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On an asymptotic formula of Srinivasa Ramanujan

by

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To Professor R. Balasubramanian on his fiftieth birthday

1. Introduction. In [16], Ramanujan records (without proof) many curious asymptotic formulae. One of them is

$$d^2(1)+d^2(2)+\ldots+d^2(n)=An(\log n)^3+Bn(\log n)^2+Cn\log n + Dn+O(n^{3/5+\varepsilon}). \tag{1.1}$$

Also he records (without proof) the result that on the assumption of the Riemann hypothesis, the error term in (1.1) can be improved to $O(n^{1/2+\varepsilon})$. In view of a method due to H. L. Montgomery and R. C. Vaughan (see [9]), it is very likely that the error term is $O(n^{1/2})$. We propose this as a conjecture (see also [15], [17]). Unconditionally, the error term related to $d^2(j)$ is known to be $O(n^{1/2+\varepsilon})$ for any positive constant ε (see for example the equation (14.30) of [6] and also [5]). Professor A. Schinzel has already considered some of the problems of Ramanujan (see [19]), namely for the arithmetic function $r^2(n)$, and he has proved that the corresponding error term is $\Omega(n^{3/8})$ and also the corresponding error term is $O(n^{1/2}(\log n)^{8/3}(\log\log n)^{1/3})$ due to an unpublished work of W. G. Nowak (see also [8] and [18]). Let

$$E(x)=\sum_{n\le x} d^2(n)-xP_3(\log x) \tag{1.2}$$

where $P_3(y)$ is a polynomial in y of degree 3. From a general theorem of M. Kühleitner and W. G. Nowak (see for example (5.4) of [8]), it follows that

$$E(x)=\Omega(x^{3/8}).$$

From Vinogradov's estimate (for $T/2\le t\le T$)

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(1.3) $$\frac{1}{\zeta(1+it)} \ll (\log T)^{2/3}(\log\log T)^{1/3},$$

it is not very difficult to prove

THEOREM A. *We have*

$$E(x) = O(x^{1/2}(\log x)^{17/3}(\log\log x)^{1/3}).$$

REMARK. We note here that an analogue of Theorem A for the "sums of two squares" function $r(n)$ was dealt with by M. Kühleitner (see [7]). We also refer to the related papers [2], [3], [12] and [20].

On the assumption of the quasi-Riemann hypothesis (namely $\zeta(s) \neq 0$ for $\sigma > \alpha$ with $1/2 < \alpha < 1$), following the proof of Theorems 14.6 and 14.8 of [21], we obtain the inequality

(1.4) $$\frac{1}{|\zeta(1+2it)|} \ll_\alpha \log\log t.$$

Hence one gets

COROLLARY. *On the assumption of the quasi-Riemann hypothesis*, *we have*

$$E(x) = O(x^{1/2}(\log x)^5(\log\log x)).$$

The main goal of this paper is to prove

MAIN THEOREM. *Unconditionally*, *we have*

$$E(x) = O(x^{1/2}(\log x)^5(\log\log x)).$$

REMARK. It is not difficult to prove an ineffective result like

$$E(x) = \Omega_\pm(x^{1/4}).$$

The ineffective version is due to E. Landau (see [4]). The general method of proving results like the one above (actually in an effective way) is due to R. Balasubramanian and K. Ramachandra (see [1]).

2. Notation and preliminaries.

2. Notation and preliminaries. C and A (with or without subscripts) denote effective positive constants unless specified otherwise; ε will always denote a sufficiently small positive constant; $T \geq T_0$ (a sufficiently large positive constant). We write $f(x) \ll g(x)$ to mean $|f(x)| < C_1 g(x)$ (sometimes we denote this by the O notation also). Let $s = \sigma + it, s_0 = 1/2 + it$ and $w = u + iv$. The notation $[x]$ denotes the integral part of x whereas $[a, b]$ denotes the interval $a \leq c \leq b$. The implied constants are all effective.

3. Some lemmas

LEMMA 3.1 (Refined version of Perron's formula). *Let* $\{\lambda_n\}$ *be a sequence of real numbers with* $0 < \lambda_1 < \ldots < \lambda_n \to \infty$ *and* $\{a_n\}$ *be any se-*

quence of complex numbers such that $f(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ (*with* $s = \sigma + it$) *is absolutely convergent in* $\sigma > 1$. *Then for* $x > 0$, $C > 1$, *we have uniformly* (*in all the parameters*) *the equality*

$$(3.1.1)\qquad \frac{1}{T}\int_0^T \dots \frac{1}{T}\int_0^T \left(\frac{1}{2\pi i}\int_{C-iT-i\phi}^{C+iT+i\phi} f(s)\,\frac{x^s}{s}\,ds\right)d\tau_1\dots d\tau_k$$

$$= \sum_{\lambda_n \le x} a_n + \frac{\theta}{\pi}\sum_{n=1}^{\infty}|a_n|\left(\frac{x}{\lambda_n}\right)^C \min(\pi + 2 + C/T, M)$$

where

$$(3.1.2)\qquad M = \min_{0\le m\in\mathbb{Z}^+\le k}\left(\frac{2^{m+1}-1}{m+1}\right)\left|T\log\left(\frac{x}{\lambda_n}\right)\right|^{-m-1},$$

θ *is a complex number with* $|\theta| \le 1$ (*moreover* θ *is real if* a_n *are all real*) *and* $T > 0, \tau_1, \dots, \tau_k$ *are real variables with* $0 \le \tau_j \le T$ $(j = 1, \dots, k)$, $\phi = \tau_1 + \dots + \tau_k$ (*we define an empty sum as zero*).

Proof. See Corollary 2 of [14]. ∎

Lemma 3.2. *Let* $T/2 < t_1 < \dots < t_R \le T$ *be well spaced points satisfying* $|t_{j+1} - t_j| \ge 1$ (*for* $j = 1, \dots, R-1$), *and suppose that for every small positive constant* ε, *the points* t_j *satisfy the inequality*

$$|\log\zeta(1 + it_j)| \gg \log\log\log T - 10\log\varepsilon.$$

Then

$$R \ll T^{2\varepsilon}.$$

Remark. This is Theorem 1 of [13]. For the sake of completeness, we present here a simple proof of Lemma 3.2.

Proof of Lemma 3.2. First of all we note that from the density estimates, we have (see [6])

$$N(\sigma, T, 2T) \ll T^{\frac{12}{5}(1-\sigma)}(\log T)^{100}.$$

Let δ be a small positive constant, say $0 < \delta < 1/100$. Suppose that the number $N(1-\delta, T, 2T)$ of zeros of $\zeta(s)$ in $\{\sigma \ge 1-\delta,\ T \le t \le 2T\}$ is $< T^{\eta}$ where $\eta > 0$ is a small positive constant (may depend on δ). Let $\varrho = \beta + i\gamma$ be any of these zeros. With each such zero, we associate the rectangle

$$\{\sigma \ge 1 - \delta,\ t \in (\gamma - (\log T)^{100},\ \gamma + (\log T)^{100})\}.$$

Let s be any point in the complement in $\{\sigma \ge 1-\delta, T \le t \le 2T\}$ of the union of all these rectangles. (Note that we have excluded a total of t-height $\le 2(\log T)^{100}T^{\eta}$ in $\{\sigma \ge 1-\delta,\ T \le t \le 2T\}$.) From the density estimate above, we observe that the region $\{\sigma \ge 1-\delta,\ s \pm (\log T)^{100}\}$ is zero-free of $\zeta(s)$. Now, we can talk of $\log\zeta(s)$ in this region. If necessary, we can exclude further $\frac{1}{2}(\log T)^{100}$ on either side of this region. The total

t-length thus excluded is $\leq 10(\log T)^{100}T^{\eta}$. Now, in the resulting region, we can not only talk of $\log\zeta(s)$ but even apply the Borel–Carathéodory theorem in $\sigma \geq 1-\delta/2$ (with centres on the line $\sigma = 2$). Therefore, in $\{\sigma \geq 1-\delta/4,\ s \pm \frac{1}{2}(\log T)^{100}\}$, we have $\log\zeta(s) = O(\log T)$. Now, for $\sigma \geq 1-\delta/8$, $T \leq t \leq 2T$, we have (with $w = u+iv$ and fixing $X = (\log T)^{8/\delta}$)

$$\frac{1}{2\pi i}\int\limits_{\Re w=\delta/4,\,|v|\leq(\log T)^3} \log\zeta(s+w)\Gamma(w)X^w\,dw = \sum_p \frac{e^{-p/X}}{p^s} + O\left(\log\left(\frac{8}{\delta}\right)e^{-C(\log T)^3}\right).$$

Now, we move the line of integration in the remaining integral above to $\sigma + u = 1-\delta/4$, that is to $u = -\delta/8$. The pole at $w = 0$ of $\Gamma(w)$ gives the residue $\log\zeta(s)$. Note that our $X = (\log T)^{8/\delta}$. The horizontal portions contribute an error which is $\ll (\log T)X^{\delta/4}e^{-(\log T)^3} \ll 1$ because of the presence of the $\Gamma(w)$ in the integrand, whereas the vertical line integral on $u = -\delta/8$ contributes an error which is $\ll (\log T)X^{-\delta/8} \ll 1$ with our choice of X. Note that

$$\sum_p \frac{e^{-p/X}}{p^s} = \sum_{p\leq X^2}\frac{1}{p} + O(1) = \log\log X^2 + O(1).$$

Therefore we obtain

$$\log\zeta(s) = \log\log X^2 + O(1) + O(\log(8/\delta)e^{-C(\log T)^3})$$

and this implies that

$$\pm\log|\zeta(s)| \leq \log\log X^2 + O(1) + O(\log(8/\delta)e^{-C(\log T)^3}).$$

So, if we exclude t-intervals of total width $\leq T^{1000\delta}$ on the line $\sigma = 1$, for the rest, we have (for $\sigma \geq 1$)

$$|\zeta(\sigma+it)|^{\pm 1} \ll \log\log T.$$

Since η and δ are arbitrary, this proves the lemma. ∎

LEMMA 3.3. *We have* (*with* $s_0 = 1/2 + it$)

$$x^{1/2}\int\limits_{|t|\leq T}\left|\frac{\zeta^4(s_0)}{\zeta(2s_0)}\right|\left|\frac{dt}{s_0}\right| \ll x^{1/2}(\log T)^5(\log\log T).$$

Proof. Let

$$\left|\frac{1}{\zeta(1+2it_j)}\right| = \max_{j<t\leq j+1}\left|\frac{1}{\zeta(1+2it)}\right|. \tag{3.3.1}$$

Also we have (for $T/2 \leq t \leq T$)

$$\zeta(1/2+it) \ll T^{1/6}(\log T) \tag{3.3.2}$$

and

$$\frac{1}{\zeta(1+2it)} \ll \log T. \tag{3.3.3}$$

It is well known that (for example see [6] or [21]) for $\sigma \geq 1/2$,

$$\int_{T/2}^{T} |\zeta^4(\sigma+it)|\, dt \ll T(\log T)^4. \tag{3.3.4}$$

We divide the interval $[[T/2]+1, [T]]$ into abutting small intervals of width 1. Below, $\sum^*$ denotes sums over odd integers, and $\sum^{**}$ denotes sums over even integers in the given interval.

We call a unit interval $[j, j+1] \subset [[T/2]+1, [T]]$ a *bad unit interval* if

$$|\log \zeta(1+it_j)| \gg \log\log\log T - 10 \log \varepsilon. \tag{3.3.5}$$

From Lemma 3.2, we observe that the number of bad unit intervals in $[[T/2]+1, [T]]$ is at most $T^{2\varepsilon}$. For the remaining *good unit intervals* in $[[T/2]+1, [T]]$, we can use the bound

$$\frac{1}{\zeta(1+it_j)} \ll \log\log T. \tag{3.3.6}$$

Therefore, we obtain

$$\begin{aligned}
x^{1/2} &\int_{|t|\leq T} \left|\frac{\zeta^4(s_0)}{\zeta(2s_0)}\right| \left|\frac{dt}{s_0}\right| \\
&\ll x^{1/2} + x^{1/2}\,\frac{\log T}{T}\left(\int_{T/2}^{T} \left|\frac{\zeta^4(s_0)}{\zeta(2s_0)}\right| dt\right) \\
&\ll x^{1/2} + x^{1/2}\,\frac{\log T}{T}\left\{\sum_{j=[T/2]+1}^{[T]-1} \int_{j}^{j+1} \left|\frac{\zeta^4(s_0)}{\zeta(2s_0)}\right| dt + T^{2/3}(\log T)^{10}\right\} \\
&\ll x^{1/2} + x^{1/2}\,\frac{\log T}{T} \\
&\qquad \times \left(\left\{\sum_{j=[T/2]+1}^{[T]-1}{}^{*} + \sum_{j=[T/2]+1}^{[T]-1}{}^{**}\right\} \int_{j}^{j+1} \left|\frac{\zeta^4(s_0)}{\zeta(2s_0)}\right| dt + T^{2/3}(\log T)^{10}\right) \\
&\ll x^{1/2} + x^{1/2}\,\frac{\log T}{T} \\
&\qquad \times \left(\left\{\sum_{j=[T/2]+1}^{[T]-1}{}^{*} + \sum_{j=[T/2]+1}^{[T]-1}{}^{**}\right\} \left|\frac{1}{\zeta(1+2it_j)}\right| \int_{j}^{j+1} |\zeta^4(s_0)|\, dt + T^{2/3}(\log T)^{10}\right)
\end{aligned}$$

$$\ll x^{1/2}+x^{1/2}\,\frac{\log T}{T}\Big(T^{2/3+10\varepsilon}(\log T)^{20}+(\log\log T)\Big(\int_{[T/2]+1}^{[T]}|\zeta^4(s_0)|\,dt\Big)\Big)$$
$$\ll x^{1/2}+x^{1/2}(\log T)^5(\log\log T).$$

This proves the lemma. ∎

Lemma 3.4. *For $\sigma\ge 1/2$, we have*

$$\int_{1/2}^{1}\int_{T/2}^{T}\left|\frac{\zeta^4(\sigma+it)}{\zeta(2\sigma+2it)}\right|\left|\frac{x^s}{s}\right|d\sigma\,dt\ll(\log T)^4(\log\log T)(x-x^{1/2})(\log x)^{-1}.$$

Proof. First of all we notice that by following the argument for Lemma 3.3, we obtain, for $\sigma\ge 1/2$,

$$\text{(3.4.1)}\qquad \int_{T/2}^{T}\left|\frac{\zeta^4(\sigma+it)}{\zeta(2\sigma+2it)}\right|dt\ll\int_{T/2}^{T}\left|\frac{\zeta^4(\sigma+it)}{\zeta(1+2it)}\right|dt$$
$$\ll(\log\log T)\int_{T/2}^{T}|\zeta^4(1/2+it)|\,dt$$
$$\ll T(\log T)^4(\log\log T).$$

Therefore, from (3.4.1), we obtain

$$\text{(3.4.2)}\qquad \int_{T/2}^{T}\left|\frac{\zeta^4(\sigma+it)}{\zeta(2\sigma+2it)}\right|\left|\frac{x^s}{s}\right|dt\ll\frac{\log\log T}{T}\int_{T/2}^{T}|\zeta^4(1/2+it)|\,|x^s|\,dt$$
$$\ll(\log T)^4(\log\log T)x^{\sigma}.$$

Hence, we get

$$\int_{1/2}^{1}\int_{T/2}^{T}\left|\frac{\zeta^4(\sigma+it)}{\zeta(2\sigma+2it)}\right|\left|\frac{x^s}{s}\right|d\sigma\,dt\ll\int_{1/2}^{1}\int_{T/2}^{T}|\zeta^4(\sigma+it)|(\log\log T)x^{\sigma}\,\frac{d\sigma\,dt}{|t|}$$
$$\ll(\log T)^4(\log\log T)(x-x^{1/2})(\log x)^{-1}.\ ∎$$

4. Proof of the Main Theorem. In Lemma 3.1, we take

$$C=1+\frac{1}{\log x},\qquad f(s)=\frac{\zeta^4(s)}{\zeta(2s)},$$

and hence we obtain

$$\text{(4.1)}\qquad \frac{1}{T}\int_0^T\cdots\frac{1}{T}\int_0^T\bigg(\frac{1}{2\pi i}\int_{1+1/\log x-iT-i\phi}^{1+1/\log x+iT+i\phi}\frac{\zeta^4(s)}{\zeta(2s)}\,\frac{x^s}{s}\,ds\bigg)\,d\tau_1\dots d\tau_k$$
$$=\sum_{n\le x}d^2(n)+\frac{\theta}{\pi}\sum_{n=1}^{\infty}d^2(n)\Big(\frac{x}{n}\Big)^{1+1/\log x}\min\Big(\pi+2+\frac{1+\log x}{T\log x},M\Big),$$

where

$$M = \min_{0 \le m \in \mathbb{Z}^+ \le k} \left(\frac{2^{m+1} - 1}{m+1} \right) |T \log(x/n)|^{-m-1}, \tag{4.2}$$

with $|\theta| \le 1$. Now, we fix $m = 0$ in (4.2) so that, from (4.1), we get

$$\frac{1}{2\pi i} \int_{1+1/\log x - iT - i\phi}^{1+1/\log x + iT + i\phi} \frac{\zeta^4(s)}{\zeta(2s)} \frac{x^s}{s} \, ds = \sum_{n \le x} d^2(n) + E + \theta_1 x^{\varepsilon}, \tag{4.3}$$

where

$$E = \frac{\theta}{\pi} \sum_{n=1}^{\infty} d^2(n) \left(\frac{x}{n} \right)^{1+1/\log x} \min \left(\pi + 2 + \frac{1 + \log x}{T \log x}, \left| T \log \left(\frac{x}{n} \right) \right|^{-1} \right) \tag{4.4}$$

and $|\theta_1| \le 1$.

Estimation of E. We choose $T = x^{1/2}$.

Case (i). Suppose that $|x - n| \le x^{\varepsilon}$. Then (since $\pi + 2 + \frac{1+\log x}{T \log x} \le 100$), we obtain

$$|E_{|x-n| \le x^{\varepsilon}}| \le \frac{100}{\pi} \sum_{|x-n| \le x^{\varepsilon}} d^2(n) \ll x^{2\varepsilon}. \tag{4.5}$$

Case (ii). Suppose that $|x - n| \ge x/2$. Therefore, we observe that

$$\left| \log \left(\frac{x}{n} \right) \right|^{-1} \le \frac{x}{|x - n|} \le 10$$

and hence

$$\begin{aligned} |E_{|x-n| \ge x/2}| &\le \frac{10}{T} \sum_{|x-n| \ge x/2} d^2(n) \left(\frac{x}{n} \right)^{1+1/\log x} \\ &\le \frac{10}{T} \sum_{n=1}^{\infty} d^2(n) \left(\frac{x}{n} \right)^{1+1/\log x} \ll \frac{x}{T} (\log x)^4. \end{aligned} \tag{4.6}$$

Case (iii). Suppose that $x^{\varepsilon} \le |x - n| \le x/2$. A result of Nair and Tenenbaum (see [11] and also [10]) states that

$$\sum_{L \le n \le L+h} d^2(n) \ll h (\log L)^3, \tag{4.7}$$

for $h \ge L^{\varepsilon}$. We notice that

$$\left(\frac{x}{n} \right)^{1+1/\log x} \le 4 \quad \text{for } n \in [x/2, x - x^{\varepsilon}] \cup [x + x^{\varepsilon}, 3x/2].$$

Therefore, from (4.7), we have

$$|E_{x^\varepsilon \le |x-n| \le x/2}| \le \frac{100x}{T} \sum_{x^\varepsilon \le |x-n| \le x/2} d^2(n) \frac{1}{|x-n|} \tag{4.8}$$
$$\le \frac{100x}{T} \sum_{U, U=2^l x^\varepsilon} \sum_{U \le |x-n| < 2U} d^2(n) \frac{1}{U}$$
$$\le \frac{100x}{T} \sum_{2x \ge U \ge x^\varepsilon} \frac{1}{U} \sum_{n \in (x-2U, x-U) \cup (x+U, x+2U)} d^2(n)$$
$$\ll \frac{x(\log x)^4}{T}.$$

So, from (4.5), (4.6) and (4.8), we conclude that

$$|E| \ll x^\varepsilon + (x(\log x)^4)/T \tag{4.9}$$

for any small positive constant ε. Now, we choose a suitable horizontal line $t = t_0 \in [T/2, T]$ and we move the line of integration appearing in (4.3) to the line $\sigma = \sigma_0 = 1/2$ along $t = t_0$. We observe that the pole at $s = 1$ (which is of order 4) contributes the main term and from Lemma 3.4, we observe that the horizontal portions contribute an error

$$\ll \frac{x}{T} (\log T)^4 (\log\log T)(\log x)^{-1}. \tag{4.10}$$

Also, from Lemma 3.3, we observe that the vertical portion (with $s_0 = 1/2 + it$) in absolute value is

$$\ll x^{1/2} \int_{|t| \le T} \left| \frac{\zeta^4(s_0)}{\zeta(2s_0)} \right| \left| \frac{dt}{s_0} \right| \ll x^{1/2} (\log T)^5 (\log\log T). \tag{4.11}$$

Now, our choice $T = x^{1/2}$ proves the Main Theorem.

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