ACTA ARITHMETICA
XCVIII.1 (2001)

Note on a theorem of Rockett and Szüsz on a diophantine equation $x^2 - dy^2 = N$

by

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1. Introduction. Let d be a non-square positive integer. We denote the simple continued fraction expansion of $\sqrt{d}$ by

$$\sqrt{d} = [a_0, a_1, \ldots] = [a_0, \overline{a_1, \ldots, a_l}],$$

where l is the length of the period, that is, the least positive integer such that $a_{k+l} = a_k$ for any $k \geq 1$. Define the sequences $\{A_k\}_{k\geq -1}$ and $\{B_k\}_{k\geq -1}$ of rational integers by

$$A_{-1} = 1, \quad A_0 = a_0, \quad A_k = a_k A_{k-1} + A_{k-2} \quad (k \geq 1),$$
$$B_{-1} = 0, \quad B_0 = 1, \quad B_k = a_k B_{k-1} + B_{k-2} \quad (k \geq 1).$$

Throughout the paper, g denotes $(1+\sqrt{5})/2$.

Rockett and Szüsz "proved" the following claim which was first announced in [3].

Claim ([4, Theorem IV.2.3]). *The positive integer solutions of* $x^2 - dy^2 = N$, *where* $2K_1\sqrt{d} < |N| < 2K_2\sqrt{d}$ *and* $K_1 < K_2$ *are positive constants, have the form*

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1},$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}*'s satisfy* $0 \leq c_{k+1} \leq a_{k+1}$ *for* $k \geq 1$, $0 \leq c_1 < a_1$, *and* $c_k = 0$ *if* $c_{k+1} = a_{k+1}$. *Moreover*, m *satisfies*

$$\tfrac{1}{2}(\log_{(1+2\sqrt{d})}(K_1) - 1) < m < \tfrac{1}{2}(\log_g(2K_2+1)+3).$$

Unfortunately, this claim has counterexamples. Two examples are given below. Example 1 shows that a solution (x, y) does not necessarily have the

2000 *Mathematics Subject Classification*: Primary 11D09; Secondary 11A55, 11Y50.
Key words and phrases: continued fraction, quadratic diophantine equation.

form as above when y is not large enough. Example 2 shows that the upper bound on m is not large enough.

EXAMPLE 1. $x^2-61y^2=705$. The table of the simple continued fraction expansion of $\sqrt{61}$ is as follows:

$$\sqrt{61}=[7,\overline{1,4,3,1,2,2,1,3,4,1,14}]$$

k	0	1	2	3	4	5	6	7	8	9	10
A_k	7	8	39	125	164	453	1070	1523	5639	24079	29718
B_k	1	1	5	16	21	58	137	195	722	3083	3805

A solution $(x,y)=(407,52)$ has the form $y=2B_4+2B_2$. But $x\neq 2A_4+2A_2=406$.

EXAMPLE 2. $x^2-2801y^2=1225$. The table of the simple continued fraction expansion of $\sqrt{2801}$ is as follows:

$$\sqrt{2801}=[45,\overline{1,1,1,1,1,1,1,1,1,1,90}]$$

k	0	1	2	3	4	5	6	7	8	9	10
A_k	45	46	91	137	228	365	593	958	1551	2509	4060
B_k	1	1	2	3	5	8	13	21	34	55	89

Since $1225<2K_2\sqrt{2801}$, we can take $K_2=12$. Then $\frac{1}{2}(\log_g(2K_2+1)+3)<5$. A solution $(x,y)=(4197,92)$ has the form $x=A_3+A_{10}$, $y=B_3+B_{10}$. So $m=8$. Hence $m<\frac{1}{2}(\log_g(2K_2+1)+3)$ does not hold.

The aim of this paper is to state and prove a correct version of the claim by Rockett and Szüsz. Section 2 is devoted to reviewing the concept of Ostrowski representation. In Section 3 we obtain the following theorem on the form of the solutions:

THEOREM 1. *All positive integer solutions of $x^2-dy^2=N$ have the form*

$$x=c_{n+1}A_n+c_{n+2}A_{n+1}+\ldots+c_{n+m}A_{n+m-1}+x',$$
$$y=c_{n+1}B_n+c_{n+2}B_{n+1}+\ldots+c_{n+m}B_{n+m-1},$$

where $c_{n+1}\neq 0$, $c_{n+m}\neq 0$, $0\le c_{k+1}\le a_{k+1}$ for $k\ge 1$, $0\le c_1<a_1$, and $c_k=0$ if $c_{k+1}=a_{k+1}$. In particular, $x'=0$ if y is sufficiently large (say, $y>|N|/(2\varepsilon\sqrt{d})$, where $\varepsilon=\min(\sqrt{d}-a_0,1/(2\sqrt{2}),(1+a_0-\sqrt{d})/\sqrt{2})$).

We discuss a “periodicity” of the solutions in Section 4. In Section 5, the length of the form of a solution is estimated. We obtain

THEOREM 2. *Under the same notation as in Theorem* 1, *assume further that $x' = 0$. Then*

$$\frac{1}{\log(1+a_0) + (\log 2)/l} \cdot \log \frac{|N|}{4\sqrt{d}} < m < \max\left(3, 3 + \log_g \sqrt{5} + \log_g \frac{|N|}{\sqrt{d}}\right).$$

In the final Section 6, we show that the above theorems are useful to examine whether $x^2 - dy^2 = N$ has an integer solution.

Throughout this paper, we use the following notations. For a real number α, $\lfloor \alpha \rfloor$ denotes the floor of α, that is, the largest integer which is not greater than α, while $\lceil \alpha \rceil$ denotes the ceiling of α, that is, the least integer which is not smaller than α.

2. Ostrowski representation of integers. The discussion in this section is valid not only for the continued fraction expansion of $\sqrt{d}$ but also for that of an arbitrary real number which is not rational (cf. [4, Chap. I and II]).

As in [4, Chap. II, §4], every positive integer y can be represented uniquely as

$$y = \sum_{k=0}^{n} c_{k+1} B_k, \tag{1}$$

where the coefficients c_{k+1} satisfy the following

COEFFICIENT CONDITION (abbrev. CC).

- $0 \le c_{k+1} \le a_{k+1}$ *for* $k \ge 1$, $0 \le c_1 < a_1$;
- *if* $c_{k+1} = a_{k+1}$, *then* $c_k = 0$.

This representation was used by Ostrowski in [1]. Hence it is natural to call (1) with CC the *Ostrowski representation* of y with respect to $\sqrt{d}$.

We define sequence $\{D_k\}_{k \ge -1}$ of real numbers, by

$$D_k = B_k\sqrt{d} - A_k.$$

We set

$$\zeta_k = [a_k, a_{k+1}, \ldots].$$

We have the following lemma on D_k's:

LEMMA 1. (i) $D_k = a_k D_{k-1} + D_{k-2}$ *for* $k \ge 1$.

(ii) $D_k = (-1)^k/(\zeta_{k+1}B_k + B_{k-1})$ *for* $k \ge 0$.

(iii) $D_{2k-1} < D_{2k+1} < 0$ *and* $0 < D_{2k+2} < D_{2k}$ *for* $k \ge 0$.

(iv) $1/(B_{k+1}+B_k) < |D_k| < 1/B_{k+1}$ *for* $k \ge 0$. *In particular*, $|D_k| < 1/2$ *for* $k \ge 1$.

Proof. See [4, Chap. I, §4]. ■

The following lemma is a more precise version of Lemma II.4.1 in [4], and plays an important role in the discussion of Sections 3 and 5.

LEMMA 2. *Assume that c_{k+1}'s satisfy CC. Suppose that integers n_0 and n_1 satisfy $0 \le n_0 \le n_1$ and $c_{n_0+1} \ne 0$. Then we have:*

(i) *If n_0 is odd, then*

$$(c_{n_0+1}-1)D_{n_0} - D_{n_0+1} > \sum_{k=n_0}^{n_1} c_{k+1}D_k > c_{n_0+1}D_{n_0} - D_{n_0+1}, \tag{2}$$

$$-\frac{1}{B_{n_0+2}+B_{n_0+1}} > \sum_{k=n_0}^{n_1} c_{k+1}D_k > -\frac{1}{B_{n_0}}, \tag{3}$$

$$0 > \sum_{k=n_0}^{n_1} c_{k+1}D_k > \begin{cases} -1/2 & \text{if } n_0 > 1, \\ -(\sqrt{d}-a_0) & \text{if } n_0 = 1. \end{cases} \tag{4}$$

(ii) *If n_0 is even, then*

$$(c_{n_0+1}-1)D_{n_0} - D_{n_0+1} < \sum_{k=n_0}^{n_1} c_{k+1}D_k < c_{n_0+1}D_{n_0} - D_{n_0+1}, \tag{5}$$

$$\frac{1}{B_{n_0+2}+B_{n_0+1}} < \sum_{k=n_0}^{n_1} c_{k+1}D_k < \frac{1}{B_{n_0}}, \tag{6}$$

$$0 < \sum_{k=n_0}^{n_1} c_{k+1}D_k < \begin{cases} 1/2 & \text{if } n_0 > 0, \\ 1-(\sqrt{d}-a_0) & \text{if } n_0 = 0. \end{cases} \tag{7}$$

Proof. We prove (i); (ii) can be proved similarly. So suppose that n_0 is odd. Since the case $n_0 = n_1$ is easy, we consider the case where $n_0 < n_1$.

By Lemma 1, we have

$$(c_{n_0+1}-1)D_{n_0} - D_{n_0+1} < -D_{n_0+1} < -\frac{1}{B_{n_0+2}+B_{n_0+1}} < 0,$$

$$c_{n_0+1}D_{n_0} - D_{n_0+1} > a_{n_0+1}D_{n_0} - D_{n_0+1} = -D_{n_0-1} > -\frac{1}{B_{n_0}},$$

and

$$c_{n_0+1}D_{n_0} - D_{n_0+1} > -D_{n_0-1} \begin{cases} > -1/2 & \text{if } n_0 > 1, \\ = -(\sqrt{d}-a_0) & \text{if } n_0 = 1. \end{cases}$$

Hence it is enough to show (2).

Let n_1' be n_1 if n_1 is even, and $n_1 - 1$ if n_1 is odd. By Lemma 1 and CC, we have

$$\sum_{k=n_0}^{n_1} c_{k+1}D_k \le c_{n_0+1}D_{n_0} + c_{n_0+2}D_{n_0+1} + c_{n_0+4}D_{n_0+3} + \ldots + c_{n_1'+1}D_{n_1'}$$

$$
\begin{aligned}
&\le c_{n_0+1}D_{n_0} + (a_{n_0+2}-1)D_{n_0+1} + a_{n_0+4}D_{n_0+3} + \ldots + a_{n_1'+1}D_{n_1'}\\
&= c_{n_0+1}D_{n_0} - D_{n_0+1} + (D_{n_0+2} - D_{n_0})\\
&\quad + (D_{n_0+4} - D_{n_0+2}) + \ldots + (D_{n_1'+1} - D_{n_1'-1})\\
&= (c_{n_0+1}-1)D_{n_0} - D_{n_0+1} + D_{n_1'+1} < (c_{n_0+1}-1)D_{n_0} - D_{n_0+1}.
\end{aligned}
$$

Next, reset $n_1' = n_1$ if n_1 is odd, and $n_1 - 1$ if n_1 is even. By Lemma 1 and CC again, we have

$$
\begin{aligned}
\sum_{k=n_0}^{n_1} c_{k+1}D_k &\ge c_{n_0+1}D_{n_0} + c_{n_0+3}D_{n_0+2} + c_{n_0+5}D_{n_0+4} + \ldots + c_{n_1'+1}D_{n_1'}\\
&\ge c_{n_0+1}D_{n_0} + a_{n_0+3}D_{n_0+2} + a_{n_0+5}D_{n_0+4} + \ldots + a_{n_1'+1}D_{n_1'}\\
&= c_{n_0+1}D_{n_0} + (D_{n_0+3} - D_{n_0+1})\\
&\quad + (D_{n_0+5} - D_{n_0+3}) + \ldots + (D_{n_1'+1} - D_{n_1'-1})\\
&= c_{n_0+1}D_{n_0} - D_{n_0+1} + D_{n_1'+1} > c_{n_0+1}D_{n_0} - D_{n_0+1}.
\end{aligned}
$$

Thus we complete the proof. ■

3. Form of the solutions. Assume that positive integers x and y satisfy $x^2 - dy^2 = N$. We have the Ostrowski representation of y with respect to $\sqrt{d}$:

$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}'s satisfy CC, $c_{n+1} \neq 0$, and $c_{n+m} \neq 0$. Let x' be the integer defined by

$$x' = x - (c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1}).$$

Then we have

$$y\sqrt{d} - x = \sum_{k=1}^{m} c_{n+k}D_{n+k-1} - x'. \tag{8}$$

We investigate the range of x'.

3.1. *Case* $N > 0$. Set $\varepsilon = \min(1/2, \sqrt{d} - a_0)$. The condition $N > 0$ implies that $x > y\sqrt{d}$. Since $x - y\sqrt{d} = N/(x + y\sqrt{d})$, we have

$$0 < N/(2x) < x - y\sqrt{d} < N/(2y\sqrt{d}). \tag{9}$$

First, assume that n is even. Then (7) yields

$$0 < \sum_{k=1}^{m} c_{n+k}D_{n+k-1} < 1 - \varepsilon.$$

This inequality and (8) show that

$$x' - (1 - \varepsilon) < x - y\sqrt{d} < x'.$$

Comparing this inequality with (9), we have $0 < x'$ and $x' - (1-\varepsilon) < N/(2y\sqrt{d})$, which implies

$$1 \le x' \le \lfloor 1 - \varepsilon + N/(2y\sqrt{d}) \rfloor.$$

In particular, it turns out that n cannot be even when $1 > 1 - \varepsilon + N/(2y\sqrt{d})$, that is, $y > N/(2\varepsilon\sqrt{d})$.

Next, assume that n is odd. Then (4) yields

$$-1 < \sum_{k=1}^{m} c_{n+k} D_{n+k-1} < 0.$$

Together with (8), this shows that

$$x' < x - y\sqrt{d} < x' + 1.$$

Comparing this inequality with (9), we have

$$0 \le x' \le \lfloor N/(2y\sqrt{d}) \rfloor.$$

In particular, it turns out that $x' = 0$ if $y > N/(2\sqrt{d})$.

Hence we obtain

THEOREM 3. *All the positive integer solutions of $x^2 - dy^2 = N$ with $N > 0$ have the form*

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1} + x',$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}'s satisfy CC, $c_{n+1} \ne 0$, $c_{n+m} \ne 0$, x' satisfies

$$0 \le x' \le \lfloor 1 - \varepsilon + N/(2y\sqrt{d}) \rfloor,$$

and $\varepsilon = \min(1/2, \sqrt{d} - a_0)$. In particular, $x' = 0$ if $y > N/(2\varepsilon\sqrt{d})$. Moreover, $x' = 0$ implies that n is odd.

3.2. *Case $N < 0$.* Set $\varepsilon' = \min(1/2, 1 - (\sqrt{d} - a_0))$. The condition $N < 0$ implies that $x < y\sqrt{d}$. Since $y\sqrt{d} - x = -N/(y\sqrt{d} + x)$, we have

$$0 < -N/(2y\sqrt{d}) < y\sqrt{d} - x < -N/(2x). \tag{10}$$

First, assume that n is odd. We combine (4), (10), and (8) to obtain

$$\lceil -(1 - \varepsilon') + N/(2x) \rceil \le x' \le -1.$$

In particular, n cannot be odd when $x > -N/(2\varepsilon')$.

Next, assume that n is even. Then (7), (10), and (8) lead to

$$\lceil N/(2x) \rceil \le x' \le 0.$$

In particular, $x' = 0$ if $x > -N/2$.

Since $x = \sqrt{dy^2 + N}$, we obtain

THEOREM 4. *All the positive integer solutions of $x^2 - dy^2 = N$ with $N < 0$ have the form*

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1} + x',$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}'s satisfy CC, $c_{n+1} \neq 0$, $c_{n+m} \neq 0$, x' satisfies

$$\lceil -(1-\varepsilon') + N/(2\sqrt{dy^2+N}) \rceil \leq x' \leq 0,$$

and $\varepsilon' = \min(1/2, 1 + a_0 - \sqrt{d})$. In particular, $x' = 0$ if $\sqrt{y^2 + N/d} > -N/(2\varepsilon'\sqrt{d})$. Moreover, $x' = 0$ implies that n is even.

COROLLARY 1. *Under the same notation as in the theorem above, we have $x' = 0$ if $y > -N/(2\varepsilon\sqrt{d})$, where*

$$\varepsilon = \varepsilon'/\sqrt{2} = \min(1/(2\sqrt{2}), (1 + a_0 - \sqrt{d})/\sqrt{2}).$$

Proof. Since $0 < \varepsilon' \leq 1/2$ and $N \leq -1$, we have

$$\sqrt{1 + 4(\varepsilon')^2/(-N)} \leq \sqrt{2}.$$

Then $y > -N/(2\varepsilon\sqrt{d})$ implies that

$$y > \frac{-N}{2\varepsilon'\sqrt{d}}\sqrt{1 + 4(\varepsilon')^2/(-N)},$$

which is equivalent to the inequality $\sqrt{y^2 + N/d} > -N/(2\varepsilon'\sqrt{d})$. ∎

Theorem 1 is a consequence of Theorems 3, 4, and Corollary 1.

4. "Periodicity" of the solutions. Before investigating the length of the form of the solutions, we discuss some property of the solutions.

We need the following lemma:

LEMMA 3. $A_{k+l} = A_kA_{l-1} + dB_kB_{l-1}$, $B_{k+l} = B_kA_{l-1} + A_kB_{l-1}$.

Proof. By equation (3) in [2, §12] and $\zeta_{l+1} = \zeta_1 = 1/(\sqrt{d} - a_0)$, we have

$$\sqrt{d} = \frac{\zeta_{l+1}A_l + A_{l-1}}{\zeta_{l+1}B_l + B_{l-1}} = \frac{A_l + (\sqrt{d} - a_0)A_{l-1}}{B_l + (\sqrt{d} - a_0)B_{l-1}},$$

which leads to

$$dB_{l-1} = A_l - a_0A_{l-1} \quad \text{and} \quad B_l - a_0B_{l-1} = A_{l-1}.$$

Hence Lemma 3 is true for $k = 0$. With the definition of $\{A_k\}$ and $\{B_k\}$, we can easily complete the proof by induction on k. ∎

The discussion of $x^2 - 77y^2 = 37$ in [4, Chap. IV, §2] suggests that the following theorem holds. Here we prove it by using the lemma above.

THEOREM 5. *Assume that $n \geq l$. Then the following statements are equivalent*:

(i) *The following* x *and* y *satisfy* $x^2 - dy^2 = N$:

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1},$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1}.$$

(ii) *The following* x *and* y *satisfy* $x^2 - dy^2 = (-1)^l N$:

$$x = c_{n+1}A_{n-l} + c_{n+2}A_{n+1-l} + \ldots + c_{n+m}A_{n+m-1-l},$$
$$y = c_{n+1}B_{n-l} + c_{n+2}B_{n+1-l} + \ldots + c_{n+m}B_{n+m-1-l}.$$

Proof. Suppose that (i) holds. Then

$$\begin{aligned} N &= \Big(\sum_k c_{n+k}A_{n+k-1}\Big)^2 - d\Big(\sum_k c_{n+k}B_{n+k-1}\Big)^2 \\ &= \sum_{k,k'} c_{n+k}c_{n+k'}(A_{n+k-1}A_{n+k'-1} - dB_{n+k-1}B_{n+k'-1}). \end{aligned}$$

By Lemma 3 and the well known fact that $A_{l-1}^2 - dB_{l-1}^2 = (-1)^l$, we have

$$\begin{aligned} & A_{n+k-1}A_{n+k'-1} - dB_{n+k-1}B_{n+k'-1} \\ &= (A_{n+k-1-l}A_{l-1} + dB_{n+k-1-l}B_{l-1})(A_{n+k'-1-l}A_{l-1} + dB_{n+k'-1-l}B_{l-1}) \\ &\quad - d(B_{n+k-1-l}A_{l-1} + A_{n+k-1-l}B_{l-1}) \\ &\quad \times (B_{n+k'-1-l}A_{l-1} + A_{n+k'-1-l}B_{l-1}) \\ &= A_{n+k-1-l}A_{n+k'-1-l}(A_{l-1}^2 - dB_{l-1}^2) \\ &\quad - dB_{n+k-1-l}B_{n+k'-1-l}(A_{l-1}^2 - dB_{l-1}^2) \\ &= (-1)^l(A_{n+k-1-l}A_{n+k'-1-l} - dB_{n+k-1-l}B_{n+k'-1-l}). \end{aligned}$$

Hence

$$\begin{aligned} N &= \sum_{k,k'} c_{n+k}c_{n+k'}(-1)^l(A_{n+k-1-l}A_{n+k'-1-l} - dB_{n+k-1-l}B_{n+k'-1-l}) \\ &= (-1)^l\Big\{\Big(\sum_k c_{n+k}A_{n+k-1-l}\Big)^2 - d\Big(\sum_k c_{n+k}B_{n+k-1-l}\Big)^2\Big\}. \ \blacksquare \end{aligned}$$

Theorem 5 will be used in the next two sections.

5. Length of the form of the solutions. In this section, j is any positive integer if l is even, and any positive even integer if l is odd.

We investigate the length of the form of the solutions, which we denote by m in the preceding theorems. The number of the cases when $x' \neq 0$ is at most finite. Hence we discuss the solutions with $x' = 0$.

Let

$$F_k = \frac{1}{\sqrt{5}}\Big\{\Big(\frac{1+\sqrt{5}}{2}\Big)^k - \Big(\frac{1-\sqrt{5}}{2}\Big)^k\Big\}$$

be the kth Fibonacci number, that is, $F_0 = 0$, $F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ $(k \geq 2)$. We have the following:

LEMMA 4. *For* $0 \leq n_0 \leq n_1$,

$$F_{n_1-n_0+1} \leq B_{n_1}/B_{n_0} \leq (1+a_0)^{n_1-n_0} \cdot 2^{\lceil (n_1-n_0)/l \rceil}.$$

Proof. We assume that $n_0 < n_1$ because the case $n_0 = n_1$ is obvious. For the upper bound, we note that

$$\frac{B_{n_1}}{B_{n_0}} = \prod_{k=n_0+1}^{n_1} \frac{B_k}{B_{k-1}} = \prod_{k=n_0+1}^{n_1} \frac{a_k B_{k-1} + B_{k-2}}{B_{k-1}} < \prod_{k=n_0+1}^{n_1} (a_k + 1).$$

Moreover, the argument on Lagrange's algorithm (cf. [4, Chap. III, §1]) shows that $a_k < \sqrt{d}$ (i.e. $a_k \leq a_0$) if $l \nmid k$ or $k = 0$, while $a_k < 2\sqrt{d}$ (i.e. $a_k \leq 2a_0 + 1$) if $l \mid k$. So we have

$$\frac{B_{n_1}}{B_{n_0}} < (a_0+1)^{n_1-n_0} \left(\frac{2a_0+1+1}{a_0+1} \right)^{\lceil (n_1-n_0)/l \rceil} = (a_0+1)^{n_1-n_0} \cdot 2^{\lceil (n_1-n_0)/l \rceil}.$$

On the other hand, we can obtain the lower bound as follows:

$$\begin{aligned}\frac{B_{n_1}}{B_{n_0}} &> \frac{B_{n_1-1} + B_{n_1-2}}{B_{n_0}} > \ldots > \frac{F_{n_1-n_0+1} B_{n_0} + F_{n_1-n_0} B_{n_0-1}}{B_{n_0}} \\ &\geq F_{n_1-n_0+1}. \ \blacksquare\end{aligned}$$

Suppose that

$$x = \sum_{k=1}^{m} c_{n+k} A_{n+k-1} \quad \text{and} \quad y = \sum_{k=1}^{m} c_{n+k} B_{n+k-1}$$

are a solution of $x^2 - dy^2 = N$. Then

$$B_n + B_{n+m-1} \leq y < B_{n+m}. \tag{11}$$

Moreover, Theorem 5 shows that

$$x_j = \sum_{k=1}^{m} c_{n+k} A_{n+k-1+jl} \quad \text{and} \quad y_j = \sum_{k=1}^{m} c_{n+k} B_{n+k-1+jl}$$

are also a solution of $x^2 - dy^2 = N$.

5.1. *Case* $N > 0$. Theorem 3 tells us that n is odd. Inequality (3) yields

$$x - y\sqrt{d} = -\sum_{k=1}^{m} c_{n+k} D_{n+k-1} > \frac{1}{B_{n+2} + B_{n+1}}.$$

By (9) and (11), we have

$$\frac{N}{2\sqrt{d}} > y(x - y\sqrt{d}) > (B_{n+m-1} + B_n) \frac{1}{B_{n+2} + B_{n+1}}$$

$$= \frac{B_{n+m-1}/B_{n+2} + B_n/B_{n+2}}{1 + B_{n+1}/B_{n+2}} > \frac{1}{2} \cdot \frac{B_{n+m-1}}{B_{n+2}},$$

that is,

$$\frac{N}{\sqrt{d}} > \frac{B_{n+m-1}}{B_{n+2}}.$$

Assume that $m \geq 3$. Then Lemma 4 leads to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{m-3} < F_{m-2} \leq \frac{N}{\sqrt{d}}.$$

Taking logarithms to base $g = (1+\sqrt{5})/2$, we obtain

$$m - 3 - \log_g \sqrt{5} < \log_g \frac{N}{\sqrt{d}}.$$

On the other hand, (3) yields

$$x - y\sqrt{d} < 1/B_n.$$

Since (9) shows that

$$N/2 < x(x - y\sqrt{d}) = (x - y\sqrt{d})^2 + y\sqrt{d}(x - y\sqrt{d}),$$

we have

$$N/2 - 1/B_n^2 < N/2 - (x - y\sqrt{d})^2 < y\sqrt{d}(x - y\sqrt{d}).$$

By (11) and Lemma 4, we have

$$\frac{N - 2/B_n^2}{2\sqrt{d}} < y(x - y\sqrt{d}) < B_{n+m}\frac{1}{B_n} \leq (1 + a_0)^m \cdot 2^{\lceil m/l \rceil}.$$

A similar discussion of (x_j, y_j) gives

$$\frac{N - 2/B_{n+jl}^2}{2\sqrt{d}} \leq (1 + a_0)^m \cdot 2^{\lceil m/l \rceil}$$

for any j. Since B_k increases to ∞ as k goes to ∞, we let $j \to \infty$ to obtain

$$N/(2\sqrt{d}) \leq (1 + a_0)^m \cdot 2^{\lceil m/l \rceil} < (1 + a_0)^m \cdot 2^{m/l+1}.$$

Taking logarithms to base e, we have

$$\log \frac{N}{4\sqrt{d}} < m \log(1 + a_0) + \frac{m}{l} \log 2.$$

Hence we obtain

THEOREM 6. *Assume that a positive integer solution of $x^2 - dy^2 = N$ with $N > 0$ has the form*

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1},$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}'s satisfy CC, $c_{n+1} \neq 0$, and $c_{n+m} \neq 0$. Then n is odd, and

$$\frac{1}{\log(1+a_0) + (\log 2)/l} \cdot \log \frac{N}{4\sqrt{d}} < m < \max\left(3, 3 + \log_g \sqrt{5} + \log_g \frac{N}{\sqrt{d}}\right).$$

5.2. *Case $N < 0$.* Theorem 4 tells us that n is even. By (6), we have

$$\frac{1}{B_{n+2} + B_{n+1}} < y\sqrt{d} - x = \sum_{k=1}^{m} c_{n+k} D_{n+k-1} < \frac{1}{B_n}.$$

Since (10) yields

$$-N/2 > (y\sqrt{d} - x)x = y\sqrt{d}(y\sqrt{d} - x) - (y\sqrt{d} - x)^2,$$

we have

$$\frac{-N}{2} + \frac{1}{B_n^2} > \frac{-N}{2} + (y\sqrt{d} - x)^2 > y\sqrt{d}(y\sqrt{d} - x).$$

By (11), we have

$$\begin{aligned} \frac{-N + (2/B_n^2)}{2\sqrt{d}} &> y(y\sqrt{d} - x) \\ &> (B_{n+m-1} + B_n)\frac{1}{B_{n+2} + B_{n+1}} > \frac{1}{2} \cdot \frac{B_{n+m-1}}{B_{n+2}}. \end{aligned}$$

Assume that $m \geq 3$. Then Lemma 4 yields

$$\frac{-N + 2/B_n^2}{\sqrt{d}} > \frac{B_{n+m-1}}{B_{n+2}} \geq F_{m-2}.$$

A similar discussion of (x_j, y_j) gives

$$\frac{-N + 2/B_{n+jl}^2}{\sqrt{d}} \geq F_{m-2}$$

for any j. We let $j \to \infty$ to obtain

$$\frac{-N}{\sqrt{d}} \geq F_{m-2} > \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{m-3}.$$

Taking logarithms to base g, we have

$$\log_g \sqrt{5} + \log_g \frac{-N}{\sqrt{d}} > m - 3.$$

On the other hand, (10), (11), and Lemma 4 show that

$$\frac{-N}{2\sqrt{d}} < y(y\sqrt{d} - x) < B_{n+m}\frac{1}{B_n} < (1 + a_0)^m \cdot 2^{m/l+1}.$$

Taking logarithms to base e, we have

$$\log \frac{-N}{2\sqrt{d}} < m \log(1 + a_0) + \frac{m}{l} \log 2.$$

Hence we obtain

THEOREM 7. *Assume that a positive integer solution of* $x^2 - dy^2 = N$ *with* $N < 0$ *has the form*

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1},$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}*'s satisfy CC*, $c_{n+1} \neq 0$, *and* $c_{n+m} \neq 0$. *Then* n *is even*, *and*

$$\frac{1}{\log(1+a_0) + (\log 2)/l} \cdot \log \frac{-N}{4\sqrt{d}} < m < \max\left(3, 3 + \log_g \sqrt{5} + \log_g \frac{-N}{\sqrt{d}}\right).$$

Theorem 2 is a consequence of Theorems 6 and 7.

6. Application. In this section, let j' be 1 if l is even, and 2 if l is odd.

It is known that only finitely many checkings are necessary to find whether $x^2 - dy^2 = N$ has a positive integer solution. That is to say, the following proposition holds:

PROPOSITION 1. *There exists a finite set* S *of integers such that the following two conditions are equivalent*:

(i) *There exists a pair* (x, y) *of positive integers satisfying* $x^2 - dy^2 = N$.

(ii) *There exists a pair* (x, y) *of positive integers satisfying* $x^2 - dy^2 = N$ *and* $y \in S$.

For example, the discussion in [5, §34] shows that we can take

$$S = \begin{cases} \{y \mid 0 < y \le B_{j'l-1}\sqrt{N}\} & \text{if } N > 0, \\ \{y \mid \sqrt{-N/d} < y \le A_{j'l-1}\sqrt{-N/d}\} & \text{if } N < 0. \end{cases}$$

The concept of Ostrowski representation yields another choice of S. Fix n_0 such that if (x, y) is a solution of $x^2 - dy^2 = N$ and $y \ge B_{n_0}$ then $x' = 0$, where x' is as in Theorem 1. If $x^2 - dy^2 = N$ has a solution, then it is well known that there exist infinitely many solutions, and Theorems 1, 2, and 5 imply that $x^2 - dy^2 = N$ has a solution

$$x = c_{n+1}A_n + c_{n+2}A_{n+1} + \ldots + c_{n+m}A_{n+m-1},$$
$$y = c_{n+1}B_n + c_{n+2}B_{n+1} + \ldots + c_{n+m}B_{n+m-1},$$

where c_{k+1}'s satisfy CC, $c_{n+1} \neq 0$, $c_{n+m} \neq 0$, m is bounded by d and N, and n satisfies $n_0 \le n < n_0 + j'l$.

Our discussion shows that we can take

$$S = \left\{ \sum_{k=1}^{m} c_{n+k}B_{n+k-1} \;\middle|\; \begin{array}{l} n_0 \le n < n_0 + j'l,\ m_{\min} < m < m_{\max}, \\ c_{n+k}\text{'s satisfy CC},\ c_{n+1} \neq 0,\ c_{n+m} \neq 0 \end{array} \right\},$$

where

$$m_{\min} = \frac{1}{\log(1 + a_0) + (\log 2)/l} \cdot \log \frac{|N|}{4\sqrt{d}},$$
$$m_{\max} = \max\left(3, 3 + \log_g \sqrt{5} + \log_g \frac{|N|}{\sqrt{d}}\right).$$

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Received on 4.8.1999
and in revised form on 3.8.2000 (3682)