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On a problem of R. C. Baker

by

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1. Introduction. For a real number y, let $\langle y\rangle$ denote its fractional part. We will use L to denote the collection of Lebesgue integrable functions on $[0,1)$ and M to denote the bounded measurable functions on $[0,1)$. Let $\mathcal{A}$ be a collection of Lebesgue measurable functions on the interval $[0,1)$. Following [B], [M] we say that a strictly increasing sequence $(a_k)_{k=1}^{\infty}$ of natural numbers is an $\widehat{\mathcal{A}}$ *sequence* if for each f in $\mathcal{A}$ we have

$$\lim_{k\to\infty}\frac{1}{a_k}\sum_{j=1}^{a_k} f(\langle x+j/a_k\rangle)=\int_0^1 f(t)\,dt$$

almost everywhere with respect to Lebesgue measure. We say that $(a_k)_{k=1}^{\infty}$ is an $\mathcal{A}^*$ *sequence* if for each f in $\mathcal{A}$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N} f(\langle a_k x\rangle)=\int_0^1 f(t)\,dt$$

almost everywhere with respect to Lebesgue measure. Examples of sequences that are both $\widehat{M}$ and M^* appear in [B], [M], though the study of each class has an independent history going back to [J] and [K] respectively. See also [E], [S]. In this paper, in answer to a question raised in [B], we show that $a_k=2^{2^k}$ is an $\widehat{L}$ sequence but not an M^* sequence. Evidently $\widehat{L}\subseteq\widehat{M}$.

As is often the case in this subject a statement's verification is straightforward given that we have isolated the right general principle, and difficult without it. To show that $a_n=2^{2^k}$ is an $\widehat{L}$ sequence, we recall B. Jessen's theorem [J].

Theorem A. *Suppose that $(a_k)_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers such that a_k divides a_{k+1} for each k. Suppose also that f*

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is a Lebesgue integrable function on $[0,1)$. *Then*

$$\lim_{k\to\infty}\frac{1}{a_k}\sum_{j=1}^{a_k} f(\langle x+j/a_k\rangle)=\int_0^1 f(t)\,dt$$

almost everywhere with respect to Lebesgue measure.

It is worthwhile to note that Jessen's theorem is a consequence of J. L. Doob's decreasing martingale theorem [D]. We now show that $a_k = 2^{2^k}$ is not an M^* sequence. Let $(\mu_N)_{N=1}^{\infty}$ denote a sequence of probability measures on the integers. We call the sequence $(\mu_N)_{N=1}^{\infty}$ *dissipative* if

$$\lim_{N\to\infty}\mu_N(k)=0 \quad \text{for all integers } k.$$

Suppose that we have a set X, a σ-algebra $\mathcal{B}$ of its subsets, and a measure μ on X which is measurable with respect to $\mathcal{B}$. Suppose that T is map from X to itself. For A in $\mathcal{B}$ set $T^{-1}A=\{x : Tx\in A\}$. We call the map T *measurable* if $T^{-1}A$ is in $\mathcal{B}$ when A is; and we call it *measure preserving* if $\mu(T^{-1}A)=\mu(A)$ for all A in $\mathcal{B}$. We call the quadruple $(X,\mathcal{B},\mu,T)$ a *dynamical system* if it is measurable and measure preserving. A dynamical system $(X,\mathcal{B},\mu,T)$ is called *ergodic* if $\mu(A\bigtriangleup T^{-1}A)=0$ implies that $\mu(A)$ is either zero or one. Here for two sets A and B we have used $A\bigtriangleup B$ to denote their symmetric difference.

For a sequence $(\mu_N)_{N=1}^{\infty}$ of probability measures on the integers and f in $L^1(X,\mathcal{B},\mu)$,

$$(\mu_N f)(x)=\sum_{k=-\infty}^{\infty}\mu_N(k)f(T^k x) \quad (N=1,2,\ldots).$$

Given $\delta>0$, a sequence of probability measures $(\mu_N)_{N=1}^{\infty}$ is called δ-*sweeping out* if for all ergodic dynamical systems $(X,\mathcal{B},\mu,T)$ and all $\varepsilon>0$ there exists E in $\mathcal{B}$ such that $\mu(E)\le\varepsilon$ and

$$\limsup_{N\to\infty}\mu_N I_E(x)\ge\delta$$

μ-almost everywhere. Here I_E denotes the indicator function of the set E.

We need the following theorem proved in [Ro].

THEOREM B. *Suppose that* $S=(b_k)_{k=1}^{\infty}$ *is a sequence of integers with*

$$\inf_{k\ge1}\frac{b_{k+1}}{b_k}>1,$$

and that each measure μ_N $(N=1,2,\ldots)$ *has support contained in* S. *Then* $(\mu_N)_{N=1}^{\infty}$ *is* δ*-sweeping out for some* $\delta>0$.

We use this theorem by applying it to the setting where

$$b_k=2^k \quad (k=1,2,\ldots),$$

$$\mu_N = \frac{1}{N}\sum_{k=1}^{N} \delta_{b_k} \qquad (N = 1, 2, \ldots)$$

for delta measures δ_a defined by

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A, \end{cases}$$

defined on the integers, $X = [0,1)$, $\mathcal{B}$ is the Lebesgue σ-algebra, μ is Lebesgue measure on $[0,1)$ and the map T is defined by $Tx = \langle 2x \rangle$. The fact that T both preserves Lebesgue measure on $[0,1)$ and is ergodic is proved in [W]. For a Lebesgue measurable set A let $|A|$ denote its Lebesgue measure. The upshot of this is that there exists $\delta > 0$ such that for any $\varepsilon > 0$, there exists a Lebesgue measurable set E contained in $[0,1)$ such that $|E| < \varepsilon$ and

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} I_E(\langle 2^{2^k} x \rangle) \ge \delta > 0$$

almost everywhere with respect to Lebesgue measure.

Thus of course $a_k = 2^{2^k}$ $(k = 1, 2, \ldots)$ is not an M^* sequence. It would be interesting to know if we could choose $\delta = 1$. Evidently in the previous argument Lebesgue measurable can be replaced by Borel measurable everywhere.

Plainly for any strictly increasing sequence $(c_k)_{k=1}^{\infty}$ of natural numbers, the sequence $a_k = 2^{c_k}$ $(k = 1, 2, \ldots)$ is an $\widehat{L}$ sequence. Given p in $[1, \infty)$ it is possible to give strictly increasing sequences $(c_k)_{k=1}^{\infty}$ of integers such that $a_k = 2^{c_k}$ $(k = 1, 2, \ldots)$ is in $(L^p)^*$ but not in $(L^q)^*$ for any $q < p$. Here L^p denotes the space of Lebesgue measurable functions on $[0,1)$ whose pth powers are Lebesgue integrable. This observation relies on a result of K. Reinhold-Larsson [RL].

Theorem C. *Given p in $[1, \infty)$, there exists a strictly increasing sequence $(c_k)_{k=1}^{\infty}$ of natural numbers such that for every dynamical system $(X, \mathcal{B}, \mu, T)$ and every function f in $L^p(X, \mathcal{B}, \mu)$ there exists $C_p > 0$ such that if*

$$Mf(x) = \Big| \sup_{N \ge 1} \sum_{k=1}^{N} f(T^{c_k} x) \Big|,$$

then

$$\mu(\{x \in X : Mf(x) > \alpha\}) \le \frac{C_p}{\alpha^p} \, \|f\|_p$$

where

$$\|f\|_p = \Big(\int_X |f|^p(x) \, d\mu \Big)^{1/p}.$$

Also if $1 \le q < p$ *then there exists* f *in* $L^q(X, \mathcal{B}, \mu)$ *such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(T^{c_k} x)$$

does not have a finite limit for almost all x *with respect to* μ.

Choosing $X = [0,1)$, $\mathcal{B}$ to be the Lebesgue σ-algebra, μ the Lebesgue measure and $Tx = \langle 2x \rangle$ and using Theorem C as before shows that $a_k = 2^{c_k}$ ($k = 1, 2, \ldots$) does not belong to $(L^q)^*$. To show that $(2^{c_k})_{k=1}^{\infty}$ is in $(L^p)^*$ we need to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\langle 2^{c_k} x \rangle) = \int_0^1 f(t)\, dt$$

almost everywhere with respect to Lebesgue measure. By a classical theorem of H. Weyl [Wy] this is known for continuous functions on $[0,1)$. Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions on $[0,1)$ converging to f in L^p norm. This means that there exists a subsequence $(n_k)_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \int_0^1 |f - f_{n_k}|^p(x)\, dx < \infty,$$

which implies that

$$\sum_{k=1}^{\infty} |f - f_{n_k}|^p(x) < \infty$$

almost everywhere with respect to Lebesgue measure on $[0,1)$. Thus for every $\varepsilon > 0$, there exists a sequence of functions $(f_{\varepsilon,k})_{k=1}^{\infty}$ such that

$$\|f - f_{\varepsilon,k}\|_p^p \le \varepsilon^{2k}$$

and $f_{\varepsilon,k}$ tends to f as k tends to infinity almost everywhere with respect to Lebesgue measure on $[0,1)$. Notice that

$$M(f+g) \le M(f) + M(g).$$

Let

$$E_{\varepsilon,k} := \{x \in [0,1) : M(f - f_{\varepsilon,k})(x) > \varepsilon^{k/p}\}$$

and note from Theorem C that

$$\mu(E_{\varepsilon,k}) \le C_p \left(\frac{1}{\varepsilon}\right)^k \int_{E_{\varepsilon,k}} |f - f_{\varepsilon,k}|^p(x)\, dx \le C_p \left(\frac{1}{\varepsilon}\right)^k \varepsilon^{2k} = C_p \varepsilon^k.$$

Let $a_N(f,x)$ denote $\frac{1}{N}\sum_{l=1}^{N} f(\langle 2^{c_l} x \rangle)$. Now

$$a_N(f,x) = a_N(f - f_{\varepsilon,k} x) + a_N(f_{\varepsilon,k}, x).$$

This means that

$$\Big|a_N(f,x)-\int_0^1 f(t)\,dt\Big| \le |a_N(f-f_{\varepsilon,k},x)| + \Big|a_N(f_{\varepsilon,k},x)-\int_0^1 f(t)\,dt\Big|$$

almost everywhere with respect to Lebesgue measure on $[0,1)$. Thus

$$\limsup_{N\to\infty}\Big|a_N(f,x)-\int_0^1 f(t)\,dt\Big| \le \limsup_{N\to\infty}|a_N(f-f_{\varepsilon,k},x)| + \Big|\int_0^1 (f-f_{\varepsilon,k})(t)\,dt\Big|,$$

which is

$$\le M(f-f_{\varepsilon,k})(x) + \int_0^1 |f-f_{\varepsilon,k}|(t)\,dt.$$

Therefore as N tends to infinity we know that $a_N(f,x)$ tends to $\int_0^1 f(t)\,dt$ for all x in $E_\varepsilon = \bigcup_{n=1}^\infty E_{\varepsilon,n}$. Let B_ε be the null set off which $f_{\varepsilon,k}$ tends to f as $k\to\infty$. This means that

$$\lambda(E_\varepsilon \cup B_\varepsilon) \le \sum_{n=1}^\infty \lambda(E_{\varepsilon,k}) \le C_p\sum_{k=1}^\infty \varepsilon^k = \frac{C_p\varepsilon}{1-\varepsilon}.$$

Letting ε tend to zero shows that $(2^{c_k})_{k=1}^\infty$ is in $(L^p)^*$ for finite p.

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