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$\Omega_{\pm}$-results of the error term in the mean square formula of the Riemann zeta-function in the critical strip

by

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1. Introduction. Let $\zeta(s)$ be the Riemann zeta-function. In 1922, Littlewood [11] established the following mean square formula for $\zeta(s)$ on the critical line:

$$\int_0^T |\zeta(1/2+iu)|^2\,du = T\log(T/(2\pi)) + (2\gamma-1)T + E(T) \qquad (T\ge 2)$$

with $E(T) \ll T^{3/4+\varepsilon}$. Here γ is the Euler constant. The upper bound for $E(T)$ is now improved but it is still quite far away from the conjectured upper bound $E(T) \ll T^{1/4+\varepsilon}$. This is believed to be a difficult problem. Nevertheless, research on $E(T)$ is still active and a lot of papers (for example, [1], [3]–[7], [11], [15], [18]) are devoted to problems concerning various properties of $E(T)$. For $T\ge 2$ and $1/2<\sigma<1$, an analogue of the above mean square formula on the line $\operatorname{Re} s=\sigma$ exists, viz.,

$$\int_0^T |\zeta(\sigma+iu)|^2\,du = \zeta(2\sigma)T + (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma} + E_\sigma(T).$$

Studies on $E_\sigma(T)$, parallel to that of $E(T)$, have been carried out by various authors (see, for instance, [8], [12], [13]). Excellent surveys are given in [10] and [14].

In this paper, we shall investigate $\Omega_{\pm}$-results of $E_\sigma(T)$ for $1/2<\sigma\le 3/4$. For the case $1/2<\sigma<3/4$, Matsumoto and Meurman [12] have proved that

$$E_\sigma(T) = \Omega_+(T^{3/4-\sigma}(\log T)^{\sigma-1/4}),$$

while Ivić and Matsumoto [8] have showed that

$$E_\sigma(T) = \Omega_-(T^{3/4-\sigma}\exp(C(\log\log T)^{\sigma-1/4}(\log\log\log T)^{\sigma-5/4}))$$

for some positive constant C. Here the Ω_--result is weaker than the Ω_+-

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result. Our purpose here is to bring the Ω_--result up to the same strength as the Ω_+-result and, furthermore, to extend the validity of these $\Omega_\pm$-results to the case $\sigma = 3/4$. We shall use two different approaches to these two cases. The case $1/2 < \sigma < 3/4$ will be treated by a method based on ideas of Szegő [17] and Hafner [2]. For the other case ($\sigma = 3/4$), we shall use the idea in Tsang [19]. This method enables us to tell more about the location of these large values.

2. Main results

THEOREM 1. *For* $1/2 < \sigma < 3/4$,

$$E_\sigma(t) = \Omega_\pm(t^{3/4-\sigma}(\log t)^{\sigma-1/4}).$$

REMARK. Unlike $E(t)$, $E_\sigma(t)$ (for $1/2 < \sigma \le 3/4$) can attain large values of the same magnitude in both the positive and negative directions.

THEOREM 2. *For all sufficiently large L and T, we have*

$$\sup_{t\in[T,T+L\sqrt{T}]} \pm E_{3/4}(t) \gg \sqrt{\log L}$$

where the implied constant is absolute.

COROLLARY. *$E_{3/4}(t)$ must have a sign change in every interval $[T, T + C\sqrt{T}]$ where $C > 0$ is a suitable constant.*

3. Some preparations. Throughout this paper, T is a sufficiently large number, $1/2 < \sigma \le 3/4$ and $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ for each natural number n.

LEMMA 3.1. *Suppose $1/2 < \sigma < 3/4$. There exist two positive constants K_1 and K_2, depending only on σ, such that*

(1) *for any $x \ge 1$,*

$$\sum_{n\le x} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} \le K_1 x^{\sigma-1/4},$$

(2) *for any $V > 1$ and for all sufficiently large $x \ge x_0(V)$,*

$$\sum_{Vx<n\le x^3} \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e^{-2\pi^2 n/x} \le K_2 (Vx)^{\sigma-1/4} e^{-2\pi^2 V}.$$

This follows from the estimate $\sum_{n\le x} \sigma_{1-2\sigma}(n) \ll_\sigma x$ and integration by parts for Stieltjes integrals.

LEMMA 3.2. *For all sufficiently large k, let $0 < x = o(k^{1/3})$ and β be any real number. Then*

$$\frac{1}{\Gamma(k+1)}\int_0^\infty e^{-u^2}u^{2k+1}\cos(4\pi\sqrt{x}u+\beta\pi)\,du = \tfrac{1}{2}e^{-2\pi^2 x}\cos(4\pi\sqrt{kx}+\beta\pi)+O(k^{-1/2})$$

where the implied constant in the O-term is absolute.

Proof. By putting $u=\sqrt{kw}$ and using

$$\Gamma(k+1)=\sqrt{2\pi}\,k^{k+1/2}e^{-k}(1+O(k^{-1})),$$

we have

$$\frac{1}{\Gamma(k+1)}\int_0^\infty e^{-u^2}u^{2k+1}\cos(4\pi\sqrt{x}u+\beta\pi)\,du = \operatorname{Re}\frac{k^{1/2}}{2\sqrt{2\pi}}e^{i\beta\pi}\int_0^\infty w^k e^{k(1-w)+4\pi i\sqrt{kxw}}\,dw\,(1+O(k^{-1})). \tag{3.1}$$

To evaluate the integral, we split it into three parts,

$$\int_0^\infty = \int_0^{1-p} + \int_{1-p}^{1+p} + \int_{1+p}^\infty = I_1+I_2+I_3, \tag{3.2}$$

say, where $p=2k^{-5/12}$. Using the trivial bound and replacing w by $(1-p)w/k$, we obtain

$$I_1 \ll \int_0^{1-p} w^k e^{k(1-w)}\,dw \ll k^{-(k+1)}e^k((1-p)e^p)^k\int_0^k w^k e^{-w}\,dw \ll k^{-1/2}((1-p)e^p)^k \ll k^{-1}. \tag{3.3}$$

Here we have used $\int_0^k w^k e^{-w}\,dw < \Gamma(k+1)$ and the estimate

$$((1-p)e^p)^k = e^{k(p+\log(1-p))} \ll e^{-kp^2/4}.$$

Similarly, by replacing w by $(1+p)w/k$, we have

$$I_3 \ll k^{-(k+1)}e^k((1+p)e^{-p})^k\int_k^\infty w^k e^{-w}\,dw \ll k^{-1/2}e^{-kp^2/4} \ll k^{-1}. \tag{3.4}$$

The second integral is evaluated as follows. We expand the integrand around $w=1$ and then apply the formula

$$\int_{-\infty}^\infty \exp(At-Bt^2)\,dt = \sqrt{\pi/B}\exp(A^2/(4B))$$

for $\operatorname{Re}B>0$. Then

$$I_2 = e^{4\pi i\sqrt{kx}}\int_{-p}^{p} e^{-(k+\pi i\sqrt{kx})\,v^2/2-2\pi i\sqrt{kx}v}(1+O(k|v|^3))\,dv \tag{3.5}$$

$$= e^{4\pi i\sqrt{kx}} \int_{-\infty}^{\infty} e^{-(k+\pi i\sqrt{kx})v^2/2-2\pi i\sqrt{kx}v}\,dv$$

$$+ O\Big(\int_{p}^{\infty} e^{-kv^2/2}\,dv + k\int_{-p}^{p} |v|^3 e^{-kv^2/2}\,dv\Big)$$

$$= e^{4\pi i\sqrt{kx}}\bigg(\frac{2\pi}{k+\pi i\sqrt{kx}}\bigg)^{1/2}\exp\bigg(-\frac{2\pi^2 kx}{k+\pi i\sqrt{kx}}\bigg) + O(k^{-1})$$

$$= \sqrt{2\pi}\,k^{-1/2}e^{-2\pi^2 x+4\pi i\sqrt{kx}} + O(k^{-1}),$$

as $x = o(k^{1/3})$. Our result follows from (3.1)–(3.5).

LEMMA 3.3. *Let a be any real number and $1/2 < \sigma < 3/4$. As $\xi \to 0+$,*

$$\sum_{n\le\xi^{-3}} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e^{-2\pi^2 n\xi}\cos(4\pi a\sqrt{\xi n}-\pi/4)$$

$$= 2^{1-2\sigma}\pi^{1/2-2\sigma}\zeta(2\sigma)\xi^{1/4-\sigma}\int_0^{\infty} e^{-2w^2}w^{2\sigma-3/2}\cos(4aw-\pi/4)\,dw$$

$$+ O(\xi^{\sigma-3/4} + |a|\xi^{1/4+\varepsilon}).$$

Proof. First we quote the following result of [9]. Define

$$\Delta_{1-2\sigma}(v,1/2)$$

$$= \sum_{n\le v}(-1)^n\sigma_{1-2\sigma}(n) - \frac{\zeta(2\sigma)}{2^{2\sigma}}v - 2^{2\sigma-2}\frac{\zeta(2-2\sigma)}{2-2\sigma}v^{2-2\sigma} + E_{1-2\sigma}(0,1/2),$$

where $E_{1-2\sigma}(0,1/2)$ is independent of v. We have

$$\Delta_{1-2\sigma}(v,1/2) \ll_\varepsilon v^{1/(1+4\sigma)+\varepsilon}. \tag{3.6}$$

Then we express the sum in the lemma in terms of integrals as

$$\sum_{n\le\xi^{-3}}(\cdots) \tag{3.7}$$

$$= \int_{1^-}^{\xi^{-3}} v^{\sigma-5/4}e^{-2\pi^2\xi v}\cos(4\pi a\sqrt{\xi v}-\pi/4)$$

$$\times (2^{-2\sigma}\zeta(2\sigma) + 2^{2\sigma-2}\zeta(2-2\sigma)v^{1-2\sigma})\,dv$$

$$+ \int_{1^-}^{\xi^{-3}} v^{\sigma-5/4}e^{-2\pi^2\xi v}\cos(4\pi a\sqrt{\xi v}-\pi/4)\,d\Delta_{1-2\sigma}(v,1/2).$$

After integrating by parts, the second integral in (3.7) is

$$\ll 1 + \int_{1^-}^{\xi^{-3}} e^{-2\pi^2\xi v}|\Delta_{1-2\sigma}(v,1/2)|(v^{\sigma-9/4} + |a|\sqrt{\xi}\,v^{\sigma-7/4} + \xi v^{\sigma-5/4})\,dv$$
$$\ll 1 + |a|\xi^{1/4+\varepsilon},$$

by (3.6). The contribution due to $v^{1-2\sigma}$ in the first integral of (3.7) is

$$\ll \int_{1^-}^{\xi^{-3}} v^{-1/4-\sigma}e^{-2\pi^2\xi v}\,dv = O(\xi^{\sigma-3/4}).$$

By the change of variable $\pi\sqrt{\xi v} = w$, we see that

$$\int_{1^-}^{\xi^{-3}} v^{\sigma-5/4}e^{-2\pi^2\xi v}\cos(4\pi a\sqrt{\xi v} - \pi/4)\,dv$$
$$= 2\pi^{1/2-2\sigma}\xi^{1/4-\sigma}\Big\{\int_0^\infty w^{2\sigma-3/2}e^{-2w^2}\cos(4aw - \pi/4)\,dw$$
$$+ O\Big(\Big(\int_0^{\pi\sqrt{\xi}} + \int_{\pi\xi^{-1}}^\infty\Big)e^{-2w^2}w^{2\sigma-3/2}\,dw\Big)\Big\}.$$

The last O-term is $O(\xi^{\sigma-1/4})$ as $\xi \to 0+$. Our result whence follows.

LEMMA 3.4. *Let h be a real-valued integrable function defined on an interval I. If*

$$|I|^{-1}\Big|\int_I h^3\Big| \le \theta\Big(|I|^{-1}\int_I h^2\Big)^{3/2}$$

for some $\theta < 1$, then

$$\sup_I(\pm h) \ge \Big(\frac{1-\theta}{2}\Big)^{1/3}\Big(|I|^{-1}\int_I h^2\Big)^{1/2}.$$

This is [19, Lemma 1].

4. A convolution of $E_\sigma(t)$. The aim of this process is to shorten the series representation for $E_\sigma(t)$ by convolving $E_\sigma(t)$ with the kernel

$$K(u) = 2B\Big(\frac{\sin 2\pi Bu}{2\pi Bu}\Big)^2$$

where $B > 0$ is large. It is easy to see that

$$K(u) = \frac{1}{2\pi}\int_{-4\pi B}^{4\pi B}\Big(1 - \frac{|y|}{4\pi B}\Big)e^{-iuy}\,dy, \tag{4.1}$$

$$\int_{-\infty}^{\infty} K(u)e^{iyu}\,du = \max\left(0, 1-\frac{|y|}{4\pi B}\right),$$

$$\int_{|u|>L} K(u)e^{iyu}\,du = -2\frac{\sin(yL)}{y}K(L) + O(y^{-2}BL^{-1}).$$

Suppose that $B \ll L^{1/4} \ll T^{1/16}$. To simplify the argument, we assume that BL is an integer (by slightly varying the value of B) so that $K(\pm L) = 0$. Hence

$$\int_{|u|>L} K(u)e^{iuy}\,du = O(y^{-2}BL^{-1}).$$

Suppose $\sqrt{T/(2\pi)} + L \le t \le \sqrt{T/\pi} - L$ and $1/2 < \sigma \le 3/4$. Proofs of both Theorems 1 and 2 are based on the following useful formula:

$$\begin{aligned} (4.2)\qquad & t^{2\sigma-3/2}\int_{-L}^{L} E_\sigma(2\pi(t+u)^2)K(u)\,du \\ &= \sqrt{2}\sum_{n\le B^2}(-1)^n\left(1-\frac{\sqrt{n}}{B}\right)\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}\cos(4\pi\sqrt{n}\,t - \pi/4) + O(1). \end{aligned}$$

To prove this, we consider separately the cases $1/2 < \sigma < 3/4$ and $\sigma = 3/4$, according to the available formulas for E_σ.

CASE (i): $1/2 < \sigma < 3/4$. We use the following Atkinson-type formula for $E_\sigma(t)$ which is given in [12]. Let

$$g(x,n) = x\log\frac{x}{2\pi n} - x + \frac{\pi}{4},$$

$$f(x,n) = 2x\,\mathrm{arsinh}\sqrt{\frac{\pi n}{2x}} + (\pi^2n^2 + 2\pi nx)^{1/2} - \frac{\pi}{4},$$

$$e(x,n) = \left(1+\frac{\pi n}{2x}\right)^{-1/4}\left(\frac{\pi n}{2x}\right)^{1/2}\left(\mathrm{arsinh}\sqrt{\frac{\pi n}{2x}}\right)^{-1},$$

where $\mathrm{arsinh}(x) = \log(x+\sqrt{x^2+1})$. Define

$$(4.3)\qquad \begin{aligned} \Sigma_{1,\sigma}(x) &= \sqrt{2}\left(\frac{x}{2\pi}\right)^{3/4-\sigma}\sum_{n\le T}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}e(x,n)\cos f(x,n), \\ \Sigma_{2,\sigma}(x) &= 2\left(\frac{x}{2\pi}\right)^{1/2-\sigma}\sum_{n\le B(x,\sqrt{T})}\frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}}\left(\log\frac{x}{2\pi n}\right)^{-1}\cos g(x,n), \end{aligned}$$

where

$$B(x,X) = \frac{x}{2\pi} + \frac{X^2}{2} - X\left(\frac{x}{2\pi}+\frac{X^2}{4}\right)^{1/2} \quad \left(= \left\{\sqrt{\frac{x}{2\pi}+\left(\frac{X}{2}\right)^2} - \frac{X}{2}\right\}^2\right).$$

By [12, Theorem 1], we have, for t in our given range and $|u| \le L$,

$$E_\sigma(2\pi(t+u)^2) = \Sigma_{1,\sigma}(2\pi(t+u)^2) - \Sigma_{2,\sigma}(2\pi(t+u)^2) + O(\log T). \tag{4.4}$$

REMARKS. The following straightforward estimates are easy to obtain. Denoting by $\partial_u = \partial/\partial u$ and $\partial_u^2 = \partial^2/\partial u^2$ the partial differential operators of the first and second order, we have

(1) $e(2\pi(t+u)^2, n) = 1 + O(nt^{-2})$ and $\partial_u e(2\pi(t+u)^2, n) \ll nt^{-3}$;

(2) for $n \ll t^2$,

$$f(2\pi(t+u)^2, n) = 4\pi\sqrt{n}(t+u) - \pi/4 + O(n^{3/2}t^{-1}),$$

$$\partial_u f(2\pi(t+u)^2, n) = 8\pi(t+u)\operatorname{arsinh}\frac{\sqrt{n}}{2(t+u)} \asymp \sqrt{n}$$

and

$$\partial_u^2 f(2\pi(t+u)^2, n) \ll n^{3/2}t^{-3};$$

(3) we have

$$\partial_u g(2\pi(t+u)^2, n) = 4\pi(t+u)\log((t+u)^2/n),$$
$$\partial_u^2 g(2\pi(t+u)^2, n) = 4\pi\log((t+u)^2/n) + 8\pi;$$

(4) $B(x, X)$ is an increasing function in x. Moreover,

$$B(2\pi(t+u)^2, \sqrt{T}) < 0.064447T \quad \text{and} \quad B(2\pi(t+u)^2, \sqrt{T}/2) < 0.135T$$

for t and u in the given range. Also, $y = B(2\pi(t+u)^2, \sqrt{T})$ is equivalent to $t + u = \sqrt{y + \sqrt{yT}}$.

In view of (4.4), in order to prove (4.2) we first evaluate

$$\int_{-L}^{L} \Sigma_{2,\sigma}(2\pi(t+u)^2)K(u)\,du.$$

We split the sum for $\Sigma_{2,\sigma}$ in (4.3) into parts with $n \le B(2\pi(t-L)^2, \sqrt{T})$, and n lying between $B(2\pi(t-L)^2, \sqrt{T})$ and $B(2\pi(t+L)^2, \sqrt{T})$. Both subsums involve the following integral. Let $F = \max(-L, \sqrt{n + \sqrt{nT}} - t)$. Applying the inversion formula (4.1), we have

$$\int_F^L (t+u)^{1-2\sigma}\left(\log\frac{(t+u)^2}{n}\right)^{-1}\cos(g(2\pi(t+u)^2, n))K(u)\,du$$
$$= \operatorname{Re}\frac{1}{2\pi}\int_{-4\pi B}^{4\pi B}\left(1 - \frac{|y|}{4\pi B}\right)$$
$$\times \int_F^L (t+u)^{1-2\sigma}\left(\log\frac{(t+u)^2}{n}\right)^{-1} e^{i(g(2\pi(t+u)^2,n)-uy)}\,du\,dy$$

$$= \operatorname{Re}\frac{1}{2\pi i}\int_{-4\pi B}^{4\pi B}\left(1-\frac{|y|}{4\pi B}\right)(t+u)^{1-2\sigma}\left(\log\frac{(t+u)^2}{n}\right)^{-1}$$

$$\times\left(4\pi(t+u)\log\frac{(t+u)^2}{n}-y\right)^{-1}e^{i(g(2\pi(t+u)^2,n)-uy)}\Big|_{u=F}^{u=L}\,dy$$

$$-\operatorname{Re}\frac{1}{2\pi i}\int_{-4\pi B}^{4\pi B}\left(1-\frac{|y|}{4\pi B}\right)\int_F^L e^{i(g(2\pi(t+u)^2,n)-uy)}$$

$$\times\frac{d}{du}\left((t+u)^{1-2\sigma}\left(\log\frac{(t+u)^2}{n}\right)^{-1}\left(4\pi(t+u)\log\frac{(t+u)^2}{n}-y\right)^{-1}\right)du\,dy.$$

Since $(t+u)^2\ge 0.159T$ and $n<0.06445T$, for $|y|\le 4\pi B$ we have

$$\frac{d}{du}\left((t+u)^{1-2\sigma}\left(\log\frac{(t+u)^2}{n}\right)^{-1}\left(4\pi(t+u)\log\frac{(t+u)^2}{n}-y\right)^{-1}\right)\ll t^{-1-2\sigma}.$$

Together with the estimates in our remarks, this integral is equal to

$$O(t^{-2\sigma})\operatorname{Re}\frac{1}{2\pi i}\int_{-4\pi B}^{4\pi B}\left(1-\frac{|y|}{4\pi B}\right)(1+O(|y|t^{-1}))e^{-iuy}\Big|_{u=F}^{u=L}dy+O(BLt^{-1-2\sigma})$$

$$\ll\begin{cases}K(L)t^{-2\sigma}+BLt^{-1-2\sigma} & \text{if } F=-L,\\ Bt^{-2\sigma} & \text{otherwise,}\end{cases}$$

$$=\begin{cases}BLt^{-1-2\sigma} & \text{if } F=-L,\\ Bt^{-2\sigma} & \text{otherwise.}\end{cases}$$

Hence, by (4.3) and according to the splitting,

$$(4.5)\qquad \int_{-L}^{L}\Sigma_{2,\sigma}(2\pi(t+u)^2)K(u)\,du$$

$$\ll\Big\{BLt^{-1-2\sigma}\sum_{n\ll T}+Bt^{-2\sigma}\sum_n{}^{*}\Big\}\sigma_{1-2\sigma}(n)n^{\sigma-1}\ll 1,$$

where the sum $\sum_n^*$ is over $B(2\pi(t-L)^2,\sqrt{T})\le n\le B(2\pi(t+L)^2,\sqrt{T})$. (Note that in this range, $n\asymp t^2$ and the number of n's is $\asymp tL$.)

We now split $\Sigma_{1,\sigma}(2\pi(t+u)^2)$ into $\sum_{n\ll B^4}+\sum_{B^4\ll n\le T}$. The second sum is handled by a similar argument as follows. Note that, for $|y|\le 4\pi B$,

$$\frac{d}{du}\left((t+u)^{3/2-2\sigma}e(2\pi(t+u)^2,n)\left(\frac{\partial}{\partial u}f(2\pi(t+u)^2,n)-y\right)^{-1}\right)$$

$$\ll n^{-1/2}t^{1/2-2\sigma}.$$

Hence, by (4.1) and integration by parts,

$$\tag{4.6} \int_{-L}^{L} (t+u)^{3/2-2\sigma} e(2\pi(t+u)^2, n) \cos(f(2\pi(t+u)^2, n)) K(u)\, du$$

$$= \operatorname{Re} \frac{1}{2\pi i} \int_{-4\pi B}^{4\pi B} \left(1 - \frac{|y|}{4\pi B}\right) (t+u)^{3/2-2\sigma} e(2\pi(t+u)^2, n)$$

$$\times \left(\frac{\partial}{\partial u} f(2\pi(t+u)^2, n)\right)^{-1} (1 + O(|y| n^{-1/2})) e^{i(f(2\pi(t+u)^2, n) - uy)} \Big|_{u=-L}^{u=L} dy$$

$$+ O(BLn^{-1/2} t^{1/2-2\sigma})$$

$$\ll B^2 n^{-1} t^{3/2-2\sigma} + BLn^{-1/2} t^{1/2-2\sigma}.$$

In the last step, we have used the fact that $K(\pm L) = 0$. Thus, the contribution of the sum over the range $B^4 \ll n \leq T$ is

$$\tag{4.7} \int_{-L}^{L} \sqrt{2}(t+u)^{3/2-2\sigma} \sum_{B^4 \ll n \leq T} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e(2\pi(t+u)^2, n) \times \cos(f(2\pi(t+u)^2, n)) K(u)\, du \ll t^{3/2-2\sigma}.$$

For $n \ll B^4$, we deduce from (4.3) together with Remarks (1) and (2) that

$$\tag{4.8} \int_{-L}^{L} \sqrt{2}(t+u)^{3/2-2\sigma} \sum_{n \ll B^4} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e(2\pi(t+u)^2, n) \times \cos(f(2\pi(t+u)^2, n)) K(u)\, du$$

$$= \sqrt{2}\, t^{3/2-2\sigma} \sum_{n \ll B^4} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} \int_{-L}^{L} \cos(4\pi\sqrt{n}(t+u) - \pi/4) K(u)\, du$$

$$+ O\Big(t^{1/2-2\sigma} \sum_{n \ll B^4} n^{1/4+\sigma} \sigma_{1-2\sigma}(n)\Big)$$

$$= \sqrt{2}\, t^{3/2-2\sigma} \sum_{n \leq B^2} (-1)^n \left(1 - \frac{\sqrt{n}}{B}\right) \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} \cos(4\pi\sqrt{n}\, t - \pi/4)$$

$$+ O(t^{3/2-2\sigma}).$$

Since $\log T \ll t^{3/2-2\sigma}$, in view of (4.3)–(4.8), the proof of (4.2) for $1/2 < \sigma < 3/4$ is complete.

Case (ii): $\sigma = 3/4$. The proof of (4.2) in this case is quite similar, but instead of (4.4) (which is not sharp enough for our purpose), we use the following result. Define

$$(4.9)\qquad \begin{aligned} \Sigma_1(x) &= \sqrt{2}\sum_{n\le T}(-1)^n\frac{\sigma_{-1/2}(n)}{\sqrt{n}}w_1(n)e(x,n)\cos f(x,n),\\ \Sigma_2(x) &= 2\left(\frac{x}{2\pi}\right)^{-1/4}\sum_n\frac{\sigma_{-1/2}(n)}{n^{1/4}}w_2(x,n)\left(\log\frac{x}{2\pi n}\right)^{-1}\cos g(x,n),\end{aligned}$$

where

$$w_1(n) = \begin{cases} 1 & \text{if } n\le T/4,\\ 2(1-\sqrt{n/T}) & \text{if } T/4 < n \le T,\end{cases}$$

$$w_2(x,n) = \begin{cases} 1 & \text{if } n \le B(x,\sqrt{T}),\\ x/(\pi\sqrt{nT}) - 2\sqrt{n/T}-1 & \text{if } B(x,\sqrt{T})\le n < B(x,\sqrt{T}/2),\\ 0 & \text{otherwise.}\end{cases}$$

Then [12, (7.1)] gives

$$E_{3/4}(2\pi(t+u)^2) = \Sigma_1(2\pi(t+u)^2) - \Sigma_2(2\pi(t+u)^2) + O(1).$$

Recall that $|u|\le L$ and $\sqrt{T/(2\pi)}\le t+u\le\sqrt{T/\pi}$. Plainly $w_2(2\pi(t+u)^2,n)$ is a continuous function in u, and, apart from the two turning points,

$$\frac{\partial}{\partial u}w_2(2\pi(t+u)^2,n) = \begin{cases} 4(t+u)/\sqrt{nT} & \text{if } \sqrt{n+\sqrt{nT}/2}-t < u < \sqrt{n+\sqrt{nT}}-t,\\ 0 & \text{otherwise.}\end{cases}$$

Hence, for $|y|\le 4\pi B$,

$$\frac{d}{du}\Bigg((t+u)^{-1/2}w_2(2\pi(t+u)^2,n)\left(\log\frac{(t+u)^2}{n}\right)^{-1} \times\left(4\pi(t+u)\log\frac{(t+u)^2}{n}-y\right)^{-1}\Bigg) \ll n^{-1/2}t^{-3/2}.$$

Thus, similarly to the proof of (4.5), we have

$$\int_{-L}^{L}(t+u)^{-1/2}w_2(2\pi(t+u)^2,n)\left(\log\frac{(t+u)^2}{n}\right)^{-1}\cos(g(2\pi(t+u)^2,n))K(u)\,du \ll BLn^{-1/2}t^{-3/2} + B^2t^{-5/2}.$$

Hence, from (4.9),

$$(4.10)\qquad \int_{-L}^{L}\Sigma_2(2\pi(t+u)^2)K(u)\,du \ll 1.$$

Next, we estimate the integral

$$\int_{-L}^{L} e(2\pi(t+u)^2, n)\cos(f(2\pi(t+u)^2, n))K(u)\,du.$$

Using the first order approximations for $e(2\pi(t+u)^2, n)$ and $f(2\pi(t+u)^2, n)$ in Remarks (1) and (2), we find that

$$\int_{-L}^{L} e(2\pi(t+u)^2, n)\cos(f(2\pi(t+u)^2, n))K(u)\,du$$
$$= \max(0, 1-\sqrt{n}/B)\cos(4\pi\sqrt{n}\,t - \pi/4) + O(BL^{-1}n^{-1} + n^{3/2}t^{-1}).$$

This is good when n is small, say $n \le B^4$. For $n \ge B^4$, we follow the argument that leads to (4.6) and prove

$$\int_{-L}^{L} e(2\pi(t+u)^2, n)\cos(f(2\pi(t+u)^2, n))K(u)\,du$$
$$\ll B^2 n^{-1} + BL\sqrt{n}\,t^{-3}.$$

Using these two estimates and in view of (4.9), we have

$$\int_{-L}^{L} \Sigma_1(2\pi(t+u)^2)K(u)\,du$$
$$= \sqrt{2}\sum_{n\le B^2}(-1)^n\frac{\sigma_{-1/2}(n)}{\sqrt{n}}\left(1-\frac{\sqrt{n}}{B}\right)\cos(4\pi\sqrt{n}\,t-\pi/4) + O(1).$$

Together with (4.10), this completes the proof of (4.2) for $\sigma = 3/4$.

5. Proof of Theorem 1. Equation (4.2) is proved under the assumption $B \ll L^{1/4} \ll T^{1/16}$. Letting $B = T^{1/6000}$ and $L = T^{1/1000}$, we may make use of (4.2) for a wide range of values of T (the value of T in (4.2)). In particular, for $T^{1/12} \le t \le T^{1/2}$, we have

$$t^{2\sigma-3/2}\int_{-L}^{L} E_\sigma(2\pi(t+u)^2)K(u)\,du = \sqrt{2}S(t) + O(1) \tag{5.1}$$

where

$$S(t) = \sum_{n\le B^2}(-1)^n\left(1-\frac{\sqrt{n}}{B}\right)\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}\cos(4\pi\sqrt{n}\,t-\pi/4).$$

Let k satisfy $T^{1/5} \le k \le T^{2/5}$. Let c be any positive constant and $\xi = c(\log T)^{-1}$. Lemma 3.2 yields

$$\frac{1}{\Gamma(k+1)}\int_0^\infty e^{-u^2}u^{2k+1}S(u\sqrt{\xi})\,du$$
$$= \frac{1}{2}\sum_{n\le B^2}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}\left(1-\frac{\sqrt{n}}{B}\right)$$
$$\times\{e^{-2\pi^2 n\xi}\cos(4\pi\sqrt{kn\xi}-\pi/4)+O(k^{-1/2})\}.$$

Note that, estimated crudely (by an argument similar to that in Lemma 3.1), we have

$$k^{-1/2}\sum_{n\le B^2}\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} \ll T^{-1/10+1/6000}\ll 1,$$
$$\sum_{\xi^{-3}<n\le B^2}\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}e^{-2\pi^2 n\xi}\ll 1,$$

and

$$\sum_{n\le \xi^{-3}}\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}\cdot\frac{\sqrt{n}}{B}\ll 1.$$

Hence

$$\frac{1}{\Gamma(k+1)}\int_0^\infty e^{-u^2}u^{2k+1}S(u\sqrt{\xi})\,du \tag{5.2}$$
$$= \frac{1}{2}\sum_{n\le \xi^{-3}}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}}e^{-2\pi^2 n\xi}\cos(4\pi\sqrt{kn\xi}-\pi/4)+O(1).$$

Define

$$g(a)=\int_0^\infty e^{-w^2}w^{2\sigma-3/2}\cos(4aw-\pi/4)\,dw.$$

It is known that (see [16]) when $\sigma>1/2$, there exist real numbers a_+ and a_- such that $g(a_+)>0$ and $g(a_-)<0$. We choose two large constants U and V such that

$$(U^{-1}K_1+e^{-2\pi^2 V}K_2)V^{\sigma-1/4} \tag{5.3}$$
$$< 2^{-3\sigma}\pi^{1/2-2\sigma}\zeta(2\sigma)\min(g(a_+),|g(a_-)|),$$

where K_1 and K_2 are those that appeared in Lemma 3.1.

Let $R=[V\xi^{-1}]$. By Dirichlet's theorem on simultaneous approximations, there exists l such that

$$T^{1/10}\le l\le (1+(4\pi U)^R)T^{1/10}\quad\text{and}\quad \|l\sqrt{n}\|<(4\pi U)^{-1}\quad\text{for } 1\le n\le R.$$

Set now the constant $c=12V\log(4\pi U)$ in the definition of ξ and put

$k_\pm = (\sqrt{2}\, a_\pm + l\xi^{-1/2})^2$. Then from the range of l, we see that $T^{1/5} < k_\pm < T^{2/5}$. Since

$$2^{\sigma-1/4} \int_0^\infty e^{-2w^2} w^{2\sigma-3/2} \cos(4\sqrt{2}\, aw - \pi/4)\, dw = g(a),$$

by Lemma 3.3,

$$\begin{aligned}(5.4)\quad &\Big| 2^{5/4-3\sigma} \pi^{1/2-2\sigma} \zeta(2\sigma) g(a_\pm) \xi^{1/4-\sigma} \\ &\qquad - \sum_{n \le \xi^{-3}} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e^{-2\pi^2 n\xi} \cos(4\pi\sqrt{k_\pm n\xi} - \pi/4) \Big| \\ &\le \Big| \sum_{n \le \xi^{-3}} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e^{-2\pi^2 n\xi} \\ &\qquad \times (\cos(4\pi\sqrt{2}\, a_\pm \sqrt{\xi n} - \pi/4) - \cos(4\pi\sqrt{k_\pm n\xi} - \pi/4)) \Big| + O(\xi^{\sigma-3/4}).\end{aligned}$$

In the last series, the subsum $\sum_{R<n\le\xi^{-3}}$, by Lemma 3.1(2), contributes no more than $K_2 V^{\sigma-1/4} e^{-2\pi^2 V} \xi^{1/4-\sigma}$. For $n \le R$, we note that

$$|\cos(4\pi\sqrt{2}\, a_\pm \sqrt{\xi n} - \pi/4) - \cos(4\pi\sqrt{k_\pm n\xi} - \pi/4)| \le U^{-1}.$$

Hence, by Lemma 3.1(1),

$$\Big| \sum_{n \le R} \cdots \Big| \le U^{-1} K_1 (V\xi^{-1})^{\sigma-1/4}.$$

Combining all these, we see that the right hand side of (5.4) is

$$\begin{aligned}&\le (U^{-1} K_1 + e^{-2\pi^2 V} K_2) V^{\sigma-1/4} \xi^{1/4-\sigma} + O(\xi^{\sigma-3/4}) \\ &< \tfrac{1}{2} \cdot 2^{5/4-3\sigma} \pi^{1/2-2\sigma} \zeta(2\sigma) \min(g(a_+), |g(a_-)|) \xi^{1/4-\sigma},\end{aligned}$$

by our choice of U and V in (5.3). In other words,

$$\pm \sum_{n \le \xi^{-3}} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{5/4-\sigma}} e^{-2\pi^2 n\xi} \cos(4\pi\sqrt{k_\pm n\xi} - \pi/4) \gg \xi^{1/4-\sigma}.$$

Hence, by (5.2),

$$\pm \frac{1}{\Gamma(k_\pm + 1)} \int_0^\infty e^{-u^2} u^{2k_\pm+1} S(u\sqrt{\xi})\, du \gg (\log T)^{\sigma-1/4}.$$

Since $S(u\sqrt{\xi}) \ll B^{2\sigma-1/2} \le B$,

$$\frac{1}{\Gamma(k_\pm+1)} \Big(\int_0^{T^{1/11}} + \int_{T^{1/3}}^\infty \Big) e^{-u^2} u^{2k_\pm+1} S(u\sqrt{\xi})\, du \ll e^{-T^{1/5}}.$$

We can, therefore, conclude that $\sup_{t\in[T^{1/12},T^{1/3}]} \pm S(t) \gg (\log T)^{\sigma-1/4}$. Then from (5.1), there exist $t_\pm \in [T^{1/12}, T^{1/3}]$ such that

$$\pm t_\pm^{2\sigma-3/2} \int_{-L}^{L} E_\sigma(2\pi(t_\pm+u)^2)K(u)\,du \gg (\log t_\pm)^{\sigma-1/4}.$$

As $L = o(t_\pm^{1/2})$ and $T \to \infty$, this completes the proof of Theorem 1.

6. Proof of Theorem 2. In this section, we take $B = [L^{7/6}]/L \approx L^{1/6}$ so that BL is an integer and $1 \ll L \le T^{1/4}$. For $\sqrt{T/(2\pi)} + L \le t \le \sqrt{T/\pi} - L$, we proved in (4.2) that

$$\int_{-L}^{L} E_{3/4}(2\pi(t+u)^2)K(u)\,du = H(t) + O(1), \tag{6.1}$$

where

$$H(t) = \sum_{n\le B^2} a_n n^{-1/2}\cos(4\pi\sqrt{n}\,t - \pi/4),$$
$$a_n = (-1)^n\sqrt{2}(1-\sqrt{n}\,B^{-1})\sigma_{-1/2}(n).$$

We shall prove below that, for any interval I inside $[\sqrt{T/(2\pi)}+L, \sqrt{T/\pi}-L]$ of length L,

$$|I|^{-1}\int_I H(t)^2\,dt \gg \log L, \tag{6.2}$$

$$|I|^{-1}\int_I H(t)^3\,dt \ll 1. \tag{6.3}$$

Then by Lemma 3.4, when L is sufficiently large,

$$\sup_{t\in I} \pm H(t) \gg \sqrt{\log L}.$$

Taking I to be the interval $[\sqrt{T/(2\pi)}+L, \sqrt{T/(2\pi)}+2L]$, we infer from (6.1) that

$$\sup_{t\in I} \pm H(t) + O(1) \le \sup_y \pm E_{3/4}(y)\int_{-L}^{L} K(u)\,du \le \sup_y \pm E_{3/4}(y),$$

where $y = 2\pi(t+u)^2$ lies in $[T, T+72L\sqrt{T}]$. This is our Theorem 2, except for the condition $L \le T^{1/4}$. However, if $T^{1/4} < L \le T^{1/2}$, then certainly

$$\sup_{t\in[T,T+L\sqrt{T}]} \ge \sup_{t\in[T,T+T^{3/4}]} \gg \sqrt{\log T^{1/4}} \ge \sqrt{\frac{1}{2}\log L}.$$

When $L > T^{1/2}$ we have, by our above result for $L = \sqrt{T}$,

$$\sup_{t\in[T,T+L\sqrt{T}]} \geq \sup_{t\in[(T+L\sqrt{T})/2,T+L\sqrt{T}]} \gg \sqrt{\frac{1}{2}\log\left(\frac{T+L\sqrt{T}}{2}\right)} \asymp \sqrt{\log L}.$$

This completes the proof of our Theorem 2.

It therefore remains to prove (6.2) and (6.3).

Consider first (6.2). By squaring out $H(t)$ and then integrating the double sum term by term, we find that

$$\int_I H(t)^2\,dt = \frac{1}{2}\sum_{m,n\leq B^2}\frac{a_n a_m}{\sqrt{nm}}\int_I \cos(4\pi(\sqrt{n}-\sqrt{m})t)\,dt + \frac{1}{2}\sum_{m,n\leq B^2}\frac{a_n a_m}{\sqrt{nm}}\int_I \sin(4\pi(\sqrt{n}+\sqrt{m})t)\,dt.$$

The diagonal terms in the first sum (that is, those with $m = n$) contribute $\frac{1}{2}|I|\sum_{n\leq B^2} a_n^2 n^{-1} \gg |I|\log B$, since $\sum_{n\leq x}\sigma_{-1/2}(n)^2 n^{-1} \sim \log x$ (see [12, p. 374]). For $m \neq n$, by a crude estimate,

$$\int_I \cos(4\pi(\sqrt{n}-\sqrt{m})t)\,dt \ll |\sqrt{n}-\sqrt{m}|^{-1} \ll \sqrt{n}+\sqrt{m}.$$

Hence the non-diagonal terms' contribution is $\ll B^3$. Since $|I| = L \geq B^3$, (6.2) follows readily.

For the third power moment of $H(t)$ in (6.3), we use similar argument. Multiplying out $H(t)^3$ and then integrating term by term, we see that the contribution of the non-diagonal terms is

$$\ll \sum_{\substack{\sqrt{m}+\sqrt{n}\neq\sqrt{k}\\ m,n,k\leq B^2}} |a_m a_n a_k|(mnk)^{-1/2}|\sqrt{m}+\sqrt{n}-\sqrt{k}|^{-1} \ll B^6,$$

by observing that $|\sqrt{m}+\sqrt{n}-\sqrt{k}| \gg \max(m,n,k)^{-3/2}$ when $\sqrt{m}+\sqrt{n}-\sqrt{k} \neq 0$, and $\sum_{n\leq B^2}|a_n| \ll B^2$.

When $\sqrt{m}+\sqrt{n}-\sqrt{k} = 0$, we must have $m = sa^2$, $n = sb^2$ and $k = s(a+b)^2$, where s is square-free and a, b are natural numbers. Hence the sum of diagonal terms is equal to

$$\frac{3\sqrt{2}}{8}|I| \sum_{\substack{m,n,k\leq B^2\\ \sqrt{m}+\sqrt{n}=\sqrt{k}}} \frac{a_m a_n a_k}{\sqrt{mnk}} \ll |I|\sum_s s^{-3/2}\sum_{a,b}|a_{sa^2}a_{sb^2}a_{s(a+b)^2}|(ab(a+b))^{-1}.$$

Since $a_n \ll n^\varepsilon$, the sums over s, a, b are all convergent and therefore $\int_I H(t)^3\,dt \ll |I|$, as desired.

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