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Modular case of Levinson's theorem

by

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Contents

1. Introduction and overview of the results. Nowadays, we know that more than 41% of the non-trivial zeros of the Riemann zeta function lie on the critical line [BCY11], [Fen12]. This is the best of a sequence of results about the percentage of zeros ρ satisfying $\Re\rho = 1/2$.

Historically, Selberg [Sel42] was the first one to show that this proportion is not zero without quantifying it. According to Titchmarsh [Tit86, Sect. 10.9], it was calculated later on in Min's dissertation that the proportion obtained by Selberg's method is very small. One may refer to the introduction of [Ste07, p. 8] for numerical values. In 1974, Levinson [Lev74] succeeded in proving that at least one-third of the non-trivial zeros lie on the critical line, by perturbing the Riemann zeta function by a linear combination of its derivatives. A significant improvement, due to Conrey [Con89], increased this proportion to more than two-fifths. In order to do this, he

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improves the general result of [BCHB85] on the asymptotic behaviour of the mollified second moment of the Riemann zeta function, when the coefficients of the mollifier are essentially given by the Möbius function, which allows him to work with a longer mollifier than Levinson's one. From this last result of Conrey and using a two-part mollifier, Bui, Conrey and Young [BCY11] proved that 41% of the non-trivial zeros ρ satisfy $\Re\rho = 1/2$.

Since the Riemann zeta function is an L-function of degree one, it is rather natural to generalise these results to L-functions of higher degrees. For instance, Hafner has extended Selberg's result to L-functions of degree two. More precisely, if f is a holomorphic cusp form of even weight and full level or an even Maaß form of full level, let $N_f(T)$ (resp. $N_{f,0}(T)$) be the number of non-trivial zeros ρ (resp. on the critical line) of $L(f,s)$ with $0 < \Im\rho \leq T$. Hafner [Haf83], [Haf87] proved that there exists a positive number A such that $N_{f,0}(T) > AN_f(T)$ for large T. Rezvyakova [Rez10] adapted [Haf83] to L-functions attached to automorphic cusp forms for congruence subgroups. Nevertheless, they do not give any explicit value for A, but by analogy with the Riemann zeta function case this constant should probably be close to zero.

In [Far94], Farmer applied Levinson's method to L-functions of a holomorphic cusp form f of even weight and full level, and succeeded in determining the asymptotic behaviour of the mollified integral second moment of $L(f,s)$ when the mollifier is a Dirichlet polynomial of length less than $T^{1/6-\varepsilon}$. From this result, Farmer obtained explicit lower bounds for the proportion of simple zeros of the jth derivative ($j \geq 1$) of the completed L-function of $L(f,s)$ which are on the critical line. Unfortunately, the length of the mollifier is too small to exhibit an explicit positive proportion of simple zeros on the critical line for $L(f,s)$ itself. Nevertheless, even though Farmer did not remark it, his result proves that at least 1.65% of the zeros of $L(f,s)$ satisfy $\Re s = 1/2$ (see Section 4).

In this paper, we exhibit a positive proportion of zeros which lie on the critical line for L-functions of holomorphic primitive cusp forms. To get this result, we study the asymptotic behaviour of the smooth mollified second moment of $L(f,s)$ following the method developed in [You10]. We choose a mollifier ψ, defined on page 206, which is a Dirichlet polynomial of length T^ν of the shape

$$\psi(s) = \sum_{n \leq T^\nu} \frac{\mu_f(n)}{n^{s+1/2-\sigma_0}} P\left(\frac{\ln(M/n)}{\ln M}\right)$$

with $M = T^\nu$, $\sigma_0 = 1/2 - R/\ln T$ where R is a positive real number. Moreover, we introduce a smooth function w compactly supported in $[T/4, 2T]$ with some conditions on its derivatives (see (9a)–(9c)). We prove the following theorem.

Theorem 1. *Let f be a holomorphic primitive cusp form of even weight, square-free level and trivial character. If $0 < \nu < \frac{1-2\theta}{4+2\theta}$ and if α, β are complex numbers satisfying $\alpha, \beta \ll L^{-1}$ with $|\alpha+\beta| \gg L^{-1}$, then*

$$\int_{-\infty}^{\infty} w(t) L(f, 1/2+\alpha+it) L(f, 1/2+\beta-it) |\psi(\sigma_0+it)|^2 \, dt = \widehat{w}(0) c(\alpha,\beta) + O(T(\ln L)^4/L) \tag{1}$$

where

$$c(\alpha,\beta) = 1 + \frac{1}{\nu} \frac{1-T^{-2(\alpha+\beta)}}{(\alpha+\beta)\ln T} \frac{d^2}{dx\,dy} \Big[M^{-\beta x - \alpha y} \int_0^1 P(x+u) P(y+u) \, du \Big] \Big|_{x=y=0} \tag{2}$$

and where $\theta = 7/64$ is the exponent in the approximation towards the Ramanujan–Petersson–Selberg conjecture (*see* (19)).

Corollary 1. *Let f be a holomorphic primitive cusp form of even weight, square-free level and trivial character. At least* 2.97% *of the non-trivial zeros of $L(f,s)$ lie on the critical line $\Re s = 1/2$. Assuming the Selberg conjecture, one can improve this percentage to* 6.93%. *In other words,*

$$\liminf_{T\to\infty} \frac{N_{f,0}(T)}{N_f(T)} \geq \begin{cases} 0.0297 & \text{unconditionally,} \\ 0.0693 & \text{under the Selberg conjecture.} \end{cases}$$

Remark 1. As Heath-Brown [HB79] and Selberg pointed out, the study of the second mollified moment allows one to obtain a lower bound for the proportion of simple non-trivial zeros lying on the critical line. Unfortunately, in our case, the length of the mollifier is too small to get a positive proportion of simple zeros satisfying the Riemann hypothesis. We plan to get back to this issue in the near future.

The method we use can also be applied to determine the asymptotic behaviour of the smooth second moment of $L(f,s)$ close to the critical line.

Theorem 2. *Let f be a holomorphic primitive cusp form of even weight, square-free level N and trivial character. If α, β are complex numbers satisfying $\alpha, \beta \ll L^{-1}$, then*

$$\int_{-\infty}^{\infty} w(t) L(f, 1/2+\alpha+it) L(f, 1/2+\beta-it) \, dt = \mathfrak{a}_f \int_{-\infty}^{\infty} w(t) \ln t \, dt + \Big[\mathfrak{b}_f + \mathfrak{a}_f \ln\Big(\frac{\sqrt{N}}{2\pi}\Big) \Big] \widehat{w}(0) + O\big(|\alpha+\beta| T (\ln T)^2 + T^{1/2+\theta+\varepsilon}\big)$$

with

$$\mathfrak{a}_f = \frac{12N}{\pi^2\nu(N)} L(\operatorname{Sym}^2 f, 1),$$

$$\mathfrak{b}_f = \frac{12N}{\pi^2\nu(N)} L(\operatorname{Sym}^2 f, 1)\left(\frac{L'(\operatorname{Sym}^2 f, 1)}{L(\operatorname{Sym}^2 f, 1)} + \gamma + \sum_{p\mid N} \frac{\ln p}{p+1} - \frac{2\zeta'(2)}{\zeta(2)}\right),$$

$$\nu(N) = N\prod_{p\mid N}\left(1+\frac{1}{p}\right).$$

REMARK 2. Our result is non-trivial only in the case $|\alpha+\beta| = o(1/\ln T)$, and furthermore we need $|\alpha+\beta| = o(1/\ln^2 T)$ to ensure that the term of order T is significant.

COROLLARY 2. *Let f be a holomorphic primitive cusp form of even weight, square-free level N and trivial character. Then*

$$\int_{-\infty}^{\infty} w(t)|L(f,1/2+it)|^2\,dt = \int_{-\infty}^{\infty} w(t)\left[\mathfrak{a}_f \ln\left(\frac{t\sqrt{N}}{2\pi}\right) + \mathfrak{b}_f\right] dt + O(T^{1/2+\theta+\varepsilon}).$$

REMARK 3. This corollary is in agreement with the conjecture in [CFK+05] about integral moments of L-functions (see Section 5.2).

In [Zha05], Zhang succeeded in determining the main term of this integral second moment of $L(f,s)$ (without the smooth function w) on the critical line. Thanks to Corollary 2, we improve his result with the following more precise asymptotic expansion. When f is a holomorphic cusp form of even weight for the full modular group, we can also refer to [Goo82] where a similar asymptotic expansion is given.

COROLLARY 3. *Let f be a holomorphic primitive cusp form of even weight, square-free level N and trivial character. Then*

$$\int_0^T |L(f,1/2+it)|^2\,dt = \mathfrak{a}_f T\ln T + \left[\mathfrak{b}_f + \mathfrak{a}_f \ln\left(\frac{\sqrt{N}}{2\pi e}\right)\right] T + O(T/\ln T).$$

NOTATION. If f and g are some functions of the real variable, then $f(x) \ll_A g(x)$ or $f(x) = O_A(g(x))$ mean that $|f(x)|$ is smaller than a constant, which only depends on A, times $|g(x)|$ for large x. Similarly, $f \asymp g$ means $f(x) \ll g(x)$ and $g(x) \ll f(x)$.

From the Riemann zeta function ζ, we define, for any positive square-free integer N,

$$\zeta^{(N)}(s) = \prod_{p\mid N}\left(1-\frac{1}{p^s}\right)\zeta(s).$$

2. Review of L-functions of primitive cusp forms. For this section, we may refer to [IK04, Chap. 14]. Throughout this paper, f denotes a holomorphic primitive cusp form of even weight k and square-free level N. The Fourier expansion of f at the cusp ∞ is given by

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} e^{2i\pi nz}$$

for every complex number z in the upper half-plane with the arithmetic normalisation $\lambda_f(1) = 1$. The Fourier coefficients $\lambda_f(n)$ satisfy the multiplicative relations

$$\lambda_f(n)\lambda_f(m) = \sum_{\substack{d\mid(m,n)\\(d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right), \tag{3}$$

$$\lambda_f(mn) = \sum_{\substack{d\mid(m,n)\\(d,N)=1}} \mu(d)\lambda_f\left(\frac{m}{d}\right)\lambda_f\left(\frac{n}{d}\right), \tag{4}$$

for all positive integers m and n. Since $\lambda_f(1) \neq 0$, we may define the convolution inverse $(\mu_f(n))$ of the sequence $(\lambda_f(n))$. This is an arithmetic multiplicative function which satisfies, for every prime number p,

$$\begin{aligned} &\mu_f(1) = 1, \qquad \mu_f(p) = -\lambda_f(p), \\ &\mu_f(p^2) = \lambda_f(p)^2 - \lambda_f(p^2) = \begin{cases} 1 & \text{if } p \nmid N, \\ 0 & \text{if } p \mid N, \end{cases} \end{aligned} \tag{5}$$

$$\mu_f(p^j) = 0 \quad \text{if } j \geq 3. \tag{6}$$

We consider

$$\begin{aligned} L(f,s) &= \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \chi_0(p)\frac{1}{p^{2s}}\right)^{-1} \\ &= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \end{aligned}$$

which is an absolutely convergent and non-vanishing Dirichlet series, an Euler product on $\Re s > 1$, where χ_0 denotes the trivial character modulo N and $\alpha_f(p)$, $\beta_f(p)$ are the complex roots of the equation $X^2 - \lambda_f(p)X + \chi_0(p) = 0$. Moreover, the function

$$\Lambda(f,s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(f,s) = L_\infty(f,s)L(f,s)$$

is the completed L-function of $L(f,s)$. It can be extended to a holomorphic function on $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(f,s) = \varepsilon(f)\Lambda(f,1-s)$$

where $\varepsilon(f) = \pm 1$. We remark that, by the duplication formula for the gamma function, the local factor at infinity can be written as

$$(7)\qquad L_\infty(f,s) = \left(\frac{2^k}{8\pi}\right)^{1/2}\left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s}{2}+\frac{k-1}{4}\right)\Gamma\left(\frac{s}{2}+\frac{k+1}{4}\right).$$

3. Mollified second moment of L-functions of modular forms. This section contains the proof of Theorem 1. We define the sequence $(\mu_f(n))$ to be the convolution inverse of $(\lambda_f(n))_{n\geq 1}$, and we define a mollifier ψ by

$$(8)\qquad \psi(s) = \sum_{n\leq M} \frac{\mu_f(n)}{n^{s+1/2-\sigma_0}} P\left(\frac{\ln(M/n)}{\ln M}\right)$$

with $M = T^\nu$, $\sigma_0 = 1/2 - R/\ln T$ where R is a positive real number and P is a real polynomial satisfying $P(0) = 0$, $P(1) = 1$. In addition, we choose a function $w : \mathbb{R} \to \mathbb{R}$ which satisfies

(9a) w is smooth,

(9b) w is compactly supported with supp $w \subset [T/4, 2T]$,

(9c) $w^{(j)}(t) \ll_j \Delta^{-j}$ for each $j \geq 0$, where $\Delta = T/L$ and $L = \ln T$.

For convenience, we set

$$I_f(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) L(f, 1/2+\alpha+it) L(f, 1/2+\beta-it) |\psi(\sigma_0+it)|^2\, dt.$$

To study the asymptotic behaviour of $I_f(\alpha,\beta)$, we need an explicit expression for $L(f,s)$ with $0 \leq \Re s \leq 1$. In Lemma 1, we get an exact formula, also called the "approximate functional equation", which gives an expression for $L(f, s+it)L(f, s-it)$ with s in the critical strip where we cannot use the Dirichlet series. Thanks to this new relation, we may split $I_f(\alpha,\beta)$ into a diagonal term (without oscillation) and an off-diagonal term (with oscillation). The off-diagonal contribution is bounded in Section 3.1, whereas the diagonal term is estimated in Section 3.2.

LEMMA 1. *Let G be any entire function which decays rapidly in vertical strips, even and normalised by $G(0) = 1$. Then for any complex numbers α, β such that $0 \leq |\Re\alpha|, |\Re\beta| \leq 1/2$, we have*

$$\begin{aligned} L(f, 1/2+\alpha+it)&L(f, 1/2+\beta-it) \\ &= \sum\sum_{m,n\geq 1} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha} n^{1/2+\beta}} \left(\frac{m}{n}\right)^{-it} V_{\alpha,\beta}(mn,t) \\ &\quad + X_{\alpha,\beta,t} \sum\sum_{m,n\geq 1} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\beta} n^{1/2-\alpha}} \left(\frac{m}{n}\right)^{-it} V_{-\beta,-\alpha}(mn,t) \end{aligned}$$

where

$$g_{\alpha,\beta}(s,t) = \frac{L_\infty(f,1/2+\alpha+s+it)L_\infty(f,1/2+\beta+s-it)}{L_\infty(f,1/2+\alpha+it)L_\infty(f,1/2+\beta-it)}, \tag{10}$$

$$V_{\alpha,\beta}(x,t) = \frac{1}{2i\pi}\int_{(1)} \frac{G(s)}{s} g_{\alpha,\beta}(s,t)x^{-s}\,ds, \tag{11}$$

$$X_{\alpha,\beta,t} = \frac{L_\infty(f,1/2-\alpha-it)L_\infty(f,1/2-\beta+it)}{L_\infty(f,1/2+\alpha+it)L_\infty(f,1/2+\beta-it)}.$$

We do not give the proof of this lemma, which is essentially the same as in [IK04, Theorem 5.3]. Nevertheless, it will be useful to have good approximations of $X_{\alpha,\beta,t}$, $g_{\alpha,\beta}(s,t)$ and $V_{\alpha,\beta}(x,t)$.

LEMMA 2. *For large t and for $s \ll t^\varepsilon$ in any vertical strip, we have*

$$X_{\alpha,\beta,t} = \left(\frac{t\sqrt{N}}{2\pi}\right)^{-2(\alpha+\beta)}\left(1+\frac{i(\alpha^2-\beta^2)}{t}+O\left(\frac{1}{t^2}\right)\right), \tag{12}$$

$$g_{\alpha,\beta}(s,t) = \left(\frac{t\sqrt{N}}{2\pi}\right)^{2s}\left(1+O\left(\frac{|s^2|}{t}\right)\right). \tag{13}$$

In addition, for each integer $j \geq 0$ and for all real $A > 0$, we have

$$t^j\frac{\partial^j}{\partial t^j}V_{\alpha,\beta}(x,t) \ll_{A,j} \left(1+\frac{|x|}{t^2}\right)^{-A}. \tag{14}$$

Proof. We may write

$$X_{\alpha,\beta,t} = \left(\frac{\sqrt{N}}{2\pi}\right)^{-2(\alpha+\beta)}\frac{\Gamma(k/2-\alpha-it)\Gamma(k/2-\beta+it)}{\Gamma(k/2+\alpha+it)\Gamma(k/2+\beta-it)},$$

$$g_{\alpha,\beta}(s,t) = \left(\frac{\sqrt{N}}{2\pi}\right)^{2s}\frac{\Gamma(k/2+\alpha+s+it)\Gamma(k/2+\beta+s-it)}{\Gamma(k/2+\alpha+it)\Gamma(k/2+\beta-it)}.$$

Then the first part of the lemma is a consequence of the following Stirling formula with $s = \sigma + i\tau$ in any vertical strip:

$$\Gamma(s) = \sqrt{2\pi}\,|\tau|^{\sigma-1/2}e^{-\frac{\pi}{2}|\tau|}e^{i(\tau\ln|\tau|-\tau+\frac{\pi}{2}(\sigma-1/2)\operatorname{sgn}(\tau))} \times\left(1-i\frac{(\sigma-1/2)^2-1/12}{2\tau}+O\left(\frac{1}{\tau^2}\right)\right).$$

We refer to [Ten95, Corollaire 0.13]. To prove (14), we move the integration line to $\Re s = A$ far to the right and by (13), we obtain the desired bound if $t^2 \ll x$. In the case $x \ll t^2$, the result follows easily from trivial bounds. ∎

Thanks to the above functional equation, we may split $I_f(\alpha,\beta)$ as a sum of diagonal terms and off-diagonal terms. More precisely, opening the mollifier ψ, we may write

$$I_f(\alpha,\beta) = \sum_{a,b\le M} \frac{\mu_f(a)\mu_f(b)}{\sqrt{ab}} P\Big(\frac{\ln(M/a)}{\ln M}\Big) P\Big(\frac{\ln(M/b)}{\ln M}\Big) \times [I_{a,b}^{D_1}(\alpha,\beta) + I_{a,b}^{D_2}(\alpha,\beta) + I_{a,b}^{ND_1}(\alpha,\beta) + I_{a,b}^{ND_2}(\alpha,\beta)] \tag{15}$$

with

$$I_{a,b}^{D_1}(\alpha,\beta) = \sum_{am=bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)V_{\alpha,\beta}(mn,t)\,dt,$$

$$I_{a,b}^{D_2}(\alpha,\beta) = \sum_{am=bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\beta}n^{1/2-\alpha}} \int_{-\infty}^{\infty} w(t)X_{\alpha,\beta,t}V_{-\beta,-\alpha}(mn,t)\,dt,$$

$$I_{a,b}^{ND_1}(\alpha,\beta) = \sum_{am\neq bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)\Big(\frac{am}{bn}\Big)^{-it} V_{\alpha,\beta}(mn,t)\,dt,$$

$$I_{a,b}^{ND_2}(\alpha,\beta) = \sum_{am\neq bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{n1/2-\beta}n^{1/2-\alpha}} \int_{-\infty}^{\infty} w(t)X_{\alpha,\beta,t}\Big(\frac{am}{bn}\Big)^{-it} V_{-\beta,-\alpha}(mn,t)\,dt.$$

3.1. Evaluation of off-diagonal terms. In this part, we evaluate the size of off-diagonal terms. More precisely, we prove the following proposition.

PROPOSITION 1. *If $0 < \nu < \frac{1-2\theta}{4+2\theta}$ and if α,β are complex numbers satisfying $\alpha,\beta \ll L^{-1}$ then there exists $\varepsilon > 0$ such that*

$$\sum_{a,b\le M} \frac{\mu_f(a)\mu_f(b)}{\sqrt{ab}} P\Big(\frac{\ln(M/a)}{\ln M}\Big) P\Big(\frac{\ln(M/b)}{\ln M}\Big) [I_{a,b}^{ND_1}(\alpha,\beta) + I_{a,b}^{ND_2}(\alpha,\beta)] \ll T^{1-\varepsilon}.$$

The main tool of the proof of this proposition is a theorem about shifted convolution sums on average.

3.1.1. *Initial lemmas.* In order to prove the previous proposition, we begin by getting rid of some harmless terms occurring in the definition of $I_{a,b}^{ND_1}(\alpha,\beta)$.

LEMMA 3. *Let $\varepsilon > 0$, $0 < \gamma < 1$, let $\alpha,\beta \ll L^{-1}$ be complex numbers and let $a,b \le T^\nu$ be positive integers. Then, for all real $A > 0$,*

$$I_{a,b}^{ND_1}(\alpha,\beta) = \sum_{\substack{am\neq bn \\ mn\ll T^{2+\varepsilon} \\ |\frac{am}{bn}-1|\ll T^{-\gamma}}} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)\Big(\frac{am}{bn}\Big)^{-it} V_{\alpha,\beta}(mn,t)\,dt + O(T^{-A}). \tag{16}$$

Proof. Firstly, by (14) with $j=0$, for all real $A>0$ we get

$$\int_{-\infty}^{\infty} w(t)\left(\frac{am}{bn}\right)^{-it} V_{\alpha,\beta}(mn,t)\,dt \ll_A T\left(\frac{T^2}{mn}\right)^A.$$

As a consequence, for $A>1/2$, since $|\lambda_f(n)|\le\tau(n)\ll n^{\epsilon}$ and $\alpha,\beta\ll L^{-1}$, we may write

$$\begin{aligned}\sum_{\substack{am\neq bn\\ mn>T^{2+\varepsilon}}} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)\left(\frac{am}{bn}\right)^{-it} V_{\alpha,\beta}(mn,t)\,dt \\ \ll T^{1+2A} \sum_{mn>T^{2+\varepsilon}} \frac{\tau(m)\tau(n)}{m^{1/2+A+\Re\alpha}n^{1/2+A+\Re\beta}} \\ \ll T^{1+2A} \sum_{h>T^{2+\varepsilon}} \frac{1}{h^{1/2+A-\epsilon}} \ll T^{2-A\varepsilon} \ll T^{-A}.\end{aligned}$$

Then, using (14) for each integer j and since $w^{(j)}(t)\ll \Delta^{-j}$, for all real $A>0$ we have, uniformly with respect to x,

$$\frac{\partial^j}{\partial t^j}[w(t)V_{\alpha,\beta}(x,t)] \ll \Delta^{-j}\left(\frac{T^2}{|x|}\right)^A.$$

Hence, if $am\neq bn$, after j integrations by parts we get

$$\begin{aligned}&\int_{-\infty}^{\infty} w(t)\left(\frac{am}{bn}\right)^{-it} V_{\alpha,\beta}(mn,t)\,dt \\ &= \frac{1}{\left(i\ln\frac{am}{bn}\right)^j} \int_{-\infty}^{\infty} \left(\frac{am}{bn}\right)^{-it} \frac{\partial^j}{\partial t^j}[w(t)V_{\alpha,\beta}(x,t)]\,dt \ll \frac{T}{\Delta^j\left|\ln\frac{am}{bn}\right|^j}\left(\frac{T^2}{mn}\right)^A.\end{aligned}$$

Therefore, with $A=1/2+\max\{\Re\alpha,\Re\beta\}+\delta$ and $\delta>0$, using the lower bound $x/2\le\ln(1+x)$ for $0<x<1$, we have

$$\begin{aligned}\sum_{\substack{am\neq bn\\ |\frac{am}{bn}-1|>T^{-\gamma}}} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)\left(\frac{am}{bn}\right)^{-it} V_{\alpha,\beta}(mn,t)\,dt \\ \ll \frac{T^{1+2A+j\gamma}}{\Delta^j} \sum_{m,n\ge1} \frac{\tau(m)\tau(n)}{m^{1/2+\Re\alpha+A}n^{1/2+\Re\beta+A}}.\end{aligned}$$

Since $\gamma<1$, the result follows easily by choosing j large. ■

We introduce a dyadic partition of unity for sums over m and n. We fix an arbitrary smooth function $\rho:]0,\infty[\,\to\mathbb{R}$, compactly supported in $[1,2]$ and with

$$\sum_{\ell=-\infty}^{\infty} \rho(2^{-\ell/2}x)=1$$

(see [Har03, Section 5]). For each integer ℓ, we define

$$\rho_\ell(x) = \rho(x/A_\ell) \quad \text{with} \quad A_\ell = 2^{\ell/2}T^\gamma.$$

In order to study the asymptotic behaviour of $I_{a,b}^{ND_1}(\alpha,\beta)$, we define

$$(17)\qquad F_{h;\ell_1,\ell_2}(x,y) = \frac{a^{1/2+\alpha}b^{1/2+\beta}}{x^{1/2+\alpha}y^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)\left(1+\frac{h}{y}\right)^{-it} V_{\alpha,\beta}\left(\frac{xy}{ab},t\right) dt \times \rho_{\ell_1}(x)\rho_{\ell_2}(y).$$

LEMMA 4. *Let $\varepsilon > 0$, $0 < \gamma < 1$, let $\alpha,\beta \ll L^{-1}$ be complex numbers and let $a,b \le T^\nu$ be positive integers. Then, for all real $A > 0$,*

$$(18)\qquad I_{a,b}^{ND_1}(\alpha,\beta) = \sum_{\substack{A_{\ell_1}A_{\ell_2} \ll abT^{2+\varepsilon} \\ A_{\ell_1} \asymp A_{\ell_2} \\ A_{\ell_1},A_{\ell_2} \gg T^\gamma}} \sum_{0<|h|\ll T^{-\gamma}\sqrt{A_{\ell_1}A_{\ell_2}}} \sum_{am-bn=h} \lambda_f(m)\lambda_f(n)F_{h;\ell_1,\ell_2}(am,bn) + O(T^{-A}).$$

Proof. For convenience, we define

$$H(x,y) = \frac{a^{1/2+\alpha}b^{1/2+\beta}}{x^{1/2+\alpha}y^{1/2+\beta}} \int_{-\infty}^{\infty} w(t)\left(\frac{x}{y}\right)^{-it} V_{\alpha,\beta}\left(\frac{xy}{ab},t\right) dt.$$

From the previous lemma and using the partition of unity, we may write

$$I_{a,b}^{ND_1}(\alpha,\beta) = \sum_{\ell_1,\ell_2}\sum_{h\neq 0} \sum_{\substack{am-bn=h \\ mn \ll T^{2+\varepsilon} \\ |\frac{am}{bn}-1| \ll T^{-\gamma}}} \lambda_f(m)\lambda_f(n)H(am,bn)\rho_{\ell_1}(am)\rho_{\ell_2}(bn) + O(T^{-A}).$$

First, if $|h| \ge \sqrt{A_{\ell_1}A_{\ell_2}}\,T^{-\gamma}$ then

$$\max\left\{\left|\frac{am}{bn}-1\right|, \left|\frac{bn}{am}-1\right|\right\}^2 \ge \left|\frac{am}{bn}-1\right|\left|\frac{bn}{am}-1\right| = \frac{h^2}{ambn} \asymp \frac{h^2}{A_{\ell_1}A_{\ell_2}} \ge T^{-2\gamma}.$$

Secondly, if $|\ell_1-\ell_2| \ge 3$, for instance if $\ell_1-\ell_2 \ge 3$, then

$$\frac{am}{bn} - 1 \ge \frac{2^{(\ell_1-\ell_2)/2}}{2} - 1 \ge \sqrt{2} - 1 \gg 1.$$

Therefore we may assume $A_{\ell_1} \asymp A_{\ell_2}$. Thirdly, if $A_{\ell_2} \le T^\gamma$ then

$$\left|\frac{am}{bn} - 1\right| = \frac{h}{bn} \ge \frac{h}{2A_{\ell_2}} \gg T^{-\gamma}.$$

Thus we may assume $A_{\ell_2} \ge T^\gamma$ and, in the same way, $A_{\ell_1} \ge T^\gamma$. Finally, since $am-bn=h$, we get $H(am,bn)\rho_{\ell_1}(am)\rho_{\ell_2}(bn) = F_{h;\ell_1,\ell_2}(am,bn)$. ∎

3.1.2. *Shifted convolution sums.* The core of the proof of our theorem is the following bound, which is a generalisation of [Blo05, Theorem 2], for shifted convolution sums on average. We define θ to be the exponent in the Ramanujan–Petersson conjecture, which claims

$$|\lambda(n)| \le \tau(n) n^{\theta} \tag{19}$$

for eigenvalues $\lambda(n)$ of the Hecke operator T_n acting on the space of weight 0 Maaß cusp forms of level N.

Theorem 3. *Let ℓ_1, ℓ_2, H and h_1 be positive integers. Let M_1, M_2, P_1, P_2 be real numbers greater than 1. Let $\{g_h\}$ be a family of smooth functions supported in $[M_1, 2M_1] \times [M_2, 2M_2]$ with $\|g_h^{(ij)}\|_\infty \ll_{i,j} (P_1/M_1)^i (P_2/M_2)^j$ for all $i, j \ge 0$. Let $(a(h))$ be a sequence of complex numbers such that*

$$a(h) \neq 0 \ \Rightarrow\ h \le H,\ h_1 \,|\, h \text{ and } (h_1, h/h_1) = 1.$$

If $\ell_1 M_1 \asymp \ell_2 M_2 \asymp A$ and if there exists $\epsilon > 0$ such that

$$H \ll \frac{A}{\max\{P_1, P_2\}^2} \frac{1}{(\ell_1 \ell_2 M_1 M_2 P_1 P_2)^{\epsilon}} \tag{20}$$

then, for all real $\varepsilon > 0$,

$$\sum_{h=1}^{H} a(h) \sum_{\substack{m_1, m_2 \ge 1 \\ \ell_1 m_1 - \ell_2 m_2 = h}} \lambda_f(m_1) \overline{\lambda_f(m_2)} g_h(m_1, m_2)$$

$$\ll A^{1/2} h_1^{\theta} \|a\|_2 (P_1 + P_2)^{3/2} \left[\sqrt{P_1 + P_2} + \left(\frac{A}{\max\{P_1, P_2\}} \right)^{\theta} \left(1 + \sqrt{\frac{(h_1, \ell_1 \ell_2) H}{h_1 \ell_1 \ell_2}} \right) \right] \times (\ell_1 \ell_2 M_1 M_2 P_1 P_2 H)^{\varepsilon}.$$

Proof. The proof is a direct generalisation of the proof of [Blo05, Theorem 2], so we only give an outline. We set

$$\Sigma(\ell_1, \ell_2, H, a) = \sum_{h=1}^{H} a(h) \sum_{\substack{m_1, m_2 \ge 1 \\ \ell_1 m_1 - \ell_2 m_2 = h}} \lambda_f(m_1) \overline{\lambda_f(m_2)} g_h(m_1, m_2).$$

We also set $\Delta = \min\{P_1/(\ell_1 M_1), P_2/(\ell_2 M_2)\}$, let $Q \ge 1/\Delta$ be a large parameter and set $\delta = 1/Q$, so that $\delta \le \Delta$. Let ϕ be a smooth function compactly supported in $[-\Delta^{-1}, \Delta^{-1}]$ with $\phi(0) = 1$ and such that j $\|\phi^{(j)}\|_\infty \ll_j \Delta^j$ for all integers. Then we define

$$W_h(x, y) = g_h(x, y) \phi(\ell_1 x - \ell_2 y - h).$$

In addition, we introduce another smooth function $w : \mathbb{R} \to \mathbb{R}$ compactly supported in $[Q, 2Q]$ and such that $\|w^{(j)}\|_\infty \ll_j Q^{-j}$. If φ denotes Euler's phi function, let

$$\Lambda = \sum_{q \equiv 0\,[N\ell_1\ell_2]} w(q)\varphi(q) \asymp \frac{Q^2}{N\ell_1\ell_2}.$$

As a result, we can rewrite

$$\Sigma(\ell_1, \ell_2, H, a) = \sum_{h=1}^{H} a(h) \int_0^1 \sum_{m_1,m_2 \geq 1} \lambda_f(m_1)\overline{\lambda_f(m_2)} e(\ell_1 m_1 \alpha) e(-\ell_2 m_2 \alpha) W_h(m_1, m_2) e(-h\alpha)\, d\alpha.$$

By Jutila's circle method (see [Blo05, Lemma 3.1]), we build an approximation $\tilde{I}$ to the characteristic function on $[0,1]$, which splits the α-integral into two parts, according to whether α is in a minor arc or a major arc. We easily estimate the contribution of the minor arcs using our bound for the $L^2([0,1])$-norm of $1 - \tilde{I}$, and we can write the contribution of the major arcs as

$$\frac{1}{2\delta\Lambda} \sum_{h=1}^{H} a(h) \sum_{q \equiv 0\,[N\ell_1\ell_2]} w(q) \sum_{d(q)}^{*} \int_{-\delta}^{\delta} \sum_{m_1,m_2 \geq 1} \lambda_f(m_1)\overline{\lambda_f(m_2)}$$
$$\times\, e\left(\ell_1 m_1 \left(\frac{d}{q} + \eta\right)\right) e\left(-\ell_2 m_2 \left(\frac{d}{q} + \eta\right)\right) W_h(m_1, m_2) e\left(-h\left(\frac{d}{q} + \eta\right)\right) d\eta.$$

We transform short sums of exponentials into long sums of exponentials by means of a Voronoï summation formula (see [Blo05, Lemma 2.2] or [KMV02, Appendix A]). If $S(m, n; q)$ denotes the classical Kloosterman sum, the major arcs contribution becomes

$$\frac{1}{2\delta\Lambda} \sum_{h=1}^{H} a(h) \sum_{q \equiv 0\,[N\ell_1\ell_2]} w(q) \int_{-\delta}^{\delta} e(-\eta h)$$
$$\times \sum_{m_1,m_2 \geq 1} \lambda_f(m_1)\overline{\lambda_f(m_2)} S(-h, \ell_2 m_2 - \ell_1 m_1; q) G_{q,h,\eta}(m_1, m_2)\, d\eta$$

where

$$G_{q,h,\eta}(x_1, x_2) = \frac{4\pi^2 \ell_1 \ell_2}{q^2} \int_0^\infty \int_0^\infty W_h(t_1, t_2) e(\ell_1 t_1 \eta - \ell_2 t_2 \eta)$$
$$\times\, J_{k-1}\left(\frac{4\pi\ell_1\sqrt{x_1 t_1}}{q}\right) J_{k-1}\left(\frac{4\pi\ell_2\sqrt{x_2 t_2}}{q}\right) dt_1\, dt_2.$$

We split this sum according to the value of $\ell_2 m_2 - \ell_1 m_1$. The diagonal contribution when $\ell_1 m_1 = \ell_2 m_2$ is easily bounded. It remains to estimate the off-diagonal contribution when $\ell_2 m_2 \neq \ell_1 m_1$, which can be rewritten as

$$\frac{\pi\ell_1\ell_2}{2\Lambda}\int_{-\delta}^{\delta}\int_0^\infty\int_0^\infty\sum_{h=1}^{H} a_{\eta,t_1,t_2}(h)\sum_{r\neq 0}\sum_{\substack{1\le m_1\le\mathcal{M}_1\\ 1\le m_2\le\mathcal{M}_2\\ \ell_2m_2-\ell_1m_1=r}}\lambda_f(m_1)\overline{\lambda_f(m_2)}$$

$$\times\sum_{q\equiv 0\,[N\ell_1\ell_2]}\frac{S(-h,r;q)}{q}\Phi_{t_1,t_2}\left(\frac{4\pi\sqrt{h|r|}}{q};m_1,m_2,r,h\right)dt_1\,dt_2\,d\eta$$

where

$$a_{\eta,t_1,t_2}(h)=a(h)e(-\eta h)W_h(t_1,t_2)e(\ell_1t_1\eta-\ell_2t_2\eta),$$

$$\Phi_{t_1,t_2}(x;m_1,m_2,r,h)=\frac{Qx}{\sqrt{h|r|}}J_{k-1}\left(\frac{x\ell_1\sqrt{m_1t_1}}{\sqrt{h|r|}}\right)J_{k-1}\left(\frac{x\ell_2\sqrt{m_2t_2}}{\sqrt{h|r|}}\right)$$
$$\times w\left(\frac{4\pi\sqrt{h|r|}}{x}\right)w_1(h)$$

and

$$\mathcal{M}_1=\frac{Q^2P_1^2k^2}{\ell_1^2M_1}(\ell_1\ell_2M_1M_2P_1P_2)^{\varepsilon},\qquad \mathcal{M}_2=\frac{Q^2P_2^2k^2}{\ell_2^2M_2}(\ell_1\ell_2M_1M_2P_1P_2)^{\varepsilon}.$$

The asymptotic behaviour of $G_{q,h,\eta}$ allows us to restrict the sums over m_1 and m_2 to $m_1\le\mathcal{M}_1$ and $m_2\le\mathcal{M}_2$. Applying the Kuznetsov trace formula (see [Blo05, Lemma 2.4] or [DI83, Theorem 1]), we decompose this off-diagonal term as a sum of three terms: the contribution of the discrete spectrum, of the continuous spectrum and of the holomorphic cusp forms. All of them will be evaluated by means of large sieve inequalities (see [Blo05, Lemma 2.5] or [DI83, Theorem 2]). In the discrete spectrum, there may be exceptional eigenvalues even though Selberg's conjecture predicts that they do not exist. We prove that the contribution of the non-exceptional eigenvalues, called the *real spectrum*, is bounded by

$$A^{1/2}\|a\|_2h_1^{\theta}(P_1+P_2)^{3/2}\left(\sqrt{P_1+P_2}+\sqrt{\frac{H(h_1,\ell_1\ell_2)}{h_1\ell_1\ell_2}}\right)Q^{\varepsilon}$$

whereas we bound the contribution of the exceptional eigenvalues by

$$A^{1/2+\theta}h_1^{\theta}\|a\|_2(P_1+P_2)^{3/2-\theta}\left(1+\sqrt{\frac{H(h_1,\ell_1\ell_2)}{h_1\ell_1\ell_2}}\right)Q^{\varepsilon}.$$

In addition, we show that the contributions of the continuous spectrum and of the holomorphic cusp forms are bounded by

$$A^{1/2}\|a\|_2(P_1+P_2)^{3/2}\left(\sqrt{P_1+P_2}+\sqrt{\frac{H(h_1,\ell_1\ell_2)}{h_1\ell_1\ell_2}}\right)Q^{\varepsilon},$$

which is smaller than the estimate of the real spectrum. The result follows easily from the last three estimates. ■

Remark 4. In [Blo05], a factor $\left(\frac{LM}{HP}\right)^\theta$ appears, which could become $\left(\frac{A}{H(P_1+P_2)}\right)^\theta$ in our theorem, and which comes from the contribution of possibly exceptional eigenvalues in the discrete spectrum. However, we only find $\left(\frac{A}{P_1+P_2}\right)^\theta$.

Let us determine the required bounds for the test function in our case.

Lemma 5. *Let $\alpha, \beta \ll L^{-1}$ be complex numbers and let σ be any positive real number. For all non-negative integers i and j, we have*

$$x^i y^j \frac{\partial^{i+j} F_{h;\ell_1,\ell_2}}{\partial x^i \partial y^j}(x,y) \ll_{i,j} \left(\frac{a}{A_{\ell_1}}\right)^{1/2+\Re\alpha+\sigma} \left(\frac{b}{A_{\ell_2}}\right)^{1/2+\Re\beta+\sigma} T^{1+2\sigma}(\ln T)^j,$$

and the implicit constant does not depend on h.

Proof. Let $P_0(Y) = 1$ and $P_j(Y) = \prod_{\ell=0}^{j-1}(Y-\ell)$ for $j \geq 1$. Let $\Re s = \sigma > 0$. Then

$$\begin{aligned} y^j \frac{\partial^j}{\partial y^j}\Big(\int_{\mathbb{R}} w(t)(1+h/y)^{-it} g_{\alpha,\beta}(s,t)\,dt\Big) \\ &= \int_{\mathbb{R}} w(t) g_{\alpha,\beta}(s,t)(1+h/y)^{-it} Q_j(t)\,dt \\ &= \int_{\mathbb{R}} \frac{(1+h/y)^{-it}}{[i\ln(1+h/y)]^j} \frac{\partial^j}{\partial t^j}[w(t)g_{\alpha,\beta}(s,t)Q_j(t)]\,dt \end{aligned}$$

where $Q_j(t) = \sum_{r=0}^j \binom{j}{r} P_r(it)P_{j-r}(-it)(1+h/y)^{-(j-r)}$. Since

$$Q_j^{(r)}(t) \ll \left|\frac{h/y}{1+h/y}\right|^j t^{j-r} \quad \text{for } r \leq j$$

and $\frac{\partial^r}{\partial t^r}[w(t)g_{\alpha,\beta}(s,t)] \ll T^{2\sigma}\Delta^{-r}$, we get

$$\frac{\partial^j}{\partial t^j}[w(t)g_{\alpha,\beta}(s,t)Q_j(t)] \ll_{s,j} T^{2\sigma}\left|\frac{h/y}{1+h/y}\right|^j (\ln T)^j$$

and the implicit constant depends polynomially on s. Since $h/y \ll T^{-\gamma}$ in our range of summation, we have

$$y^j \frac{\partial^j}{\partial y^j}\Big(\int_{\mathbb{R}} w(t)(1+h/y)^{-it} g_{\alpha,\beta}(s,t)\,dt\Big) \ll_{s,j} T^{1+2\sigma}(\ln T)^j \tag{21}$$

and the implicit constant does not depend on h. Writing

$$F_{h;\ell_1,\ell_2}(x,y) = \frac{\rho_{\ell_1}(x)\rho_{\ell_2}(y)}{2i\pi}\int_{(\sigma)}\frac{G(s)}{s}\left(\frac{a}{x}\right)^{1/2+\alpha+s}\left(\frac{b}{y}\right)^{1/2+\beta+s}\int_{\mathbb{R}} w(t)\left(1+\frac{h}{y}\right)^{-it} g_{\alpha,\beta}(s,t)\,dt\,ds,$$

the result follows easily from the Leibniz formula and the bound (21). ∎

Remark 5. The trivial bound for shifted convolution sums, namely taking absolute values and applying the Ramanujan–Petersson bound on average, is given, for all $\varepsilon > 0$, by

$$\sum_{\ell_1 m_1 - \ell_2 m_2 = h} \lambda_f(m_1)\lambda_f(m_2) g_h(m_1,m_2) \ll_\varepsilon \min\{M_1,M_2\}(M_1M_2)^{\varepsilon}. \tag{22}$$

The trivial bound (22) and Lemma 5 imply the following corollary.

Corollary 4. *For all $\varepsilon > 0$, we have*

$$I_{a,b}^{ND_1}(\alpha,\beta) \ll_\varepsilon \min\{a,b\}T^{1+\varepsilon}.$$

Remark 6. This trivial bound of $I_{a,b}^{ND_1}(\alpha,\beta)$ fails to prove Proposition 1. In other words, taking care of the oscillations of the Hecke eigenvalues is required. First, we apply the following bound, proved by Blomer [Blo04], for shifted convolution sums.

Theorem 4 (Blomer [Blo04]). *Let $\varepsilon > 0$, and let ℓ_1, ℓ_2 and h be positive integers. Let M_1, M_2, P_1 and P_2 be real numbers greater than 1. Let g_h be a smooth function supported in $[M_1, 2M_1] \times [M_2, 2M_2]$ such that $\|g_h^{(ij)}\|_\infty \ll_{i,j} (P_1/M_1)^i (P_2/M_2)^j$ for all $i, j \geq 0$. Then*

$$\sum_{\ell_1 m_1 - \ell_2 m_2 = h} \lambda_f(m_1)\lambda_f(m_2) g_h(m_1,m_2) \ll_{\varepsilon,P_1,P_2,N,k} (\ell_1 M_1 + \ell_2 M_2)^{1/2+\theta+\varepsilon}.$$

This bound is uniform in ℓ_1, ℓ_2, h, and the dependence on P_1, P_2, N and k is polynomial.

Remark 7. Remembering the trivial bound (22), this theorem agrees with the square-root cancellation philosophy.

Theorem 4 and Lemma 5 imply the following proposition, which gives a first admissible bound, and which will be improved in Proposition 3.

Proposition 2. *Let $\alpha, \beta \ll L^{-1}$ be complex numbers and a, b be positive integers. For all $\varepsilon > 0$, we have*

$$I_{a,b}^{ND_1}(\alpha,\beta) \ll_\varepsilon (ab)^{3/4+\theta/2} T^{1/2+\theta+\varepsilon}.$$

Proof. By Theorem 4, Lemma 5 gives

$$\sum_{am-bn=h} \lambda_f(m)\lambda_f(n)F_{h;\ell_1,\ell_2}(am,bn) \ll \left(\frac{a}{A_{\ell_1}}\right)^{1/2+\Re\alpha+\sigma}\left(\frac{b}{A_{\ell_2}}\right)^{1/2+\Re\beta+\sigma} T^{1+2\sigma}(\ln T)^{\kappa}(A_{\ell_1}+A_{\ell_2})^{1/2+\theta+\varepsilon}$$

where κ is a constant. Thus, thanks to Lemma 4 and with $1/2+\theta+\varepsilon-\Re\alpha-\Re\beta-2\sigma>0$, we get

$$\begin{aligned}
&I_{a,b}^{ND_1}(\alpha,\beta)\\
&\ll T^{1-\gamma+2\sigma}(\ln T)^{\kappa}\\
&\quad\times \sum_{\substack{A_{\ell_1}A_{\ell_2}\ll abT^{2+\varepsilon}\\ A_{\ell_1}\asymp A_{\ell_2}\\ A_{\ell_1},A_{\ell_2}\gg T^{\gamma}}} \left(\frac{a}{A_{\ell_1}}\right)^{1/2+\Re\alpha+\sigma}\left(\frac{b}{A_{\ell_2}}\right)^{1/2+\Re\beta+\sigma}\sqrt{A_{\ell_1}A_{\ell_2}}(A_{\ell_1}+A_{\ell_2})^{1/2+\theta+\varepsilon}\\
&\ll T^{1-\gamma+2\sigma}(ab)^{1/2+\sigma}(\ln T)^{\kappa}\sum_{T^{\gamma}\ll A_{\ell_1}\ll\sqrt{ab}\,T^{1+\varepsilon/2}} A_{\ell_1}^{1/2+\theta+\varepsilon-\Re\alpha-\Re\beta-2\sigma}\\
&\ll T^{1-\gamma}T^{1/2+\theta+\varepsilon}(ab)^{3/4+\theta/2}\sum_{T^{\gamma}\ll A_{\ell_1}\ll\sqrt{ab}\,T^{1+\varepsilon/2}} 1.
\end{aligned}$$

Hence $\sum_{T^{\gamma}\ll A_{\ell_1}\ll\sqrt{ab}\,T^{1+\varepsilon/2}} 1 = \sum_{1\le 2^{\ell_1/2}\ll\sqrt{ab}\,T^{1-\gamma+\varepsilon/2}} 1 \ll \ln T$. Finally, the result easily follows from the choice $\gamma=1-\varepsilon$. ■

Remark 8. We are tempted to solve the shifted convolution problem on average (over h) and to take care of the resulting additional oscillations of the Hecke eigenvalues. For instance, using [Ric06, Theorem 6.3], one can check that, for all $\varepsilon>0$,

$$I_{a,b}^{ND_1}(\alpha,\beta) \ll_{\varepsilon} (ab)^{3/4+\theta/2}T^{3/2+\theta+\varepsilon}.$$

It turns out that this bound is not admissible. This is due to the fact that the length of the h-sum is very small. That is why we need a bound for short sums of shifted convolution sums, which is given by Theorem 3.

Proposition 3. *Let $\alpha,\beta\ll L^{-1}$ be complex numbers and a,b be positive integers. For all $\varepsilon>0$, we have*

$$I_{a,b}^{ND_1}(\alpha,\beta) \ll_{\varepsilon} (ab)^{(1+\theta)/2}T^{1/2+\theta+\varepsilon}.$$

Proof. We apply Theorem 3 with $H=T^{-\gamma}\sqrt{A_{\ell_1}A_{\ell_2}}$, $h_1=1$ and

$$a(h)=\begin{cases}1 & \text{if } h\le H,\\ 0 & \text{otherwise.}\end{cases}$$

From Lemma 4, we get

$$I_{a,b}^{ND_1}(\alpha,\beta) \ll \sum_{\substack{A_{\ell_1}A_{\ell_2} \ll abT^{2+\varepsilon} \\ A_{\ell_1} \asymp A_{\ell_2} \\ A_{\ell_1}, A_{\ell_2} \gg T^{\gamma}}} \left(\frac{a}{A_{\ell_1}}\right)^{1/2+\Re\alpha+\sigma} \left(\frac{b}{A_{\ell_2}}\right)^{1/2+\Re\beta+\sigma} T^{1+2\sigma+\varepsilon}\sqrt{A_{\ell_1}H}$$
$$\times \left[\sqrt{\ln T} + \left(\frac{A_{\ell_1}}{\ln T}\right)^{\theta}\left(1+\sqrt{\frac{H}{ab}}\right)\right]$$
$$\ll (ab)^{1/2+\theta}T^{1+2\sigma+\varepsilon-\gamma/2} \sum_{T^{\gamma} \ll A_{\ell} \ll \sqrt{ab}\,T^{1+\varepsilon}} A_{\ell}^{\theta-(\Re\alpha+\Re\beta+2\sigma)}$$
$$\ll (ab)^{(1+\theta)/2}T^{1-\gamma/2+\theta+\varepsilon}.$$

Finally, the result follows for $\gamma = 1-\varepsilon$. ∎

Corollary 5. *Let $\alpha,\beta \ll L^{-1}$ be complex numbers and a,b be positive integers. For all $\varepsilon > 0$, we have*

$$I_{a,b}^{ND_2}(\alpha,\beta) \ll_{\varepsilon} (ab)^{(1+\theta)/2}T^{1/2+\theta+\varepsilon}.$$

Proof. Set

$$w_1(t) = w(t)\left(\frac{t\sqrt{N}}{2\pi}\right)^{-2(\alpha+\beta)}\left(1+\frac{i(\alpha^2-\beta^2)}{t}\right).$$

Thanks to (12), we may write

$$I_{a,b}^{ND_2}(\alpha,\beta) = \sum_{am\neq bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\beta}n^{1/2-\alpha}} \int_{-\infty}^{\infty} w_1(t)\left(\frac{am}{bn}\right)^{-it} V_{-\beta,-\alpha}(mn,t)\,dt$$
$$+ O\left(\frac{1}{T}\sum_{mn \ll T^{2+\varepsilon}} \frac{|\lambda_f(m)\lambda_f(n)|}{m^{1/2-\Re\beta}n^{1/2-\Re\alpha}}\right).$$

The error term becomes $O(T^{\varepsilon})$ and since w_1 satisfies (9a)–(9c) we may apply Proposition 3 up to replacing w by w_1 and (α,β) by $(-\beta,-\alpha)$. ∎

Proof of Proposition 1. Using Proposition 3 and Corollary 5, we trivially bound the off-diagonal term by

$$\sum_{a,b\leq M} \frac{\mu_f(a)\mu_f(b)}{\sqrt{ab}} P\left(\frac{\ln(M/a)}{\ln M}\right)P\left(\frac{\ln(M/b)}{\ln M}\right)[I_{a,b}^{ND_1}(\alpha,\beta) + I_{a,b}^{ND_2}(\alpha,\beta)]$$
$$\ll T^{1/2+\theta+\varepsilon}\sum_{a,b\leq M}(ab)^{\theta/2} \ll T^{1/2+\theta+\varepsilon}T^{\nu(2+\theta)}.$$

Thus, if $\nu < \frac{1-2\theta}{4+2\theta}$, the off-diagonal part of $I_f(\alpha,\beta)$ is bounded by $T^{1-\varepsilon}$. ∎

3.2. Evaluation of diagonal terms. For $i = 1$ or $i = 2$, let

$$I_f^{D_i}(\alpha,\beta) = \sum_{a,b\le M} \frac{\mu_f(a)\mu_f(b)}{\sqrt{ab}} P\Big(\frac{\ln(M/a)}{\ln M}\Big) P\Big(\frac{\ln(M/b)}{\ln M}\Big) I_{a,b}^{D_i}(\alpha,\beta). \tag{23}$$

We consider the diagonal part $I_f^D(\alpha,\beta)$ of the mollified second moment. Thus,

$$I_f^D(\alpha,\beta) = I_f^{D_1}(\alpha,\beta) + I_f^{D_2}(\alpha,\beta). \tag{24}$$

In this section, we prove the following proposition:

PROPOSITION 4. *Let $0<\nu<1$. For complex numbers $\alpha,\beta \ll L^{-1}$ such that $|\alpha+\beta| \gg L^{-1}$, we have*

$$I_f^D(\alpha,\beta) = \widehat{w}(0)c(\alpha,\beta) + O(T(\ln L)^4/L)$$

where $c(\alpha,\beta)$ is defined in (2).

3.2.1. *Initial lemmas.*

LEMMA 6. *Let $\Omega_{\alpha,\beta}$ be the set of vectors (u,v,s) in $\mathbb{C}^3$ satisfying*

$$\begin{cases} \Re u + \Re v > -1/2, \\ \Re s > -1/4 - \Re\alpha/2 - \Re\beta/2, \\ \Re u + \Re s > -1/2 - \Re\alpha, \\ \Re v + \Re s > -1/2 - \Re\beta. \end{cases}$$

Then

$$\sum_{\substack{a,b,m,n\ge 1\\ am=bn}} \frac{\mu_f(a)\mu_f(b)\lambda_f(m)\lambda_f(n)}{a^{1/2+v}b^{1/2+u}m^{1/2+\alpha+s}n^{1/2+\beta+s}}$$
$$= \frac{L(f\times f, 1+\alpha+\beta+2s)L(f\times f,1+u+v)}{L(f\times f,1+\alpha+u+s)L(f\times f,1+\beta+v+s)} A_{\alpha,\beta}(u,v,s)$$

where $A_{\alpha,\beta}(u,v,s)$ is given by an absolutely convergent Euler product on $\Omega_{\alpha,\beta}$.

Proof. Set

$$\mathcal{P} = \sum_{\substack{a,b,m,n\ge 1\\ am=bn}} \frac{\mu_f(a)\mu_f(b)\lambda_f(m)\lambda_f(n)}{a^{1/2+v}b^{1/2+u}m^{1/2+\alpha+s}n^{1/2+\beta+s}}.$$

Using (4), for any prime number p such that $p\nmid N$, we get

$$\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+1})}{p^{\ell s}} = \lambda_f(p)\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell s}} - \frac{1}{p^s}\sum_{\ell\ge 0}\frac{\lambda_f(p^{\ell+1})\lambda_f(p^\ell)}{p^{\ell s}},$$

$$\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+2})}{p^{\ell s}} = \lambda_f(p^2)\sum_{\ell\ge 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell s}} - \frac{\lambda_f(p)}{p^s}\sum_{\ell\ge 0}\frac{\lambda_f(p^{\ell+1})\lambda_f(p^\ell)}{p^{\ell s}}.$$

Thus, we deduce

$$\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+1})}{p^{\ell s}}=\lambda_f(p)\Big(1+\frac{1}{p^s}\Big)^{-1}\sum_{l\geq 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell s}},\tag{25}$$

$$\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+2})}{p^{\ell s}}=\Big(1+\frac{1}{p^s}\Big)^{-1}\Big(\lambda_f(p^2)-\frac{1}{p^s}\Big)\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell s}}.\tag{26}$$

In addition, since

$$\mathcal{P}=\prod_p\Big(\sum_{\substack{\ell_1,\ell_2,\ell_3,\ell_4\geq 0\\ \ell_1+\ell_3=\ell_2+\ell_4}}\frac{\mu_f(p^{\ell_1})\mu_f(p^{\ell_2})\lambda_f(p^{\ell_3})\lambda_f(p^{\ell_4})}{p^{\ell_1(1/2+v)}p^{\ell_2(1/2+u)}p^{\ell_3(1/2+\alpha+s)}p^{\ell_4(1/2+\beta+s)}}\Big),$$

using (5) and (6) we get

$$\begin{aligned}\mathcal{P}=\prod_{p\nmid N}\Big[&\Big(1+\frac{\lambda_f(p)^2}{p^{1+u+v}}+\frac{1}{p^{2(1+u+v)}}\Big)\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell(1+\alpha+\beta+2s)}}\\ &-\lambda_f(p)\Big(\frac{1}{p^{1+v+\beta+s}}+\frac{1}{p^{1+u+\alpha+s}}\Big)\Big(1+\frac{1}{p^{1+u+v}}\Big)\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+1})}{p^{\ell(1+\alpha+\beta+2s)}}\\ &+\Big(\frac{1}{p^{2(1+v+\beta+s)}}+\frac{1}{p^{2(1+u+\alpha+s)}}\Big)\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)\lambda_f(p^{\ell+2})}{p^{\ell(1+\alpha+\beta+2s)}}\Big]\\ &\times\prod_{p\mid N}\Big[\Big(1+\frac{\lambda_f(p)^2}{p^{1+u+v}}-\frac{\lambda_f(p)^2}{p^{1+\alpha+u+s}}-\frac{\lambda_f(p)^2}{p^{1+\beta+v+s}}\Big)\sum_{\ell\geq 0}\frac{\lambda_f(p^\ell)^2}{p^{\ell(1+\alpha+\beta+2s)}}\Big].\end{aligned}$$

Thus, it follows from (25) and (26) that

$$\begin{aligned}\mathcal{P}=&L(f\times f,1+\alpha+\beta+2s)\prod_{p\mid N}\Big[1+\frac{\lambda_f(p)^2}{p^{1+u+v}}-\frac{\lambda_f(p)^2}{p^{1+\alpha+u+s}}-\frac{\lambda_f(p)^2}{p^{1+\beta+v+s}}\Big]\\ &\times\prod_{p\nmid N}\Big[\Big(1+\frac{\lambda_f(p)^2}{p^{1+u+v}}+\frac{1}{p^{2(1+u+v)}}\Big)\Big(1-\frac{1}{p^{2(1+\alpha+\beta+2s)}}\Big)\\ &\qquad-\lambda_f(p)^2\Big(1-\frac{1}{p^{1+\alpha+\beta+2s}}\Big)\Big(\frac{1}{p^{1+v+\beta+s}}+\frac{1}{p^{1+u+\alpha+s}}\Big)\Big(1+\frac{1}{p^{1+u+v}}\Big)\\ &+\Big(1-\frac{1}{p^{1+\alpha+\beta+2s}}\Big)\Big(\lambda_f(p^2)-\frac{1}{p^{1+\alpha+\beta+2s}}\Big)\Big(\frac{1}{p^{2(1+v+\beta+s)}}+\frac{1}{p^{2(1+u+\alpha+s)}}\Big)\Big]\\ =&L(f\times f,1+\alpha+\beta+2s)\\ &\times\prod_p\Big[1+\frac{\lambda_f(p)^2}{p^{1+u+v}}-\frac{\lambda_f(p)^2}{p^{1+\alpha+u+s}}-\frac{\lambda_f(p)^2}{p^{1+\beta+v+s}}+\chi_0(p)E_p\Big]\end{aligned}$$

where

$$E_p = \frac{1}{p^2}\Bigg[\frac{1}{p^{2(u+v)}} - \frac{1}{p^{2(\alpha+\beta+2s)}} - \frac{\lambda_f(p)^2}{p^s}\Big(\frac{1}{p^{u+\alpha}} + \frac{1}{p^{v+\beta}}\Big)\Big(\frac{1}{p^{u+v}} - \frac{1}{p^{\alpha+\beta+2s}}\Big) + \frac{\lambda_f(p^2)}{p^{2s}}\Big(\frac{1}{p^{2(u+\alpha)}} + \frac{1}{p^{2(v+\beta)}}\Big)\Bigg]$$
$$+\frac{\lambda_f(p)^2}{p^{3+\alpha+\beta+3s}}\Bigg[\frac{1}{p^{u+v}}\Big(\frac{1}{p^{u+\alpha}}+\frac{1}{p^{v+\beta}}\Big)-\frac{1}{p^s}\Big(\frac{1}{p^{2(u+\alpha)}}+\frac{1}{p^{2(v+\beta)}}\Big)-\frac{1}{p^{u+v+\alpha+\beta+s}}\Bigg]$$
$$+\frac{1}{p^{4+2(\alpha+\beta+2s)}}\Bigg[\frac{1}{p^{2(u+\alpha+s)}} + \frac{1}{p^{2(v+\beta+s)}} - \frac{1}{p^{2(u+v)}}\Bigg].$$

Since the Rankin–Selberg L-function $L(f \times f, z)$ admits, for $\Re z > 1$, the Euler product

$$L(f \times f, z) = \prod_p L_p(f \times f, z)$$

with

$$L_p(f\times f, z) = \Big(1 - \frac{\alpha_f(p)^2}{p^z}\Big)^{-1}\Big(1 - \frac{\alpha_f(p)\beta_f(p)}{p^z}\Big)^{-2}\Big(1 - \frac{\beta_f(p)^2}{p^z}\Big)^{-1}$$
$$= \Big(1 - \frac{\lambda_f(p)^2}{p^z} + \chi_0(p)\Big[\frac{2+\lambda_f(p)^2}{p^{2z}} - \frac{\lambda_f(p)^2}{p^{3z}} + \frac{1}{p^{4z}}\Big]\Big)^{-1},$$

we may write

$$1 + \frac{\lambda_f(p)^2}{p^{1+u+v}} - \frac{\lambda_f(p)^2}{p^{1+\alpha+u+s}} - \frac{\lambda_f(p)^2}{p^{1+\beta+v+s}} + \chi_0(p)E_p$$
$$= \frac{L_p(f\times f, 1+u+v)}{L_p(f\times f, 1+u+\alpha+s)L_p(f\times f, 1+v+\beta+s)}\times$$
$$\Big[1 + L_p(f\times f, 1+u+\alpha+s)L_p(f\times f, 1+v+\beta+s)\sum_{r=2}^{8}\sum_{\ell}\frac{a_{r,\ell}(p)}{p^{r+X_{r,\ell}(u,v,\alpha,\beta,s)}}\Big]$$

where the sum over ℓ is finite, $X_{r,\ell}$ are linear forms in u, v, α, β, s, and $a_{r,\ell}(p)$ are complex numbers with $|a_{r,\ell}(p)| \ll 1$. As a result, we obtain

$$\mathcal{P} = \frac{L(f\times f, 1+\alpha+\beta+2s)L(f\times f, 1+u+v)}{L(f\times f, 1+\alpha+u+s)L(f\times f, 1+\beta+v+s)}A_{\alpha,\beta}(u,v,s)$$

where

$$A_{\alpha,\beta}(u,v,s) = \prod_p\Big[1 + \sum_{r,\ell} O\Big(\frac{1}{p^{r+X_{r,\ell}(\Re u,\Re v,\Re\alpha,\Re\beta,\Re s)}}\Big)\Big].$$

Then $A_{\alpha,\beta}(u,v,s)$ is an absolutely convergent Euler product in $\{\Re(\alpha+u+s) > -1\}\cap\{\Re(\beta+v+s) > -1\}\cap\bigcap_{r,\ell}\{X_{r,\ell}(\Re u, \Re v, \Re\alpha, \Re\beta, \Re s) > 1-r\}$. Making explicit all the linear forms $X_{2,\ell}$, we obtain $\{X_{2,\ell}(\Re u, \Re v, \Re\alpha, \Re\beta, \Re s) > -1\} = \Omega_{\alpha,\beta}$. Similarly, writing out all the linear forms $X_{r,\ell}$, we prove that for (u,v,s) in $\Omega_{\alpha,\beta}$ we have $X_{r,\ell}(\Re u, \Re v, \Re\alpha, \Re\beta, s) > -r/2 \geq 1-r$. As a result, $A_{\alpha,\beta}(u,v,s)$ is an absolutely convergent Euler product on $\Omega_{\alpha,\beta}$ and defines a holomorphic function on $\Omega_{\alpha,\beta}$. ∎

LEMMA 7. *We have* $A_{0,0}(0,0,0)=1$.

Proof. Thanks to Lemma 6, if $\Re s>0$, we may write

$$A_{0,0}(s,s,s)=\sum_{\substack{a,b,m,n\ge 1\\ am=bn}}\frac{\mu_f(a)\mu_f(b)\lambda_f(m)\lambda_f(n)}{(ambn)^{1/2+s}}$$
$$=\sum_{a,m\ge 1}\frac{\mu_f(a)\lambda_f(m)}{(am)^{1+2s}}\sum_{n\mid am}\mu_f(am/n)\lambda_f(n).$$

Since $(\mu_f(n))$ is the convolution inverse of $(\lambda_f(n))$, we have

$$\sum_{n\mid d}\mu_f(d/n)\lambda_f(n)=\delta(d).$$

Thus $A_{0,0}(s,s,s)=\mu_f(1)\lambda_f(1)=1$. To conclude, we extend this relation to $s=0$ by continuity in the half-plane $\Re s>0$. ∎

LEMMA 8. *For all non-negative integers a, we have*

$$\sum_{n\le M}\frac{\lambda_f(n)^2}{n}\left(\ln\frac{M}{n}\right)^a=\frac{\operatorname{Res}_{s=1}L(f\times f,s)}{\zeta^{(N)}(2)}\int_1^M\frac{1}{r}\left(\ln\frac{M}{r}\right)^a dr+O((\ln M)^a).$$

Proof. We may find in [Ran39] the following asymptotic behaviour:

$$\sum_{n\le x}\lambda_f(n)^2=x\frac{\operatorname{Res}_{s=1}L(f\times f,s)}{\zeta^{(N)}(2)}+O(x^{3/5}).$$

After one integration by parts, we get

$$\sum_{n\le M}\frac{\lambda_f(n)^2}{n}\left(\ln\frac{M}{n}\right)^a=\frac{\operatorname{Res}_{s=1}L(f\times f,s)}{\zeta^{(N)}(2)}\sum_{n\le M}\frac{1}{n}\left(\ln\frac{M}{n}\right)^a+O((\ln M)^a).$$

Finally, the Euler–Maclaurin formula gives

$$\sum_{n\le M}\frac{1}{n}\left(\ln\frac{M}{n}\right)^a=\int_1^M\frac{1}{r}\left(\ln\frac{M}{r}\right)^a dr+O((\ln M)^a).\ ∎$$

3.2.2. *Estimation of $I_f^{D_1}(\alpha,\beta)$.* For positive integers i and j, and for any positive real δ, let

$$(27)\quad J_{\alpha,\beta}(i,j)=\frac{1}{(2i\pi)^2}\int_{(\delta)}\int_{(\delta)}M^{u+v}\frac{L(f\times f,1+u+v)A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)L(f\times f,1+\beta+v)}\frac{du}{u^{i+1}}\frac{dv}{v^{j+1}}.$$

LEMMA 9. *Let $\alpha, \beta \ll 1/L$ be complex numbers. For positive integers i and j, we have*

$$J_{\alpha,\beta}(i,j) = \frac{(\ln M)^{i+j-1}}{i!j!\,\mathrm{Res}_{s=1} L(f\times f,s)}\,\frac{d^2}{dxdy}\Big[M^{\alpha x+\beta y}\int_0^1 (x+u)^i(y+u)^j\,du\Big]\Big|_{x=y=0} + O\bigg(L^{i+j-2}\Big(1+\frac{(\ln L)^2}{L^{i-1}}\Big)\Big(1+\frac{(\ln L)^2}{L^{j-1}}\Big)\bigg).$$

Proof. We use the Dirichlet series of $L(f\times f,s)$. Moving either the u or the v integration line far to the right, we obtain

$$J_{\alpha,\beta}(i,j) = \sum_{n\le M}\frac{\lambda_f(n)^2}{n} \times \frac{1}{(2i\pi)^2}\int_{(\delta)}\int_{(\delta)}\Big(\frac{M}{n}\Big)^{u+v}\frac{\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)L(f\times f,1+\beta+v)}\,\frac{du}{u^{i+1}}\,\frac{dv}{v^{j+1}}.$$

Let

$$r^{(i,j)}_{\alpha,\beta}(u,v) = \Big(\frac{M}{n}\Big)^{u+v}\frac{\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)L(f\times f,1+\beta+v)}\,\frac{1}{u^{i+1}v^{j+1}}.$$

We consider the contour $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3$ with $c>0$, $Y\gg 1$ large and

$$\begin{aligned}\gamma_1 &= \{i\tau : |\tau|\ge Y\},\\ \gamma_2 &= \{\sigma\pm iY : -c/\ln Y\le\sigma\le 0\},\\ \gamma_3 &= \{-c/\ln Y+i\tau : |\tau|\le Y\}.\end{aligned}$$

By the standard zero-free region of $L(f\times f,s)$, we replace integration over $\Re u=\Re v=\delta$ by integration over γ. Thus,

$$\begin{aligned}\frac{1}{(2i\pi)^2}&\int_{(\delta)}\int_{(\delta)} r^{(i,j)}_{\alpha,\beta}(u,v)\,du\,dv\\ &= \mathrm{Res}_{u=0}\frac{1}{2i\pi}\int_{\Re v=\delta} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv + \frac{1}{(2i\pi)^2}\int_{u\in\gamma}\int_{\Re v=\delta} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv\,du\\ &= \mathrm{Res}_{u=v=0}\, r^{(i,j)}_{\alpha,\beta}(u,v) + \mathrm{Res}_{u=0}\frac{1}{2i\pi}\int_{v\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv\\ &\quad + \mathrm{Res}_{v=0}\frac{1}{2i\pi}\int_{u\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,du + \frac{1}{(2i\pi)^2}\int_\gamma\int_\gamma r^{(i,j)}_{\alpha,\beta}(u,v)\,du\,dv.\end{aligned}$$

• We begin with the estimation of $\mathrm{Res}_{u=0} \frac{1}{2i\pi} \int_{v\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv$. We express the residue as a contour integral over a circle of radius $1/L$. We get

$$\mathrm{Res}_{u=0} \frac{1}{2i\pi} \int_{v\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv = \frac{1}{(2i\pi)^2} \int_{v\in\gamma} \frac{(M/n)^v}{L(f\times f, 1+\beta+v)} \times \oint_{D(0,L^{-1})} \left(\frac{M}{n}\right)^u \frac{\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)} \frac{du}{u^{i+1}} \frac{dv}{v^{j+1}}.$$

Furthermore, since $|\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)| \ll 1$ in our range of integration and since

$$\frac{1}{L(f\times f,1+\alpha+u)} \ll \alpha+u \ll L^{-1} \quad \text{because } u \asymp 1/L,$$

we obtain

$$\mathrm{Res}_{u=0} \frac{1}{2i\pi} \int_{v\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv \ll L^{i-1} \int_{v\in\gamma} \frac{(M/n)^{\Re v}}{|L(f\times f,1+\beta+v)|} \frac{dv}{|v|^{j+1}}.$$

As $-L'(f\times f,z)/L(f\times f,z) = \sum_{n\ge 1} \Lambda_f(n)/n^z$ with $\Lambda_f(n)\ge 0$, we deduce (cf. [Ten95, Section 3.10] and [IK04, Section 5.3]) that

$$\frac{1}{L(f\times f,\sigma+i\tau)} \ll \ln|\tau|,$$

and it follows that

$$\int_{v\in\gamma} \frac{(M/n)^{\Re v}}{|L(f\times f,1+\beta+v)|} \frac{dv}{|v|^{j+1}} \ll \int_{|\tau|\ge Y} \frac{\ln\tau}{|\tau|^{j+1}}\,d\tau + \ln Y \cdot \int_{-c/\ln Y}^{0} \frac{d\sigma}{|\sigma+iY|^{j+1}} + \left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}} \ln Y \cdot \int_{|\tau|\le Y} \frac{d\tau}{|\tau - ic/\ln Y|^{j+1}} \ll \frac{\ln Y}{Y^j} + (\ln Y)^{j+1}\left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}. \tag{28}$$

As a result, we get the bound

$$\mathrm{Res}_{u=0} \frac{1}{2i\pi} \int_{v\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,dv \ll L^{i-1}\ln Y \cdot \left(\frac{1}{Y^j} + (\ln Y)^j \left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\right). \tag{29}$$

• Since $r^{(i,j)}_{\alpha,\beta}(u,v) = r^{(j,i)}_{\beta,\alpha}(v,u)$, the previous bound immediately yields

$$\mathrm{Res}_{v=0} \frac{1}{2i\pi} \int_{u\in\gamma} r^{(i,j)}_{\alpha,\beta}(u,v)\,du \ll L^{j-1}\ln Y \cdot \left(\frac{1}{Y^i} + (\ln Y)^i \left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\right). \tag{30}$$

- By (28), we bound $\int_\gamma\int_\gamma r_{\alpha,\beta}^{(i,j)}(u,v)\,du\,dv$:

$$\begin{aligned}
(31)\qquad &\frac{1}{(2i\pi)^2}\int_\gamma\int_\gamma r_{\alpha,\beta}^{(i,j)}(u,v)\,du\,dv\\
&=\frac{1}{(2i\pi)^2}\int_\gamma\int_\gamma\left(\frac{M}{n}\right)^{u+v}\frac{\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)L(f\times f,1+\beta+v)}\,\frac{du\,dv}{u^{i+1}v^{j+1}}\\
&\ll\int_\gamma\frac{(M/n)^{\Re v}}{|L(f\times f,1+\beta+v)|}\,\frac{dv}{|v|^{j+1}}\int_\gamma\frac{(M/n)^{\Re u}}{|L(f\times f,1+\alpha+u)|}\,\frac{du}{|u|^{i+1}}\\
&\ll\left(\frac{\ln Y}{Y^j}+(\ln Y)^{j+1}\left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\right)\left(\frac{\ln Y}{Y^i}+(\ln Y)^{i+1}\left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\right)\\
&\ll(\ln Y)^2\left(\frac{1}{Y^{i+j}}+(\ln Y)^{i+j}\left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\right).
\end{aligned}$$

- In addition, for any positive integer ℓ, let

$$W_\ell=\sum_{n\le M}\frac{\lambda_f(n)^2}{n}\left(\frac{1}{Y^\ell}+(\ln Y)^\ell\left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\right).$$

We can bound

$$\begin{aligned}
W_\ell&\ll\frac{1}{Y^\ell}\sum_{n\le\frac{M}{(Y\ln Y)^{\ell(\ln Y)/c}}}\frac{\lambda_f(n)^2}{n}+(\ln Y)^\ell\sum_{\frac{M}{(Y\ln Y)^{\ell(\ln Y)/c}}\le n\le M}\frac{\lambda_f(n)^2}{n}\left(\frac{M}{n}\right)^{-\frac{c}{\ln Y}}\\
&\ll\frac{L}{Y^\ell}+(\ln Y)^\ell\sum_{0\le d\le\frac{\ln(Y\ln Y)}{\ln 2}}2^{-d\ell}\sum_{\frac{M}{(2^{d+1})^{\ell(\ln Y)/c}}\le n\le\frac{M}{(2^d)^{\ell(\ln Y)/c}}}\frac{\lambda_f(n)^2}{n}\\
&\ll\frac{L}{Y^\ell}+(\ln Y)^{\ell+1}.
\end{aligned}$$

As a consequence, by (29)–(31), we obtain

$$\begin{aligned}
J_{\alpha,\beta}(i,j)&=\sum_{n\le M}\left[\frac{\lambda_f(n)^2}{n}\operatorname{Res}_{u=v=0}r_{\alpha,\beta}^{(i,j)}(u,v)\right]\\
&\quad+O\big(L^{i-1}W_j\ln Y+L^{j-1}W_i\ln Y+W_{i+j}(\ln Y)^2\big)\\
&=\sum_{n\le M}\left[\frac{\lambda_f(n)^2}{n}\operatorname{Res}_{u=v=0}r_{\alpha,\beta}^{(i,j)}(u,v)\right]\\
&\quad+O\left[L^{i-1}\ln Y\cdot\left(\frac{L}{Y^j}+(\ln Y)^j\right)+L^{j-1}\ln Y\cdot\left(\frac{L}{Y^i}+(\ln Y)^i\right)\right.\\
&\quad\left.+(\ln Y)^2\left(\frac{L}{Y^{i+j}}+(\ln Y)^{i+j}\right)\right].
\end{aligned}$$

Choosing $Y = L$, which is allowed, yields the error term $O((L^{i-1} + (\ln L)^2) \times (L^{j-1} + (\ln L)^2))$. Then

$$J_{\alpha,\beta}(i,j) = \sum_{n\le M}\left[\frac{\lambda_f(n)^2}{n}\operatorname{Res}_{u=v=0} r_{\alpha,\beta}^{(i,j)}(u,v)\right] + O\left(L^{i+j-2}\left(1+\frac{(\ln L)^2}{L^{i-1}}\right)\left(1+\frac{(\ln L)^2}{L^{j-1}}\right)\right). \tag{32}$$

• We finish with the estimation of $\operatorname{Res}_{u=v=0} r_{\alpha,\beta}^{(i,j)}(u,v)$. To do so, we again express the residue as a contour integral over a circle of radius $1/L$. Thus

$$\operatorname{Res}_{u=v=0} r_{\alpha,\beta}^{(i,j)}(u,v) = \frac{1}{(2i\pi)^2}\oint_{D(0,L^{-1})}\oint_{D(0,L^{-1})}\left(\frac{M}{n}\right)^{u+v} \times \frac{\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)L(f\times f,1+\beta+v)}\,\frac{du}{u^{i+1}}\,\frac{dv}{v^{j+1}}.$$

Furthermore, since $u \asymp v \asymp 1/L$, we have

$$\begin{aligned}
\zeta^{(N)}(2(1+u+v)) &= \zeta^{(N)}(2) + O(1/L),\\
A_{\alpha,\beta}(u,v,0) &= A_{0,0}(0,0,0) + O(1/L),\\
\frac{1}{L(f\times f,1+\alpha+u)} &= \frac{\alpha+u}{\operatorname{Res}_{s=1} L(f\times f,s)}(1+O(1/L)),\\
\frac{1}{L(f\times f,1+\beta+v)} &= \frac{\beta+v}{\operatorname{Res}_{s=1} L(f\times f,s)}(1+O(1/L)).
\end{aligned}$$

With Lemma 7, we obtain

$$\frac{\zeta^{(N)}(2(1+u+v))A_{\alpha,\beta}(u,v,0)}{L(f\times f,1+\alpha+u)L(f\times f,1+\beta+v)} = (\alpha+u)(\beta+v)\frac{\zeta^{(N)}(2)}{[\operatorname{Res}_{s=1} L(f\times f,s)]^2} + O(1/L^3).$$

Then

$$\operatorname{Res}_{u=v=0} r_{\alpha,\beta}^{(i,j)}(u,v) = \frac{\zeta^{(N)}(2)}{[\operatorname{Res}_{s=1} L(f\times f,s)]^2} \times\left[\frac{1}{2i\pi}\oint_{D(0,L^{-1})}\left(\frac{M}{n}\right)^{u}\frac{\alpha+u}{u^{i+1}}\,du\right]\left[\frac{1}{2i\pi}\oint_{D(0,L^{-1})}\left(\frac{M}{n}\right)^{v}\frac{\beta+v}{v^{j+1}}\,dv\right] + O(L^{i+j-3}).$$

In addition, thanks to the Cauchy formula, for any positive integer ℓ,

$$\frac{1}{2i\pi}\oint\limits_{D(0,L^{-1})}\left(\frac{M}{n}\right)^u\frac{\alpha+u}{u^{\ell+1}}\,du$$
$$=\frac{d}{dx}\left[\frac{e^{\alpha x}}{2i\pi}\oint\limits_{D(0,L^{-1})}\left(\frac{M}{n}e^x\right)^u\frac{du}{u^{\ell+1}}\right]\Bigg|_{x=0}=\frac{d}{dx}\left[e^{\alpha x}\frac{1}{\ell!}\frac{d^\ell}{du^\ell}\left[\left(\frac{M}{n}e^x\right)^u\right]\Bigg|_{u=0}\right]\Bigg|_{x=0}$$
$$=\frac{1}{\ell!}\frac{d}{dx}\left[e^{\alpha x}\left(x+\ln\frac{M}{n}\right)^\ell\right]\Bigg|_{x=0}=\frac{1}{\ell!}\left[\alpha\left(\ln\frac{M}{n}\right)^\ell+\ell\left(\ln\frac{M}{n}\right)^{\ell-1}\right].$$

Then we can write

$$\operatorname{Res}_{u=v=0} r^{(i,j)}_{\alpha,\beta}(u,v)=\frac{\zeta^{(N)}(2)}{i!j![\operatorname{Res}_{s=1}L(f\times f,s)]^2}$$
$$\times\left[\alpha\beta\left(\ln\frac{M}{n}\right)^{i+j}+(\alpha i+\beta j)\left(\ln\frac{M}{n}\right)^{i+j-1}+ij\left(\ln\frac{M}{n}\right)^{i+j-2}\right]$$
$$+O(L^{i+j-3}).$$

From (32) and by Lemma 8, we get

$$J_{\alpha,\beta}(i,j)=\frac{[\operatorname{Res}_{s=1}L(f\times f,s)]^{-1}}{i!j!}$$
$$\times\int_1^M\left(\alpha\left(\ln\frac{M}{r}\right)^i+i\left(\ln\frac{M}{r}\right)^{i-1}\right)\left(\beta\left(\ln\frac{M}{r}\right)^j+j\left(\ln\frac{M}{r}\right)^{j-1}\right)\frac{dr}{r}$$
$$+O\left(L^{i+j-2}\left(1+\frac{(\ln L)^2}{L^{i-1}}\right)\left(1+\frac{(\ln L)^2}{L^{j-1}}\right)\right).$$

Changing the variable r to u with $r=M^{1-u}$ concludes the proof. ∎

Lemma 10. *If $0<\nu<1$ and $\alpha,\beta\ll L^{-1}$ are complex numbers with $|\alpha+\beta|\gg L^{-1}$ then*

$$I_f^{D_1}(\alpha,\beta)=\frac{\widehat{w}(0)}{(\alpha+\beta)\ln M}\frac{d^2}{dxdy}\left[M^{\alpha x+\beta y}\int_0^1P(x+u)P(y+u)\,du\right]\Bigg|_{x=y=0}$$
$$+O\left(\frac{T(\ln L)^4}{L}\right).$$

Proof. We use the Mellin transformation to write

$$\left(\frac{\ln(M/a)}{\ln M}\right)^i=\begin{cases}\dfrac{i!}{(\ln M)^i}\dfrac{1}{2i\pi}\displaystyle\int_{(1)}\left(\frac{M}{a}\right)^v\frac{dv}{v^{i+1}} & \text{if } 1\le a\le M,\\ 0 & \text{if } a>M.\end{cases}$$

Set $P(X) = \sum_{i=1}^{\deg P} a_i X^i$. Thus, from (23),

$$I_f^{D_1}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j\geq 1} \frac{a_i a_j i! j!}{(\ln M)^{i+j}} \frac{1}{(2i\pi)^3} \int_{(1)}\int_{(1)}\int_{(1)} \frac{M^{u+v}G(s)}{s} g_{\alpha,\beta}(s,t)$$
$$\times \sum_{\substack{a,b,m,n\geq 1\\ am=bn}} \frac{\mu_f(a)\mu_f(b)\lambda_f(m)\lambda_f(n)}{a^{1/2+v}b^{1/2+u}m^{1/2+\alpha+s}n^{1/2+\beta+s}}\, ds\, \frac{du}{u^{j+1}}\, \frac{dv}{v^{i+1}}\, dt.$$

Due to Lemma 6, we can write

$$I_f^{D_1}(\alpha,\beta)$$
$$= \int_{-\infty}^{\infty} w(t) \sum_{i,j\geq 1} \frac{a_i a_j i! j!}{(\ln M)^{i+j}} \frac{1}{(2i\pi)^3} \int_{(1)}\int_{(1)}\int_{(1)} \frac{M^{u+v}G(s)}{s} g_{\alpha,\beta}(s,t) A_{\alpha,\beta}(u,v,s)$$
$$\times \frac{L(f\times f, 1+\alpha+\beta+2s)L(f\times f,1+u+v)}{L(f\times f,1+\alpha+u+s)L(f\times f,1+\beta+v+s)}\, ds\, \frac{du}{u^{j+1}}\, \frac{dv}{v^{i+1}}\, dt.$$

We specialize G to

$$G(s) = e^{s^2}\frac{(\alpha+\beta)^2-(2s)^2}{(\alpha+\beta)^2}.$$

As a result, $G(s)L(f\times f, 1+\alpha+\beta+2s)$ is an entire function. First, we move the integration lines from $\Re u = \Re v = 1$ to $\Re u = \Re v = \delta$ with δ small, to ensure the absolute convergence of $A_{\alpha,\beta}(u,v,s)$. Secondly, we move the integration line from $\Re s = 1$ to $\Re s = -\delta+\epsilon$ with $0<\epsilon<\delta$, crossing a pole at $s=0$. Since $t \asymp T$, $\nu < 1$ and $g_{\alpha,\beta}(s,t) \ll T^{2s}$, we can bound

$$\int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{a_i a_j i! j!}{(\ln M)^{i+j}} \frac{1}{(2i\pi)^3} \int_{\Re u=\delta}\int_{\Re v=\delta}\int_{\Re s=-\delta+\epsilon} \frac{M^{u+v}G(s)}{s} g_{\alpha,\beta}(s,t)$$
$$\times A_{\alpha,\beta}(u,v,s)\frac{L(f\times f, 1+\alpha+\beta+2s)L(f\times f,1+u+v)}{L(f\times f,1+\alpha+u+s)L(f\times f,1+\beta+v+s)}\, ds\, \frac{du}{u^{j+1}}\frac{dv}{v^{i+1}}\, dt$$
$$\ll \int_{-\infty}^{\infty} |w(t)|\, dt\, T^{2(-\delta+\epsilon)}M^{2\delta} \ll T^{1-(2-2\nu)\delta+\epsilon} \ll T^{1-\varepsilon}$$

for sufficiently small ε. Then, using some previous notation, this estimate gives

$$I_f^{D_1}(\alpha,\beta) = \widehat{w}(0)L(f\times f, 1+\alpha+\beta)\sum_{i,j}\frac{a_i a_j i! j!}{(\ln M)^{i+j}} J_{\alpha,\beta}(i,j) + O(T^{1-\varepsilon}).$$

Thanks to Lemma 9 and since

$$L(f\times f, 1+\alpha+\beta) = \frac{\operatorname{Res}_{s=1} L(f\times f, s)}{\alpha+\beta} + O(1),$$

we get

$$I_f^{D_1}(\alpha,\beta) = \widehat{w}(0)\bigg[\frac{\mathrm{Res}_{s=1} L(f\times f,s)}{\alpha+\beta} + O(1)\bigg]\bigg[\frac{1}{\mathrm{Res}_{s=1} L(f\times f,s)\ln M}$$
$$\times \frac{d^2}{dxdy}\Big[M^{\alpha x+\beta y}\int_0^1 P(x+u)P(y+u)\,du\Big]\Big|_{x=y=0} + O\bigg(\frac{(\ln L)^4}{L^2}\bigg)\bigg] + O(T^{1-\varepsilon})$$
$$= \frac{\widehat{w}(0)}{(\alpha+\beta)\ln M}\,\frac{d^2}{dxdy}\Big[M^{\alpha x+\beta y}\int_0^1 P(x+u)P(y+u)\,du\Big]\Big|_{x=y=0}$$
$$+ O\bigg(\frac{T(\ln L)^4}{(\alpha+\beta)L^2}\bigg) + O(T/L).$$

We conclude using the assumption $|\alpha+\beta| \gg L^{-1}$. ■

REMARK 9. Sometimes, it may be useful to consider the relation

$$(33)\qquad \frac{d^2}{dxdy}\Big[M^{\alpha x+\beta y}\int_0^1 P(x+u)P(y+u)\,du\Big]\Big|_{x=y=0}$$
$$= \int_0^1 (P'(u)+\alpha\ln M P(u))\big(P'(u)+\beta\ln M P(u)\big)\,du.$$

LEMMA 11. *If $0<\nu<1$ and $\alpha,\beta \ll L^{-1}$ are complex numbers with $|\alpha+\beta| \gg L^{-1}$ then*

$$I_f^{D_2}(\alpha,\beta) = T^{-2(\alpha+\beta)} I_f^{D_1}(-\beta,-\alpha) + O(T/L).$$

Proof. We write

$$I_{a,b}^{D_2}(\alpha,\beta) = \sum_{am=bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\beta}n^{1/2-\alpha}} \int_{-\infty}^{\infty} w(t)\bigg(\frac{t\sqrt{N}}{2\pi}\bigg)^{-2(\alpha+\beta)} V_{-\beta,-\alpha}(mn,t)\,dt$$
$$+ O\bigg(\sum_{\substack{am=bn\\ mn\ll T^{2+\varepsilon}}} \frac{|\lambda_f(m)\lambda_f(n)|}{m^{1/2-\beta}n^{1/2-\alpha}}\bigg).$$

Let $a' = a/(a,b)$ and $b' = b/(a,b)$. Then, for all $\delta>0$, the above error term becomes

$$\sum_{\substack{am=bn\\ mn\ll T^{2+\varepsilon}}} \frac{|\lambda_f(m)\lambda_f(n)|}{m^{1/2-\beta}n^{1/2-\alpha}} \ll \frac{1}{(a'b')^{1/2-\delta}} \sum_{k\ll T^{1+\varepsilon/2}/\sqrt{a'b'}} \frac{1}{k^{1-\alpha-\beta-2\delta}} \ll \frac{T^{\varepsilon}}{\sqrt{a'b'}}.$$

Therefore, if $w_2(t) = w(t)\big(\frac{t\sqrt{N}}{2\pi}\big)^{-2(\alpha+\beta)}$, we may write

$$I_f^{D_2}(\alpha,\beta) = \sum_{a,b\le M} \frac{\mu_f(a)\mu_f(b)}{\sqrt{ab}} P\Big(\frac{\ln(M/a)}{\ln M}\Big) P\Big(\frac{\ln(M/b)}{\ln M}\Big)$$
$$\times \Big[\sum_{am=bn} \frac{\lambda_f(m)\lambda_f(n)}{m^{1/2-\beta} n^{1/2-\alpha}} \int_{-\infty}^{\infty} w_2(t) V_{-\beta,-\alpha}(mn,t)\,dt\Big]$$
$$+ O\Big(T^{\varepsilon} \sum_{a,b\le M} \frac{1}{\sqrt{ab}\sqrt{a'b'}}\Big).$$

We also have

$$\sum_{a,b\le M} \frac{1}{\sqrt{ab}\sqrt{a'b'}} \ll \sum_{k\le M} \frac{1}{k} \sum_{a,b\le M/k} \frac{1}{ab} \ll \sum_{k\le M} \frac{1}{k}\Big(\ln \frac{M}{k}\Big)^2 \ll (\ln M)^3$$

and since w_2 satisfies (9a)–(9c), up to replacing w by w_2 and (α,β) by $(-\beta,-\alpha)$, Lemma 10 gives

$$I_f^{D_2}(\alpha,\beta) = \frac{\widehat{w}_2(0)}{(-\alpha-\beta)\ln M} \frac{d^2}{dxdy}\Big[M^{-\alpha x-\beta y} \int_0^1 P(x+u)P(y+u)\,du\Big]\Big|_{x=y=0}$$
$$+ O(T(\ln L)^4/L).$$

Finally, due to the support of w, we can write $\big(\frac{t\sqrt{N}}{2\pi}\big)^{-2(\alpha+\beta)} = T^{-2(\alpha+\beta)} + O(1/L)$, which gives $\widehat{w}_2(0) = T^{-2(\alpha+\beta)}\widehat{w}(0) + O(T/L)$. ∎

Proof of Proposition 4. From (24) and using Lemma 11, we can write

$$\begin{aligned} I_f^{D}(\alpha,\beta) &= I_f^{D_1}(\alpha,\beta) + T^{-2(\alpha+\beta)} I_f^{D_1}(-\beta,-\alpha) + O(T(\ln L)^4/L) \\ &= I_f^{D_1}(\alpha,\beta) + I_f^{D_1}(-\beta,-\alpha) + I_f^{D_1}(-\beta,-\alpha)[T^{-2(\alpha+\beta)} - 1] \\ &\quad + O(T(\ln L)^4/L). \end{aligned}$$

Finally, using (33) and Lemma 10, we have

$$I_f^{D_1}(\alpha,\beta) + I_f^{D_1}(-\beta,-\alpha) = \widehat{w}(0) + O(T(\ln L)^4/L).$$

Combining these relations, we get the result. ∎

4. Effective proportion of zeros on the critical line. In this section, we prove Corollary 1. From Theorem 1, we may deduce the following theorem about the mollified second moment of $L(f,s)$ and its derivative.

THEOREM 5. *Let Q be a polynomial with complex coefficients satisfying $Q(0) = 1$. Let*

$$V(s) = Q\Big(-\frac{1}{2\ln T}\frac{d}{ds}\Big)L(f,s).$$

Then, if $\nu < \frac{1-2\theta}{4+2\theta}$, *we have*

$$\frac{1}{T}\int_1^T |V\psi(\sigma_0+it)|^2\,dt = c(P,Q,2R,\nu/2)+o(1)$$

where

$$c(P,Q,r,\xi) = 1+\frac{1}{\xi}\int_0^1\int_0^1 e^{2rs}\left[\frac{d}{dx}\left(e^{r\xi x}Q(s+\xi x)P(x+u)\right)\bigg|_{x=0}\right]^2 du\,ds.$$

We do not give the proof of this theorem, which is essentially the same as the one in [You10, Theorem 1]. We refer to [You10, Sections 2 and 3] for more details. Now, Q is a polynomial with complex coefficients of the shape

$$Q(x) = 1+\sum_{n=1}^{M} i^{n+1}\lambda_n[(1-2x)^n-1] \tag{34}$$

where M is a positive integer and $(\lambda_1,\dots,\lambda_M)\in\mathbb{R}^M$.

Let $N_f(T)$ (resp. $N_{f,0}(T)$) be the number of non-trivial zeros ρ (resp. on the critical line) of $L(f,s)$ with $0<\Im\rho\le T$ for f a holomorphic primitive cusp form of even weight, square-free level and trivial character.

Proposition 5. *For f a holomorphic primitive cusp form of even weight, square-free level and trivial character, and Q as in* (34), *we have*

$$\liminf_{T\to\infty}\frac{N_{f,0}(T)}{N_f(T)} \ge \limsup_{T\to\infty}\left[1-\frac{1}{2R}\ln\left(\frac{1}{T}\int_1^T |V\psi(\sigma_0+it)|^2\,dt\right)\right].$$

Hence, using Theorem 5, we get

$$\liminf_{T\to\infty}\frac{N_{f,0}(T)}{N_f(T)} \ge 1-\inf_{P,Q,R}\frac{1}{R}\ln c(P,Q,R,\nu/2).$$

The work of Kim and Sarnak [Kim03] gives $\theta = 7/64$, so Theorem 5 gives $\nu = 5/27$. The Ramanujan–Petersson conjecture ($\theta = 0$) gives $\nu = 1/4$.

Lemma 12 ([Con89, Sect. 4]). *We have*

$$\inf_P\frac{1}{R}\ln c(P,Q,R,\nu/2) = \frac{1}{R}\ln\left(\frac{1+|w(1)|^2}{2}+\frac{A\alpha}{\tanh\frac{\nu\alpha}{2}}\right)$$

where $w(x) = e^{Rx}Q(x)$, $A = \int_0^1 |w(x)|^2\,dx$, $B = \int_0^1 w(x)\overline{w'(x)}\,dx$, $C = \int_0^1 |w'(x)|^2\,dx$ *and* $\alpha = \sqrt{(B-\overline{B})^2+4AC}/(2A)$.

For empirical reasons, we restrict ourselves to polynomials Q with real coefficients of the shape

$$Q(x) = 1+\sum_{n=1}^{N} h_n[(1-2x)^{2n-1}-1] \tag{35}$$

where N is a positive integer and $(h_1,\dots,h_N)\in\mathbb{R}^N$. Then we get

$$\liminf_{T\to\infty}\frac{N_{f,0}(T)}{N_f(T)}\ge 1-\inf_{Q\,\mathrm{real},\,R}\frac{1}{R}\ln\left(\frac{1+w(1)^2}{2}+\frac{\sqrt{AC}}{\tanh\left(\frac{\nu}{2}\sqrt{\frac{C}{A}}\right)}\right).\tag{36}$$

To obtain Corollary 1, we choose $N=4$, R and Q as in (35) where R, h_1, h_2, h_3, h_4 are given in the following table.

	$\nu=1/6$	$\nu=5/27$
R	6.6838894702116801322	6.4278834168344993342
h_1	1.6017785744634898860	1.5898336242677838745
h_2	−3.0362512753510924917	−2.8999828229132398066
h_3	3.0757757634512927939	3.0171733454035522056
h_4	−1.1407980564855935531	−1.1164150244992046552

	$\nu=1/4$
R	5.6503610091685135131
h_1	1.5369390514358411982
h_2	−2.7929104872905007806
h_3	2.7758193765120241770
h_4	−1.0187870607687957034

5. Non-mollified second integral moment. This section contains the proof of Theorem 2. For convenience, we set

$$M_{f,2}(\alpha,\beta)=\int_{-\infty}^{\infty}w(t)L(f,1/2+\alpha+it)L(f,1/2+\beta-it)\,dt$$

where w satisfies (9a)–(9c). Applying the approximate equation (Lemma 1) and using previous notation, we can write

$$M_{f,2}(\alpha,\beta)=I_{1,1}^{D_1}(\alpha,\beta)+I_{1,1}^{D_2}(\alpha,\beta)+I_{1,1}^{ND_1}(\alpha,\beta)+I_{1,1}^{ND_2}(\alpha,\beta).$$

Proposition 2 and Corollary 5 allow us to bound the off-diagonal contribution so that

$$M_{f,2}(\alpha,\beta)=I_{1,1}^{D_1}(\alpha,\beta)+I_{1,1}^{D_2}(\alpha,\beta)+O(T^{1/2+\theta+\varepsilon}).\tag{37}$$

5.1. Diagonal contribution. We begin with a useful lemma.

Lemma 13. *The Laurent series of the meromorphic function* $s\mapsto\frac{L(f\times f,s)}{\zeta^{(N)}(2s)}$ *about* $s=1$ *can be written as*

$$\frac{L(f\times f,1+s)}{\zeta^{(N)}(2(1+s))}=\frac{\mathfrak{a}_f/2}{s}+\mathfrak{b}_f/2+O(s).$$

Proof. We have

$$\frac{L(f\times f,1+s)}{\zeta^{(N)}(2(1+s))}=\frac{1}{\prod_{p|N}\left(1+\frac{1}{p^{1+s}}\right)}\frac{\zeta(1+s)}{\zeta(2(1+s))}L(\mathrm{Sym}^2 f,1+s)$$
$$=\frac{N}{\nu(N)}\left(1+s\sum_{p|N}\frac{\ln p}{p+1}+O(s^2)\right)\left(L(\mathrm{Sym}^2 f,1)+sL'(\mathrm{Sym}^2 f,1)+O(s^2)\right)$$
$$\times\frac{\left(1-2\frac{\zeta'(2)}{\zeta(2)}+O(s^2)\right)}{\zeta(2)}\left(\frac{1}{s}+\gamma+O(s)\right).$$

An easy calculation gives the result. ■

LEMMA 14. *Let $\alpha,\beta\ll\ln T$ be complex numbers. Then*

$$I_{1,1}^{D_1}(\alpha,\beta)=\int_{\mathbb{R}}w(t)\,\mathrm{Res}_{s=0}\left[\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)}{\zeta^{(N)}(2(1+\alpha+\beta+2s))}\right]dt$$
$$+O(T^{1/2}).$$

Proof. By the definition of $I_{1,1}^{D_1}(\alpha,\beta)$, we can write

$$I_{1,1}^{D_1}(\alpha,\beta)=\sum_{m=n}\frac{\lambda_f(m)\lambda_f(n)}{m^{1/2+\alpha}n^{1/2+\beta}}\int_{\mathbb{R}}w(t)V_{\alpha,\beta}(mn,t)\,dt$$
$$=\int_{\mathbb{R}}w(t)\frac{1}{2i\pi}\int_{(\sigma)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\sum_{m\ge1}\frac{\lambda_f(m)^2}{m^{1+\alpha+\beta+2s}}\,ds\,dt.$$

Since $L(f\times f,s)=\zeta^{(n)}(2s)\sum_{m\ge1}\lambda_f(m)^2/m^s$, for all positive real σ we get

$$I_{1,1}^{D_1}(\alpha,\beta)=\int_{\mathbb{R}}w(t)\frac{1}{2i\pi}\int_{(\sigma)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)}{\zeta^{(N)}(2(1+\alpha+\beta+2s))}\,ds\,dt.$$

If $\alpha+\beta\neq0$, we specialise

$$G(s)=e^{s^2}\frac{(\alpha+\beta)^2-(2s)^2}{(\alpha+\beta)^2}$$

in order to ensure $G\left(-\frac{\alpha+\beta}{2}\right)=0$. We move the integration line from $\Re s=\sigma$ to $\Re s=-A$ with $A=1/4+(\alpha+\beta)/2$, crossing a pole at $s=0$. Thus,

$$I_{1,1}^{D_1}(\alpha,\beta)=\int_{\mathbb{R}}w(t)\,\mathrm{Res}_{s=0}\left[\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)}{\zeta^{(N)}(2(1+\alpha+\beta+2s))}\right]dt$$
$$+O(T^{1-2A}).\ ■$$

In order to calculate the residue at $s=0$ in the previous lemma, we split our proof according to the multiplicity of this pole.

Double pole case. In this subsection, we assume $\alpha+\beta=0$. Then the pole at $s=0$ in the previous lemma has multiplicity 2.

LEMMA 15. *We have*

$$\mathrm{Res}_{s=0}\left[\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)}{\zeta^{(N)}(2(1+\alpha+\beta+2s))}\right]=\frac{\mathfrak{a}_f}{2}\ln\left(\frac{t\sqrt{N}}{2\pi}\right)+\frac{\mathfrak{b}_f}{2}.$$

Proof. We compute the following asymptotic behaviour at $s=0$:

$$\begin{aligned}
&G(s)\left(\frac{t\sqrt{N}}{2\pi}\right)^{2s}\frac{L(f\times f,1+2s)}{\zeta^{(N)}(2(1+2s))}\\
&\quad=[1+O(s^2)]\left[1+2s\ln\left(\frac{t\sqrt{N}}{2\pi}\right)+O(s^2)\right]\left[\frac{\mathfrak{a}_f/2}{2s}+\mathfrak{b}_f/2+O(s)\right]\\
&\quad=\frac{\mathfrak{a}_f/2}{s}+\frac{\mathfrak{a}_f}{2}\ln\left(\frac{t\sqrt{N}}{2\pi}\right)+\frac{\mathfrak{b}_f}{2}+O(s).\ \blacksquare
\end{aligned}$$

These results prove Theorem 2 when $\alpha+\beta=0$. More precisely, by (37) and since $I_{1,1}^{D_1}(\alpha,\beta)=I_{1,1}^{D_2}(\alpha,\beta)$ in the case under consideration, the following corollary follows from Lemmas 14 and 15.

COROLLARY 6. *Let $\alpha\ll L^{-1}$ be a complex number. For all $\varepsilon>0$,*

$$M_{f,2}(\alpha,-\alpha)=a_f\int_{\mathbb{R}}w(t)\ln t\,dt+b_f\int_{\mathbb{R}}w(t)\,dt+O(T^{1/2+\theta+\varepsilon}).$$

Simple pole case. In this subsection, we assume that $\alpha+\beta\neq 0$ and $G(s)=e^{s^2\frac{(\alpha+\beta)^2-(2s)^2}{(\alpha+\beta)^2}}$. We also set $w_2(t)=w(t)\left(\frac{t\sqrt{N}}{2\pi}\right)^{-2(\alpha+\beta)}$.

LEMMA 16. *Let $\alpha,\beta\ll L^{-1}$ be complex numbers. For all $\varepsilon>0$, we have*

$$\begin{aligned}
M_{f,2}(\alpha,\beta)&=\frac{L(f\times f,1+\alpha+\beta)}{\zeta^{(N)}(2(1+\alpha+\beta))}\widehat{w}(0)+\frac{L(f\times f,1-\alpha-\beta)}{\zeta^{(N)}(2(1-\alpha-\beta))}\widehat{w}_2(0)\\
&\quad+O(T^{1/2+\theta+\varepsilon}).
\end{aligned}$$

Proof. Since the pole at $s=0$, which appears in Lemma 14, is simple, we have

$$\mathrm{Res}_{s=0}\left[\frac{G(s)}{s}g_{\alpha,\beta}(s,t)\frac{L(f\times f,1+\alpha+\beta+2s)}{\zeta^{(N)}(2(1+\alpha+\beta+2s))}\right]=\frac{L(f\times f,1+\alpha+\beta)}{\zeta^{(N)}(2(1+\alpha+\beta))}.$$

Then, by Lemma 14,

$$I_{1,1}^{D_1}(\alpha,\beta)=\frac{L(f\times f,1+\alpha+\beta)}{\zeta^{(N)}(2(1+\alpha+\beta))}\widehat{w}(0)+O(T^{1/2}).$$

Up to replacing w by w_2, we have $I_{1,1}^{D_2}(\alpha,\beta)=I_{1,1}^{D_1}(-\beta,-\alpha)$, thus

$$I_{1,1}^{D_2}(\alpha,\beta)=\frac{L(f\times f,1-\alpha-\beta)}{\zeta^{(N)}(2(1-\alpha-\beta))}\widehat{w}_2(0)+O(T^{1/2}).\ \blacksquare$$

As a result, we obtain Theorem 2 when $\alpha+\beta\neq 0$.

COROLLARY 7. *Let $\alpha, \beta \ll L^{-1}$ be complex numbers. For all $\varepsilon > 0$, we have*

$$M_{f,2}(\alpha,\beta) = \mathfrak{a}_f \int_{-\infty}^{\infty} w(t) \ln t \, dt + \left[\mathfrak{b}_f + \mathfrak{a}_f \ln\left(\frac{\sqrt{N}}{2\pi}\right)\right] \widehat{w}(0) + O\big(|\alpha+\beta| T (\ln T)^2 + T^{1/2+\theta+\varepsilon}\big).$$

Proof. Thanks to Lemma 13, the previous lemma gives

$$M_{f,2}(\alpha,\beta) = \frac{\mathfrak{a}_f/2}{\alpha+\beta}(\widehat{w}(0) - \widehat{w}_2(0)) + \frac{\mathfrak{b}_f}{2}(\widehat{w}(0) + \widehat{w}_2(0)) + O(T|\alpha+\beta| + T^{1/2+\theta+\varepsilon}).$$

Furthermore,

$$\begin{aligned}
\widehat{w}_2(0) &= \int_{\mathbb{R}} w(t) \left(\frac{t\sqrt{N}}{2\pi}\right)^{-2(\alpha+\beta)} dt \\
&= \int_{\mathbb{R}} w(t)\left(1 - 2(\alpha+\beta)\ln\left(\frac{t\sqrt{N}}{2\pi}\right) + O(|\alpha+\beta|^2 \ln^2 t)\right) \\
&= \widehat{w}(0) - 2(\alpha+\beta)\int_{\mathbb{R}} w(t) \ln\left(\frac{t\sqrt{N}}{2\pi}\right) dt + O(|\alpha+\beta|^2 T \ln^2 T).
\end{aligned}$$

An easy calculation gives the result. ∎

We remark that, up to replacing $\Delta = T/\ln T$ by $T/(\ln T)^2$, we obtain Corollary 3.

5.2. Conjecture of Conrey, Farmer, Keating, Rubinstein and Snaith. We can find in [CFK$^+$05] numerous conjectures related to integral moments of L-functions. In particular, Conjecture 2.5.4 predicts the asymptotic behaviour of any even integral moment of a primitive L-function on the critical line.

CONJECTURE 1. *Let $\mathcal{L}(s)$ be a primitive L-function. Let k be a positive integer. Then for any "suitable" weight function g, we have*

$$\int_{-\infty}^{\infty} |\mathcal{L}(1/2+it)|^{2k} g(t)\, dt = \int_{-\infty}^{\infty} P_k\big(w \ln(Q^{2/w} t/2)\big)(1 + O(t^{-1/2+\varepsilon}))g(t)\, dt$$

where w and Q are respectively the degree and the conductor of $\mathcal{L}$, and P_k is an explicit polynomial of degree k^2.

In this paper, we consider L-functions of holomorphic primitive cusp forms of even weight, square-free level N and trivial character. The degree of such an L-function is $w = 2$ and the conductor is $Q = \sqrt{N}/\pi$ (cf. (7)). The following conjecture is a simple rewriting of Conjecture 1 in this case for the second integral moment and when $g(t) = r(t/T)$ with r a smooth function compactly supported in $[1, 2]$.

Conjecture 2. *Let f be a holomorphic primitive cusp forms of even weight, square-free level N and trivial character. Then, for any $\varepsilon > 0$, we have*

$$\int_{-\infty}^{\infty} |L(f, 1/2 + it)|^2 g(t)\, dt = \int_{-\infty}^{\infty} P_1\left(2\ln\left(\frac{t\sqrt{N}}{2\pi}\right)\right) g(t)\, dt + O(T^{1/2+\varepsilon})$$

with

$$P_1(x) = \frac{-1}{(2i\pi)^2} \oint_{|z_1|=r_1} \oint_{|z_2|=r_2} \frac{L(f \times f, 1 + z_1 - z_2)}{\zeta^{(N)}(2(1 + z_1 - z_2))} \frac{(z_2 - z_1)^2}{z_1^2 z_2^2} e^{\frac{x}{2}(z_1 - z_2)}\, dz_1\, dz_2$$

for any small positive real numbers r_1 and r_2 (namely $r_1 + r_2 < 1$).

In order to compare our Corollary 2 with this conjecture, we have to compute P_1. Choosing $r_1 \neq r_2$, and since $\frac{(z_2-z_1)^2}{z_1^2 z_2^2} = \frac{1}{z_1^2} - \frac{2}{z_1 z_2} + \frac{1}{z_2^2}$, we have

$$\begin{aligned} P_1(x) &= \frac{2}{(2i\pi)^2} \oint_{|z_1|=r_1} \oint_{|z_2|=r_2} \frac{L(f \times f, 1 + z_1 - z_2)}{\zeta^{(N)}(2(1 + z_1 - z_2))} \frac{1}{z_1 z_2} e^{\frac{x}{2}(z_1 - z_2)}\, dz_1\, dz_2 \\ &= \frac{2}{2i\pi} \oint_{|z_2|=r_2} \frac{L(f \times f, 1 - z_2)}{\zeta^{(N)}(2(1 - z_2))} e^{-\frac{x}{2} z_2} \frac{dz_2}{z_2}. \end{aligned}$$

Moreover, since

$$\begin{aligned} \frac{L(f \times f, 1 - z_2)}{\zeta^{(N)}(2(1 - z_2))} e^{-\frac{x}{2} z_2} &= \left(\frac{-\mathfrak{a}_f/2}{z_2} + \mathfrak{b}_f + O(z_2^2)\right)\left(1 - \frac{x}{2} z_2 + O(z_2^2)\right) \\ &= \frac{-\mathfrak{a}_f/2}{z_2} + \left(\frac{\mathfrak{b}_f}{2} + \frac{x\mathfrak{a}_f}{4}\right) + O(z_2^2), \end{aligned}$$

we obtain

$$P_1(x) = \frac{\mathfrak{a}_f}{2} x + \mathfrak{b}_f.$$

To conclude, we can see that the main terms in Corollary 2 and Conjecture 2 are similar and, assuming the Ramanujan–Petersson conjecture, the error terms are also equal.

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