.1}$$

i.e., $\Psi^t(g) = g\Psi_0^t$, and the partially hyperbolic flow Φ^t as right translation by

$$\Phi_0^t = \left(\begin{pmatrix}\mathrm{e}^{-t/2} & 0\\ 0 & \mathrm{e}^{t/2}\end{pmatrix};\boldsymbol{0}\right), \tag{3.2}$$

i.e., $\Phi^t(g) = g\Phi_0^t$.

Let us assume for a moment that $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$. Then, for

$$g_0 = \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right), \quad \boldsymbol{x} \in \mathbb{R}^k, \tag{3.3}$$

we have

$$\Phi^t \circ \Psi^{u+1}(\Gamma g_0) = \Phi^t \circ \Psi^u(\Gamma g_0), \tag{3.4}$$

since g_0 commutes with Ψ_0, and $\Psi_0 \in \Gamma$. Hence $\Omega_t = \{\Phi^t \circ \Psi^u(\Gamma g_0) : u \in [0,1)\}$ represents a closed orbit for every $t \in \mathbb{R}$.

If Γ is a subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$, the manifold $\Gamma\backslash G^k$ is a finite covering of $(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k$. Therefore the orbit

$\Omega_t \subset (\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k$ lifts to a closed orbit $\widetilde{\Omega}_t = \{\Phi^t \circ \Psi^u(\Gamma g_0) : u \in [0,r)\}$ in $\Gamma\backslash G^k$, for a suitable integer $r = r(\Gamma) \geq 1$. In Theorem 3.1 we will show that $\widetilde{\Omega}_t$ becomes equidistributed as $t \to \infty$. This result may be viewed as a special case of Theorem 1.4 by Shah [14] which is based on Ratner's classification of measures invariant under a unipotent flow. We give a more elementary proof, which exploits the simple arithmetic nature of the lattice $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$, but still relies on Ratner's theory.

THEOREM 3.1. *Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ of finite index, and set $r = r(\Gamma)$. Fix some point*

$$g_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{pmatrix}\right) \in G^k \tag{3.5}$$

such that the components of the vector $({}^{\mathrm{t}}\boldsymbol{x}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Let h be a piecewise continuous function $\mathbb{R}/r\mathbb{Z} \to \mathbb{C}$. Then, for any bounded continuous function F on $\Gamma\backslash G^k$, we have

$$\lim_{t\to\infty} \frac{1}{r}\int_0^r F \circ \Phi^t \circ \Psi^u(\Gamma g_0) h(u)\, du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\, d\mu\, \frac{1}{r}\int_0^r h(u)\, du \tag{3.6}$$

where μ is the Haar measure of G^k.

Proof. Due to the linearity of the above expressions in h, we can assume without loss of generality that h is a probability density. Then

$$\varrho_t(F) = \frac{1}{r}\int_0^r F \circ \Phi^t \circ \Psi^u(\Gamma g_0) h(u)\, du \tag{3.7}$$

defines a family of probability measures for bounded continuous functions F on $\Gamma\backslash G^k$. Following the proof of [8, Prop. 5.4] one shows that the family of probability measures $\{\varrho_t : t \geq 0\}$ is relatively compact, that is, every sequence contains a subsequence which weakly converges to a probability measure on $\Gamma\backslash G^k$. Furthermore every limiting measure is invariant under the unipotent flow Ψ^u (compare the proof of [8, Prop. 5.5]). The most obvious invariant measure is of course the suitably normalized Haar measure μ. Ratner's theory [11], [12] yields that all other ergodic invariant measures are localized on smooth embedded subvarieties. (A detailed description of the relevant measures in the case of $(\Gamma\backslash G^k, \Psi^t)$ can be found in [8].) These measures are, however, excluded as possible limits by the following lemma (compare the analogous argument in [8]). ∎

LEMMA 3.2. *Under the assumptions of Theorem* 3.1 *with $h \geq 0$ we have, for any continuous function $F : \Gamma\backslash G^k \to \mathbb{R}_+$ with compact support*,

$$\limsup_{t\to\infty} \frac{1}{r}\int_0^r F \circ \Phi^t \circ \Psi^u(g_0)h(u)\,du \leq \frac{\max(h)}{2} \int_{\Gamma\backslash G^k} F\,d\mu. \tag{3.8}$$

Let us first consider the special test function

$$F_\square(M;\boldsymbol{\xi}) = \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z})} f(\gamma M)\chi_\square(\gamma\boldsymbol{\xi}), \tag{3.9}$$

with (in the Iwasawa parametrization (2.8))

$$f(M) = f(\tau,\phi) = \chi_1(u + v\cot\phi)\chi_2(v^{-1/2}\cos\phi)\chi_3(v^{-1/2}\sin\phi) \tag{3.10}$$

where χ_j $(j = 1,2,3)$ is the characteristic function of the interval $I_j \subset \mathbb{R}$. The function $\chi_\square : \mathbb{T}^{2k} \to \mathbb{R}$ is the characteristic function of a cube in $\mathbb{T}^{2k}$. Clearly, $F_\square$ may be viewed as a function on $(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k$ and hence on $\Gamma\backslash G^k$.

LEMMA 3.3. *Suppose the components of the vector* $({}^{\mathrm{t}}\boldsymbol{x}, 1) \in \mathbb{R}^{k+1}$ *are linearly independent over* $\mathbb{Q}$. *Then*

$$\limsup_{v\to 0}\int_0^1 F_\square\biggl(u + \mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \mathbf{0}\end{pmatrix}\biggr)\,du \leq |I_1|\,|I_2|\,|I_3|\,\mathrm{Vol}(\square). \tag{3.11}$$

Proof. For $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ we have

$$\begin{aligned} &F_\square\biggl(u + \mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \mathbf{0}\end{pmatrix}\biggr)\\ &\quad= \sum_{\gamma\in\mathrm{SL}(2,\mathbb{Z})} f\biggl(\frac{a(u+\mathrm{i}v)+b}{c(u+\mathrm{i}v)+d}, \arg(c\tau+d)\biggr)\eta_1(a\boldsymbol{x})\eta_2(c\boldsymbol{x}), \end{aligned} \tag{3.12}$$

where η_1, η_2 are the characteristic functions of cubes in $\mathbb{T}^k$. This simplifies to (cf. [8, Sec. 5.10.2])

$$\begin{aligned} &F_\square\biggl(u + \mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \mathbf{0}\end{pmatrix}\biggr)\\ &\quad= \sum_{\substack{\gamma\in\mathrm{SL}(2,\mathbb{Z})\\ c\neq 0}} \chi_1\Bigl(\frac{a}{c}\Bigr)\,\chi_2(v^{-1/2}(cu+d))\chi_3(cv^{1/2})\eta_1(a\boldsymbol{x})\eta_2(c\boldsymbol{x}). \end{aligned} \tag{3.13}$$

Given a, c with $\gcd(a,c) = 1$, we can find a pair (b_0, d_0) such that $ad_0 - b_0c = 1$. All solutions $(b,d) \in \mathbb{Z}^2$ of the equation $ad - bc = 1$ are then given by

$b = b_0 + ma$, $d = d_0 + mc$ where $m \in \mathbb{Z}$. Hence

$$(3.14)\quad F_\square\left(u + \mathrm{i}v, 0; \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{pmatrix}\right) = \sum_{\substack{a,c,m\in\mathbb{Z} \\ \gcd(a,c)=1 \\ c\neq 0}} \chi_1\left(\frac{a}{c}\right)\chi_2\left(v^{-1/2}c\left(u + m + \frac{d_0}{c}\right)\right)\chi_3(cv^{1/2})\eta_1(a\boldsymbol{x})\eta_2(c\boldsymbol{x}).$$

We integrate over u and drop the condition $\gcd(a,c) = 1$; this yields

$$(3.15)\quad \int_0^1 F_\square\left(u + \mathrm{i}v, 0; \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{pmatrix}\right) du \le v^{1/2}|I_2| \sum_{\substack{a,c\in\mathbb{Z} \\ c\neq 0}} \frac{1}{|c|}\chi_1\left(\frac{a}{c}\right)\chi_3(cv^{1/2})\eta_1(a\boldsymbol{x})\eta_2(c\boldsymbol{x}).$$

Terms with $|c| < v^{-1/4}$ are of subleading order and can thus be dropped. For $|c| \ge v^{-1/4}$ we have, by Weyl's equidistribution theorem [16, Satz 4],

$$(3.16)\quad \frac{1}{|c|}\chi_1\left(\frac{a}{c}\right)\eta_1(a\boldsymbol{x}) = |I_1| \int \eta_1 + o(1)$$

as $v \to 0$ where the implied constant is independent of c. Applying Weyl's theorem a second time to the c-sum on the right-hand side of (3.15), we find that the latter converges to $|I_1|\,|I_2|\,|I_3|\,\mathrm{Vol}(\square)$. ∎

Proof of Lemma 3.2. We have

$$(3.17)\quad \int_{(\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k})} F_\square\, d\mu$$
$$= \mathrm{Vol}(\square) \int_0^{2\pi}\int_0^{\infty}\int_{\mathbb{R}} \chi_1(u + v\cot\phi)\chi_2(v^{-1/2}\cos\phi)\chi_3(v^{-1/2}\sin\phi)\,\frac{du\,dv\,d\phi}{v^2}$$
$$= \mathrm{Vol}(\square)|I_1| \int_0^{2\pi}\int_0^{\infty} \chi_2(v^{-1/2}\cos\phi)\chi_3(v^{-1/2}\sin\phi)\,\frac{dv\,d\phi}{v^2}$$
$$= 2\,\mathrm{Vol}(\square)|I_1| \int_0^{2\pi}\int_0^{\infty} \chi_2(r\cos\phi)\chi_3(r\sin\phi)r\,dr\,d\phi$$
$$= 2\,\mathrm{Vol}(\square)|I_1|\,|I_2|\,|I_3|.$$

Hence the statement of Lemma 3.2 holds for $F = F_\square$ and $\Gamma = \mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k}$. If $\widetilde{F} : (\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k})\backslash G^k \to \mathbb{R}_+$ is continuous and has compact support, it can be arbitrarily well approximated from above by finite linear combinations of functions of the type $F_{\square}$. That is, for every $\varepsilon > 0$ there are finitely many cubes $\square_1, \square_2, \ldots$ and positive coefficients $\sigma_1, \sigma_2, \ldots$ such that

$$\widetilde{F} \le \sum_j \sigma_j F_{\square_j}, \qquad \int\limits_{(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k} \Big(\sum_j \sigma_j F_{\square_j} - \widetilde{F} \Big) \, d\mu < \varepsilon. \tag{3.18}$$

So

$$\begin{aligned} &\limsup_{v\to 0} \int\limits_0^1 \widetilde{F}\left(u + \mathrm{i}v, 0; \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{pmatrix}\right) du \\ &\le \limsup_{v\to 0} \int\limits_0^1 \left(\sum_j \sigma_j F_{\square_j}\left(u + \mathrm{i}v, 0; \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{pmatrix}\right)\right) du \\ &\le \frac{1}{2} \int\limits_{(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k} \Big(\sum_j \sigma_j F_{\square_j} \Big) \, d\mu \le \frac{1}{2} \int\limits_{(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k} \widetilde{F} \, d\mu + \frac{\varepsilon}{2} \end{aligned} \tag{3.19}$$

for any $\varepsilon > 0$, i.e.,

$$\limsup_{v\to 0} \int\limits_0^1 \widetilde{F}\left(u + \mathrm{i}v, 0; \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{pmatrix}\right) du \le \frac{1}{2} \int\limits_{(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k} \widetilde{F} \, d\mu. \tag{3.20}$$

To conclude the proof for general $F : \Gamma\backslash G^k \to \mathbb{R}_+$, we note that for $\widetilde{F}$: $(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k \to \mathbb{R}_+$ defined by

$$\widetilde{F}(g) = \sum_{\gamma \in \Gamma\backslash(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})} F(\gamma g) \tag{3.21}$$

we have $F \le \widetilde{F}$. Then

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{r} \int\limits_0^r F \circ \Phi^t \circ \Psi^u(g_0) \, h(u) \, du \\ &\qquad \le \max(h) \limsup_{t\to\infty} \int\limits_0^1 \widetilde{F} \circ \Phi^t \circ \Psi^u(g_0) \, du \\ &\qquad \le \frac{\max(h)}{2} \int\limits_{(\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k})\backslash G^k} \widetilde{F} \, d\mu = \frac{\max(h)}{2} \int\limits_{\Gamma\backslash G^k} F \, d\mu. \ \blacksquare \end{aligned} \tag{3.22}$$

4. Diophantine conditions. The following lemma is the key to extend the equidistribution theorem (Theorem 3.1) to unbounded test functions.

LEMMA 4.1. *Let $\boldsymbol{\alpha}$ be diophantine of type κ, and $f \in \mathrm{C}(\mathbb{R}^k)$ of rapid decay. Then, for any fixed $A > 1$ and $0 < \varepsilon < 1/(\kappa - 1)$,*

$$\sum_{D \le c \le 2D} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f(T(c\boldsymbol{\alpha} + \boldsymbol{m})) \ll \begin{cases} T^{-A} & (D \le T^{\varepsilon}), \\ 1 & (T^{\varepsilon} \le D \le T^{1/(\kappa-1)}), \\ DT^{-1/(\kappa-1)} & (D \ge T^{1/(\kappa-1)}) \end{cases} \tag{4.1}$$

uniformly for all $D > 0$, $T \ge 1$.

Proof. The proof is almost identical to the one of [9, Lem. 6.5]. We divide the sum over c into blocks of the form

$$\sum_{0 \le c \le T^{1/(\kappa-1)}} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f(T((b+c)\boldsymbol{\alpha} + \boldsymbol{m})). \tag{4.2}$$

The number of such blocks is of the order $DT^{-1/(\kappa-1)} + 1$. In view of the diophantine condition on $\boldsymbol{\alpha}$ there is a constant C such that, for all $0 < |q| \le T^{1/(\kappa-1)}$, we have

$$\frac{C}{|q|T} \le \frac{C}{|q|^{\kappa}} \le \left| \boldsymbol{\alpha} + \frac{\boldsymbol{m}}{q} \right|, \tag{4.3}$$

and therefore

$$|q\boldsymbol{\alpha} + \boldsymbol{m}| \ge \frac{C}{T}. \tag{4.4}$$

For b fixed, the minimal distance (with respect to the maximum norm) between the points $(b+c)\boldsymbol{\alpha} + \boldsymbol{m}$ ($0 \le c \le T^{1/(\kappa-1)}$, $\boldsymbol{m} \in \mathbb{Z}^k$) is bounded from below by

$$\min_{\substack{0<|q|\le T^{1/(\kappa-1)} \\ \boldsymbol{m}\in\mathbb{Z}^k}} |q\boldsymbol{\alpha} + \boldsymbol{m}| \ge \frac{C}{T}. \tag{4.5}$$

Any cube with sides of length $1/T$ contains hence at most $(C^{-1}+1)^k$ points. Therefore, with f fixed and rapidly decreasing,

$$\sum_{0 \le c \le T^{1/(\kappa-1)}} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f(T((b+c)\boldsymbol{\alpha} + \boldsymbol{m})) \ll 1, \tag{4.6}$$

independently of b, which proves the second and third bounds. As to the first bound, note that

$$\|c\boldsymbol{\alpha} + \boldsymbol{m}\| \ge |c\boldsymbol{\alpha} + \boldsymbol{m}| \ge \frac{C}{c^{\kappa-1}} \ge \frac{C}{(2D)^{\kappa-1}}, \tag{4.7}$$

which holds for all $c \le 2D$. Since f decreases faster than any inverse polynomial, we have

$$\sum_{D \le c \le 2D} \sum_{\boldsymbol{m} \in \mathbb{Z}^k} f(T(c\boldsymbol{\alpha} + \boldsymbol{m})) \ll D \left(\frac{D^{\kappa-1}}{T} \right)^B \tag{4.8}$$

for any $B > 1$. ■

For $f \in \mathrm{C}(\mathbb{R}^k)$ of rapid decay, $R > 1$ and $\beta \in \mathbb{R}$, let us consider the function

$$F_R(\tau;\boldsymbol{\xi}) = \sum_{\gamma\in\Gamma_\infty\backslash\,\mathrm{SL}(2,\mathbb{Z})} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f((\boldsymbol{y}_\gamma+\boldsymbol{m})v_\gamma^{1/2})\, v_\gamma^\beta\, \chi_{[R,\infty)}(v_\gamma), \tag{4.9}$$

where $\chi_{[R,\infty)}$ is the characteristic function of $[R,\infty)$, and $v_\gamma > 0$, $\boldsymbol{y}_\gamma \in \mathbb{R}^k$ are defined by

$$v_\gamma = \mathrm{Im}(\gamma\tau), \qquad \begin{pmatrix}\boldsymbol{x}_\gamma\\ \boldsymbol{y}_\gamma\end{pmatrix} = \gamma\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}. \tag{4.10}$$

The function F_R is thus, by construction, invariant under $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$. Further properties of F_R are discussed in [9, Sec. 6]. In particular, we later use the formula

$$\int\limits_{\Gamma\backslash G} F_R\, d\mu = 2\pi\, \frac{R^{-(3/2-\beta)}}{3/2-\beta} \int\limits_{\mathbb{R}} f(w)\, dw \tag{4.11}$$

in the case $\beta = 1$. We assume in the following that $f \ge 0$.

Proposition 4.2. *Let $\boldsymbol{x}$ be diophantine of type κ, and set $\beta = k/2$. Then, for any $\varepsilon < 1/(\kappa-1)-(k-2)$,*

$$\limsup_{v\to 0} \int\limits_0^1 F_R\left(u+\mathrm{i}v; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right) du \ll_\varepsilon R^{-\varepsilon/2}. \tag{4.12}$$

Proof. Assume without loss of generality that (a) f is even, and that (b) for any $r \ge 1$ we have $f(r\boldsymbol{x}) \le f(\boldsymbol{x})$. Property (a) implies

$$F_R(\tau;\boldsymbol{\xi}) = 2\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f((\boldsymbol{y}+\boldsymbol{m})v^{1/2})v^\beta\chi_{[R,\infty)}(v) \tag{4.13}$$

$$+2\sum_{\substack{(c,d)\in\mathbb{Z}^2\\ \gcd(c,d)=1\\ c>0}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\left((c\boldsymbol{x}+d\boldsymbol{y}+\boldsymbol{m})\, \frac{v^{1/2}}{|c\tau+d|}\right) \frac{v^\beta}{|c\tau+d|^{2\beta}}\, \chi_{[R,\infty)}\left(\frac{v}{|c\tau+d|^2}\right).$$

We are interested in the average

$$\int\limits_0^1 F_R\left(u+\mathrm{i}v; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right) du = 2\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(\boldsymbol{m}v^{1/2})v^\beta\chi_{[R,\infty)}(v) \tag{4.14}$$

$$+ 2v^{1-\beta}\sum_{c=1}^\infty \frac{\tau(c)}{c^{2\beta}} \int\limits_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\left((c\boldsymbol{x}+\boldsymbol{m})\, \frac{1}{c\sqrt{v(t^2+1)}}\right)$$

$$\times\, \chi_{[\sqrt{R},\infty)}\left(\frac{1}{c\sqrt{v(t^2+1)}}\right) \frac{dt}{(t^2+1)^\beta}$$

where

$$\tau(c) = \sum_{\substack{d=0 \\ \gcd(c,d)=1}}^{c-1} 1 \leq c. \tag{4.15}$$

The first term on the right-hand side of (4.14) is identically zero for $v < R$. For the second term we introduce a dyadic covering of $[\sqrt{R}, \infty)$ by the set $\bigcup_j [2^j, 2^{j+1})$, with $j \in \mathbb{Z}$ and $2^{j+1} \geq \sqrt{R}$. Hence an upper bound for (4.14) is obtained by summing over j the expression (up to a factor of 2)

$$\begin{aligned}
(4.16)\qquad & v^{1-k/2} \sum_{c=1}^{\infty} \frac{1}{c^{k-1}} \int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f\left((c\boldsymbol{x}+\boldsymbol{m}) \frac{1}{c\sqrt{v(t^2+1)}} \right) \\
&\qquad \times \chi_{[2^j,2^{j+1})}\left(\frac{1}{c\sqrt{v(t^2+1)}} \right) \frac{dt}{(t^2+1)^{k/2}} \\
&\leq v^{1-k/2} \sum_{c=1}^{\infty} \frac{1}{c^{k-1}} \int_{\mathbb{R}} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(2^j(c\boldsymbol{x}+\boldsymbol{m})) \\
&\qquad \times \chi_{[2^j,2^{j+1})}\left(\frac{1}{c\sqrt{v(t^2+1)}} \right) \frac{dt}{(t^2+1)^{k/2}} \\
&\leq 2^{(j+1)(k-1)} v^{1/2} \int_{\mathbb{R}} \bigg\{ \sum_{c=1}^{\infty} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(2^j(c\boldsymbol{x}+\boldsymbol{m})) \\
&\qquad \times \chi_{[2^j,2^{j+1})}\left(\frac{1}{c\sqrt{v(t^2+1)}} \right) \bigg\} \frac{dt}{\sqrt{t^2+1}}
\end{aligned}$$

where we have set $\beta = k/2$, and used property (b) in the first inequality. Comparing this with the expressions in Lemma 4.1 suggests

$$D = \frac{2^{-(j+1)}}{\sqrt{v(t^2+1)}}, \qquad T = 2^j. \tag{4.17}$$

Note that the range of integration is always restricted to $t^2 + 1 \leq v^{-1}$ since $2^j c \geq 1$. Lemma 4.1 yields now in the first domain ($D \leq T^\varepsilon$)

$$\begin{aligned}
(4.18)\qquad & 2^{(j+1)(k-1)} v^{1/2} \int_{D\leq T^\varepsilon} \bigg\{ \sum_{D\leq c\leq 2D} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(T(c\boldsymbol{x}+\boldsymbol{m})) \bigg\} \frac{dt}{\sqrt{t^2+1}} \\
&\ll 2^{j(k-1-A)} v^{1/2} \int_{t^2+1\leq v^{-1}} \frac{dt}{\sqrt{t^2+1}} \ll 2^{j(k-1-A)} v^{1/2} |\log v|,
\end{aligned}$$

where we choose $A > k-1$. In the second domain ($T^\varepsilon \leq D \leq T^{1/(\kappa-1)}$) the condition $T^\varepsilon \leq D$ implies

$$2^{j(k-1)} \leq \frac{2^{-j(2-k+\varepsilon)}}{2\sqrt{v(t^2+1)}}, \tag{4.19}$$

and so

$$(4.20)\qquad 2^{(j+1)(k-1)}v^{1/2}\int\limits_{D\le T^{\varepsilon}}\Big\{\sum_{D\le c\le 2D}\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(T(c\boldsymbol{x}+\boldsymbol{m}))\Big\}\frac{dt}{\sqrt{t^2+1}} \ll 2^{-j(2-k+\varepsilon)}\int\limits_{\mathbb{R}}\frac{dt}{t^2+1}.$$

We choose ε in such a way that $2-k+\varepsilon>0$ (this is possible since $\varepsilon<1/(\kappa-1)$ and $\kappa<(k-1)/(k-2)$). In the third domain ($D\ge T^{1/(\kappa-1)}$) we have

$$(4.21)\qquad 2^{(j+1)(k-1)}v^{1/2}\int\limits_{D\ge T^{1/(\kappa-1)}}\Big\{\sum_{D\le c\le 2D}\sum_{\boldsymbol{m}\in\mathbb{Z}^k} f(T(c\boldsymbol{x}+\boldsymbol{m}))\Big\}\frac{dt}{\sqrt{t^2+1}} \ll 2^{-j(2-k+1/(\kappa-1))}\int\limits_{\mathbb{R}}\frac{dt}{t^2+1}.$$

Clearly contribution (4.20) from the second domain dominates the other two. Summation over $j\in\mathbb{Z}$ with $2^{j+1}\ge\sqrt{R}$ yields an error $\ll R^{-(2-k+\varepsilon)/2}$. The bound (4.12) is obtained by redefining ε in the obvious way. ∎

5. Equidistribution and unbounded test functions. We say a function F on $\Gamma\backslash G^k$ is *dominated by* F_R if, for some fixed constant $L>1$, we have

$$(5.1)\qquad |F(\tau,\phi;\boldsymbol{\xi})|X_R(\tau)\le L+F_R(\tau;\boldsymbol{\xi})$$

for all sufficiently large $R>1$, uniformly for all $(\tau,\phi;\boldsymbol{\xi})\in G^k$. Here

$$(5.2)\qquad X_R(\tau)=\sum_{\gamma\in\{\Gamma_\infty\cup(-1)\Gamma_\infty\}\backslash\mathrm{SL}(2,\mathbb{Z})}\chi_{[R,\infty)}(v_\gamma).$$

THEOREM 5.1. *Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{Z})\ltimes\mathbb{Z}^{2k}$ of finite index, set $r=r(\Gamma)$, and let $h\ge 0$ be a piecewise continuous function $\mathbb{R}/r\mathbb{Z}\to\mathbb{R}_+$. Fix some $\boldsymbol{x}\in\mathbb{T}^k$ such that the components of the vector $({}^{\mathrm{t}}\boldsymbol{x},1)\in\mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Then, for any continuous function $F\ge 0$ dominated by F_R,*

$$(5.3)\qquad \liminf_{v\to 0}\frac{1}{r}\int_0^r F\Big(u+\mathrm{i}v,0;\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\Big)h(u)\,du \ge \frac{1}{\mu(\Gamma\backslash G^k)}\int\limits_{\Gamma\backslash G^k}F\,d\mu\,\frac{1}{r}\int_0^r h(u)\,du.$$

If $\boldsymbol{x}$ *is diophantine of type* $\kappa < (k-1)/(k-2)$, *then*

$$\limsup_{v\to 0} \frac{1}{r}\int_0^r F\Big(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\Big) h(u)\,du \le \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\,d\mu\,\frac{1}{r}\int_0^r h(u)\,du. \tag{5.4}$$

Proof. The proof follows from the same argument as in [8, Th. 7.3], cf. also [9, Th. 6.8]. We may assume without loss of generality that $r^{-1}\int_0^r h(u)\,du = 1$. For the lower bound define

$$G_R(\tau,\phi;\boldsymbol{\xi}) := F(\tau,\phi;\boldsymbol{\xi})(1-X_R(\tau)) \le F(\tau,\phi;\boldsymbol{\xi}), \tag{5.5}$$

which is a bounded function. Therefore by Theorem 3.1 (we may ignore the fact that G_R is only piecewise continuous, cf. the footnote on [8, p. 454]),

$$\lim_{v\to 0}\frac{1}{r}\int_0^r G_R(u+\mathrm{i}v,0;\boldsymbol{\xi})h(u)\,du = \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} G_R\,d\mu. \tag{5.6}$$

Because $0 \le FX_R \le LX_R + F_R$ for R sufficiently large,

$$\int_{\Gamma\backslash G^k} FX_R\,d\mu \le \int_{\Gamma\backslash G^k} (LX_R+F_R)\,d\mu \ll LR^{-1} + R^{-1/2} \tag{5.7}$$

from (4.11) and a similar formula for the integral over X_R, and thus

$$\int_{\Gamma\backslash G^k} G_R\,d\mu = \int_{\Gamma\backslash G^k} F\,d\mu + O(LR^{-1}+R^{-1/2}), \tag{5.8}$$

which yields

$$\liminf_{v\to 0}\frac{1}{r}\int_0^r F(u+\mathrm{i}v,0;\boldsymbol{\xi})h(u)\,du \ge \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} F\,d\mu + O(R^{-1/2}), \tag{5.9}$$

for all R large enough. This proves the lower bound since R can be chosen arbitrarily large.

Let us now turn to the upper bound. For R large enough,

$$F(\tau,\phi;\boldsymbol{\xi}) \le F(\tau,\phi;\boldsymbol{\xi})(1-X_R(\tau)) + LX_R(\tau) + F_R(\tau;\boldsymbol{\xi}). \tag{5.10}$$

In view of the lower bound and Proposition 4.2, we find that

$$\limsup_{v\to 0}\frac{1}{r}\int_0^r F(u+\mathrm{i}v,0;\boldsymbol{\xi})h(u)\,du \le \frac{1}{\mu(\Gamma\backslash G^k)}\int_{\Gamma\backslash G^k} F\,d\mu + O(R^{-1/2}) + O(R^{-\eta}) \tag{5.11}$$

for some small constant $\eta > 0$, which holds for arbitrarily large R. This concludes the proof. ∎

The above theorem can easily be rephrased for functions F which are invariant under a subgroup of $\mathrm{SL}(2,\mathbb{Z}) \ltimes \left(\frac{1}{2}\mathbb{Z}\right)^{2k}$ rather than $\mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^{2k}$ (compare the proof of [8, Cor. 7.6]). The special choice $F = \Theta_f \overline{\Theta_g}$ then leads to the following corollary.

COROLLARY 5.2. *Suppose that $f(\boldsymbol{w}) = \psi(\|\boldsymbol{w}\|^2)$ with $\psi \in \mathcal{S}(\mathbb{R}_+)$ real-valued, and let $h \geq 0$ be a piecewise continuous function $\mathbb{R}/2\mathbb{Z} \to \mathbb{R}_+$. Fix some $\boldsymbol{x} \in \mathbb{T}^k$ such that the components of the vector $({}^{\mathrm{t}}\boldsymbol{x}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$. Then*

$$(5.12)\qquad \liminf_{v\to 0} \frac{1}{2}\int_0^2 \left|\Theta_f\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right)\right|^2 h(u)\,du \geq \frac{k}{2} B_k \int_0^\infty \psi(r)^2 r^{k/2-1}\,dr\, \frac{1}{2}\int_0^2 h(u)\,du.$$

If $\boldsymbol{x}$ is diophantine of type $\kappa < (k-1)/(k-2)$, then

$$(5.13)\qquad \limsup_{v\to 0} \frac{1}{2}\int_0^2 \left|\Theta_f\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right)\right|^2 h(u)\,du \leq \frac{k}{2} B_k \int_0^\infty \psi(r)^2 r^{k/2-1}\,dr\, \frac{1}{2}\int_0^2 h(u)\,du.$$

Proof. Apply Theorem 5.1. Proposition 2.1 ensures that $\Theta_f \overline{\Theta_g}$ is dominated by F_R (for a suitable choice of f in the definition of F_R). The right-hand sides of (5.12) and (5.13) follow from the identity ([9, Lem. 7.2])

$$(5.14)\qquad \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} \Theta_f \overline{\Theta_g}\,d\mu = \frac{k}{2} B_k \int_0^\infty \psi_1(r)\psi_2(r) r^{k/2-1}\,dr$$

for $f(\boldsymbol{w}) = \psi_1(\|\boldsymbol{w}\|^2)$ and $g(\boldsymbol{w}) = \psi_2(\|\boldsymbol{w}\|^2)$. ∎

COROLLARY 5.3. *Assume Γ, r, h, $\boldsymbol{x}$ are as in Theorem 5.1. If $\boldsymbol{x}$ is diophantine of type $\kappa < (k-1)/(k-2)$, then, for any continuous function $F : \Gamma\backslash G^k \to \mathbb{C}$ dominated by F_R,*

$$(5.15)\qquad \lim_{v\to 0} v^{k/2-1} \frac{1}{r}\int_0^r F\left(u+\mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right) h(u)\,du = \frac{1}{\mu(\Gamma\backslash G^k)} \int_{\Gamma\backslash G^k} F\,d\mu\, \frac{1}{r}\int_0^r h(u)\,du.$$

Proof. Compare the proof of [8, Cor. 7.4]. ∎

COROLLARY 5.4. *Suppose that $f(\boldsymbol{w}) = \psi(\|\boldsymbol{w}\|^2)$ with $\psi \in \mathcal{S}(\mathbb{R}_+)$ real-valued, and let $h \geq 0$ be a piecewise continuous function $\mathbb{R}/2\mathbb{Z} \to \mathbb{R}_+$. Fix some $\boldsymbol{x} \in \mathbb{T}^k$ such that the components of the vector $({}^{\mathrm{t}}\boldsymbol{x}, 1) \in \mathbb{R}^{k+1}$ are linearly independent over $\mathbb{Q}$ and that $\boldsymbol{x}$ is diophantine of type $\kappa < (k-1)/(k-2)$. Then*

$$\lim_{v\to 0} \frac{1}{2}\int_0^2 \Theta_f\left(u + \mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right)\overline{\Theta_g\left(u + \mathrm{i}v, 0; \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{0}\end{pmatrix}\right)} h(u)\, du = \frac{k}{2} B_k \int_0^\infty \psi_1(r)\psi_2(r) r^{k/2-1}\, dr\, \frac{1}{2}\int_0^2 h(u)\, du. \tag{5.16}$$

Proof. Compare the proof of [8, Cor. 7.5]. ∎

The main results of this paper, Theorems 1.1 and 1.2, now follow immediately from the above Corollaries 5.2 and 5.4, respectively.

6. Proof of the main theorems

Proof of Theorem 1.1. Corollary 5.2 yields, for every $\psi \in \mathcal{S}(\mathbb{R}_+)$,

$$\liminf_{M\to\infty} \frac{1}{M^{k/2}} \sum_{\mu=0}^{\infty} \psi\left(\frac{\mu}{M}\right)^2 |r_{\boldsymbol{\alpha}}(\mu)|^2 \geq \frac{k}{2} B_k \int_0^\infty \psi(r)^2 r^{k/2-1}\, dr, \tag{6.1}$$

and, under the usual diophantine conditions,

$$\limsup_{M\to\infty} \frac{1}{M^{k/2}} \sum_{\mu=0}^{\infty} \psi\left(\frac{\mu}{M}\right)^2 |r_{\boldsymbol{\alpha}}(\mu)|^2 \leq \frac{k}{2} B_k \int_0^\infty \psi(r)^2 r^{k/2-1}\, dr. \tag{6.2}$$

Given any $\varepsilon > 0$, we can find $\psi_+, \psi_- \in \mathcal{S}(\mathbb{R}_+)$ such that $\psi_-^2 \leq \chi_{[0,1]} \leq \psi_+^2$, where $\chi_{[0,1]}$ is the characteristic function of the unit interval, and

$$\frac{k}{2} B_k \int_0^\infty [\psi_+(r)^2 - \psi_-(r)^2] r^{k/2-1}\, dr < \varepsilon. \tag{6.3}$$

Therefore

$$\liminf_{M\to\infty} \frac{1}{M^{k/2}} \sum_{\mu=0}^{M} |r_{\boldsymbol{\alpha}}(\mu)|^2 \geq \frac{k}{2} B_k \int_0^1 r^{k/2-1}\, dr - \varepsilon, \tag{6.4}$$

$$\limsup_{M\to\infty} \frac{1}{M^{k/2}} \sum_{\mu=0}^{M} |r_{\boldsymbol{\alpha}}(\mu)|^2 \leq \frac{k}{2} B_k \int_0^1 r^{k/2-1}\, dr + \varepsilon \tag{6.5}$$

for any $\varepsilon > 0$. ∎

Proof of Theorem 1.2. By Parseval's identity,

$$(6.6)\qquad \frac{1}{2}\int_0^2 \Theta_f\Big(u+\mathrm{i}\,\frac{1}{M},0;\begin{pmatrix}\boldsymbol{\alpha}\\ \mathbf{0}\end{pmatrix}\Big)\overline{\Theta_g\Big(u+\mathrm{i}\,\frac{1}{M},0;\begin{pmatrix}\boldsymbol{\alpha}\\ \mathbf{0}\end{pmatrix}\Big)}h(u)\,du$$

$$= \frac{1}{M^{k/2}}\sum_{\mu_1,\mu_2=0}^{\infty}\psi_1\Big(\frac{\mu_1}{M}\Big)\psi_2\Big(\frac{\mu_2}{M}\Big)r_{\boldsymbol{\alpha}}(\mu_1)\,\overline{r_{\boldsymbol{\alpha}}(\mu_2)}\,\widehat{h}(\mu_1-\mu_2),$$

with the Fourier coefficients of h defined by

$$(6.7)\qquad \widehat{h}(\mu)=\frac{1}{2}\int_0^2 h(u)e\Big(\frac{1}{2}\,\mu u\Big)\,du.$$

Thus Corollary 5.4 is equivalent to Theorem 1.2. ■

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