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Transcendence results on the generating functions of the characteristic functions of certain self-generating sets, II

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1. Introduction and results. A few years ago, Dilcher and Stolarsky [4, Definition 5.1] introduced the two power series

$$\begin{aligned} F(z) &:= 1 + z + z^2 + z^5 + z^6 + z^8 + z^9 + z^{10} + z^{21} + z^{22} + \cdots, \\ G(z) &:= 1 + z + z^3 + z^4 + z^5 + z^{11} + z^{12} + z^{13} + z^{16} + z^{17} + \cdots, \end{aligned} \tag{1.1}$$

with coefficients 0 and 1 only, defining holomorphic functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here the infinite sequences of integers appearing in the exponents are examples of so-called self-generating sequences.

The main aim of the present paper is to study the transcendence degree of the field $\mathbb{C}(z)(F(z), F(z^4), G(z), G(z^4))$ over $\mathbb{C}(z)$, and the analogous question over $\mathbb{Q}$ if the variable z is specialized to non-zero algebraic points α in $\mathbb{D}$. By virtue of the relation

$$F(z)G(z^4) - zF(z^4)G(z) = 1, \tag{1.2}$$

which is formula (5.8) in [4], we immediately obtain the right-hand inequality in

$$2 \le \operatorname{trdeg}_{\mathbb{C}(z)} \mathbb{C}(z)(F(z), F(z^4), G(z), G(z^4)) \le 3.$$

The left-hand inequality is a consequence of the algebraic independence over $\mathbb{C}(z)$ of the functions $G(z)$ and $G(z^4)$ proved first by Adamczewski [1], or of that of $F(z)$ and $F(z^4)$ obtained in [3, Theorem 1.11].

As our main result, we shall subsequently establish the following sharpening of the algebraic independence results just quoted.

THEOREM 1.1. *The functions $F(z), G(z), G(z^4)$ are algebraically independent over $\mathbb{C}(z)$.*

Clearly, by (1.2), the same statement holds if we interchange F and G. Also, equivalent to Theorem 1.1 is the following symmetrical formulation.

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Theorem 1.2. *One has*

$$\operatorname{trdeg}_{\mathbb{C}(z)} \mathbb{C}(z)(F(z), F(z^4), G(z), G(z^4)) = 3.$$

Combining this second formulation with a suitable algebraic independence criterion, we can demonstrate the following arithmetical application of our main result.

Theorem 1.3. *For any non-zero algebraic $\alpha \in \mathbb{D}$,*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(F(\alpha), F(\alpha^4), G(\alpha), G(\alpha^4)) = 3.$$

This theorem together with (1.2) implies that, for every non-zero algebraic $\alpha \in \mathbb{D}$, any three of the four numbers $F(\alpha), F(\alpha^4), G(\alpha), G(\alpha^4)$ are algebraically independent. Moreover, it simultaneously improves on Adamczewski's result in [1, Proposition 3.1] that, for the same α's, the numbers $G(\alpha)$ and $G(\alpha^4)$ are algebraically independent, as well as on our F-analogue in [3, Theorem 1.12].

Our paper is organized in such a way that we first prove Theorem 1.3 in Sec. 2. Next, in Sec. 3 we provide several preliminary results for the proper proof of Theorem 1.1 in Sec. 4. This proof uses an extension to *three* functions of the elementary (or *poor man's*) method we already applied in [2, Theorem 1.3] and in [3, Theorem 1.11] to show the algebraic independence of *two* functions over $\mathbb{C}(z)$. In the case of the function pairs $F(z), F(z^4)$, or $G(z), G(z^4)$, one could use instead (as done in [1] for the second pair) the procedure of Nishioka [5, Theorem 5.2] which is limited to only two functions.

To conclude this section, we briefly comment on some of the questions we asked in [3, Problem 1.15]. Clearly, the algebraic independence of $F(z)$ and $G(z)$ over $\mathbb{C}(z)$ is contained in our Theorem 1.1, and its arithmetical analogue in Theorem 1.3. Nevertheless, one may be interested in direct proofs, i.e., without detour via algebraic independence of more than two objects as in the above theorems.

2. Proof of Theorem 1.3. This proof essentially depends on an algebraic independence criterion of Nishioka [5, Theorem 4.2.1] from whose formulation we quote here only the homogeneous version.

Lemma 2.1. *Let K denote an algebraic number field, and let $t \in \mathbb{Z}_{\geq 2}$. Suppose that $f_1, \ldots, f_m \in K[[z]]$ converge in some disc $U \subset \mathbb{D}$ about the origin, where they satisfy the matrix functional equation*

$$^{\tau}(f_1(z^t), \ldots, f_m(z^t)) = \mathcal{A}(z) \cdot {}^{\tau}(f_1(z), \ldots, f_m(z))$$

with $\mathcal{A}(z) \in \operatorname{Mat}_{m,m}(K(z))$, τ denoting the matrix transpose. If α is a non-zero algebraic number in U such that none of the α^{t^j} $(j = 0, 1, \ldots)$ is a pole of the entries of $\mathcal{A}(z)$, then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(f_1(\alpha), \ldots, f_m(\alpha)) \geq \operatorname{trdeg}_{K(z)} K(z)(f_1(z), \ldots, f_m(z)). \tag{2.1}$$

Proof of Theorem 1.3. According to [4, Proposition 5.1], the functions F and G satisfy the system of linear homogeneous functional equations

$$\begin{aligned} F(z) &= p(z)F(z^4) - z^4F(z^{16}), \\ zG(z) &= p(z)G(z^4) - G(z^{16}) \end{aligned} \tag{2.2}$$

in $\mathbb{D}$ with

$$p(z) := 1 + z + z^2. \tag{2.3}$$

Set $F(z^4) =: I(z)$, $G(z^4) =: H(z)$. Then (2.2) is equivalent to the matrix functional equation

$$^\tau(F(z^4), G(z^4), H(z^4), I(z^4)) = \mathcal{A}(z) \cdot {}^\tau(F(z), G(z), H(z), I(z))$$

with

$$\mathcal{A}(z) := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -z & p(z) & 0 \\ -z^{-4} & 0 & 0 & z^{-4}p(z) \end{pmatrix}.$$

Thus, we may apply Lemma 2.1 with $K = \mathbb{Q}$, $m = 4$, $(f_1, f_2, f_3, f_4) = (F, G, H, I)$, $t = 4$, $U = \mathbb{D}$ to obtain, for every non-zero algebraic $\alpha \in \mathbb{D}$,

$$\begin{aligned} &\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(F(\alpha), F(\alpha^4), G(\alpha), G(\alpha^4)) \\ &\qquad\qquad \geq \operatorname{trdeg}_{\mathbb{Q}(z)} \mathbb{Q}(z)(F(z), F(z^4), G(z), G(z^4)). \end{aligned} \tag{2.4}$$

This concludes our proof of Theorem 1.3 if we observe first that, according to (1.2), the left-hand side of (2.4) cannot exceed 3, while the right-hand side remains unchanged if we replace $\mathbb{Q}(z)$ by $\mathbb{C}(z)$ in both places, and subsequently use Theorem 1.2. ■

3. Some preliminaries to the proof of Theorem 1.1. Recall that, by the definition of $H(z)$ after (2.3), the second equation in (2.2), and equation (1.2), we have

$$\begin{aligned} &G(z^4) = H(z), \qquad H(z^4) = -zG(z) + p(z)H(z), \\ &F(z^4) = \frac{H(z)F(z) - 1}{zG(z)} \end{aligned} \tag{3.1}$$

with $p(z)$ defined in (2.3). In this section, we consider the polynomial sequences $(a_m(z))_{m\geq -1}$ and $(b_m(z))_{m\geq -1}$ defined by

$$\begin{aligned} &a_{-1}(z) = 1, \qquad b_{-1}(z) = 0, \\ &a_{m+1}(z) = -zb_m(z^4), \quad b_{m+1}(z) = a_m(z^4) + p(z)b_m(z^4) \qquad (m \geq -1). \end{aligned} \tag{3.2}$$

We note that if $\ell(z) = a_m(z)G(z) + b_m(z)H(z)$, then $\ell(z^4) = a_{m+1}(z)G(z) + b_{m+1}(z)H(z)$, by (3.5). In fact, this is the motivation for the above definition. We also introduce the notation

$$\ell_m = a_m(z)x + b_m(z)y \in \mathbb{C}[z,x,y] \quad (m \geq -1), \tag{3.3}$$

in particular, $\ell_{-1} = x$ and $\ell_0 = y$. Moreover, we note that the substitution $z \to z^4$, $x \to y$, $y \to -zx + p(z)y$ in ℓ_m gives ℓ_{m+1}.

The following three lemmas will be useful.

Lemma 3.1. *We have* $\gcd(a_m(z), b_m(z)) = 1$ *for all* $m \geq -1$. *Moreover, the degrees of* $a_m(z)$ *and* $b_m(z)$ *are strictly increasing with* m (≥ 0).

Proof. The claim concerning the degrees follows immediately from (3.2). Further, z does not divide any $b_m(z)$ $(m \geq 0)$, and, by assuming

$$\gcd(a_m(z), b_m(z)) = 1,$$

we therefore obtain

$$\begin{aligned}\gcd(a_{m+1}(z), b_{m+1}(z)) &= \gcd\big(-zb_m(z^4), a_m(z^4) + p(z)b_m(z^4)\big)\\ &= \gcd(a_m(z^4), b_m(z^4)) = \gcd(a_m(z), b_m(z)) \quad (m \geq 0).\end{aligned}$$

Thus $\gcd(a_m(z), b_m(z)) = 1$ for all $m \geq -1$ by induction. ∎

Before formulating the next lemmas, we note that if $P(z,x,y) \in \mathbb{C}[z,x,y]$ is homogeneous in x, y and non-zero, then also $P(z^4, y, \ell_1) \neq 0$. Indeed, if

$$P(z,x,y) = \sum_{k=0}^{L} p_k(z)x^{L-k}y^k \neq 0,$$

then, after some computations,

$$\begin{aligned}P(z^4, y, \ell_1) &= \sum_{k=0}^{L}\sum_{j=0}^{k}(-1)^j\binom{k}{j}p_k(z^4)z^j p(z)^{k-j}x^j y^{L-j}\\ &= \sum_{k=0}^{L}\bigg(\sum_{j=0}^{k}(-1)^{L-k}\binom{L-k+j}{L-k}p_{L-k+j}(z^4)z^{L-k}p(z)^j\bigg)x^{L-k}y^k,\end{aligned}$$

and so the assumption $P(z^4, y, \ell_1) = 0$ implies $p_L(z^4) = p_{L-1}(z^4) = \cdots = p_0(z^4) = 0$, leading to $P(z,x,y) = 0$, a contradiction. Thus $P(z^4, y, \ell_1) \neq 0$.

Lemma 3.2. *Assume that the polynomial* $P(z,x,y) \in \mathbb{C}[z,x,y] \setminus \{0\}$ *is homogeneous in* x, y. *Then* $\ell_m \mid P(z,x,y)$ *if and only if* $\ell_{m+1} \mid P(z^4, y, \ell_1)$.

Proof. If $P(z,x,y) = \ell_m P_1(z,x,y)$, then $P(z^4, y, \ell_1) = \ell_{m+1}P_1(z^4, y, \ell_1)$ as noted above. Thus, each factor ℓ_m of $P(z,x,y)$ produces a factor ℓ_{m+1} to $P(z^4, y, \ell_1)$. Assume now that $P(z,x,y)$ does not have a factor ℓ_m and $P(z^4, y, \ell_1) = \ell_{m+1}P_2(z,x,y)$ with some $P_2(z,x,y) \in \mathbb{C}[z,x,y]$.

Let

$$P(z,x,y) = \sum_{j=0}^{N} p_j(z)x^j y^{N-j} \quad \text{with all } p_j(z) \in \mathbb{C}[z].$$

If x is not a factor of $P(z,x,y)$, then $p_0(z) \neq 0$. But this means that in

$$P(z^4, y, \ell_1) = \sum_{j=0}^{N} p_j(z^4) y^j \bigl(-zx + p(z)y\bigr)^{N-j}$$

the term x^N has a non-zero coefficient, and so y does not divide $P(z^4, y, \ell_1)$. Thus $m = -1$ leads to a contradiction. Further, if y is not a factor of $P(z,x,y)$, then $p_N(z) \neq 0$, and this means that all terms in the above sum except $p_N(z^4)y^N \neq 0$ are divisible by ℓ_1. So we have a contradiction also if $m = 0$.

We may now assume that $m \geq 1$ and

$$P(z,x,y) = x^t y^u \sum_{j=0}^{M} q_j(z) x^j y^{M-j},$$

where $q_j(z) \in \mathbb{C}[z]$ satisfy $q_0(z) q_M(z) \neq 0$ for $M = N - t - u$. Since we assume that $\ell_{m+1} \mid P(z^4, y, \ell_1)$, we necessarily have $M > 0$. We may now write

$$P(z,x,y) = x^t y^u q_M(z) \prod_{j=1}^{M} (x + A_j(z) y),$$

where the A_j's are non-zero algebraic functions. Since ℓ_m is not a factor of $P(z,x,y)$, we have

$$A_j(z) \neq \frac{b_m(z)}{a_m(z)} \qquad (j = 1, \ldots, M). \tag{3.4}$$

Now we get

$$P(z^4, y, \ell_1) = y^t \ell_1^u q_M(z^4) \Bigl(\prod_{j=1}^{M} -zA_j(z^4) \Bigr) \prod_{j=1}^{M} \Bigl(x - \Bigl(\frac{p(z)}{z} + \frac{1}{zA_j(z^4)} \Bigr) y \Bigr).$$

Since $\ell_{m+1} \mid P(z^4, y, \ell_1)$, there must exist an index j such that

$$x - \Bigl(\frac{p(z)}{z} + \frac{1}{zA_j(z^4)} \Bigr) y = x + \frac{b_{m+1}(z)}{a_{m+1}(z)} y = x - \Bigl(\frac{p(z)}{z} + \frac{a_m(z^4)}{zb_m(z^4)} \Bigr) y,$$

where we used definition (3.2). This gives $A_j(z^4) = b_m(z^4)/a_m(z^4)$ and therefore $A_j(z) = b_m(z)/a_m(z)$, contradicting (3.4). ∎

The following result is a consequence of the proof of [3, Theorem 1.11] as we explain below.

LEMMA 3.3. *Assume that $P(z,x,y) \in \mathbb{C}[z,x,y] \setminus \{0\}$ is homogeneous in x, y and*

$$P(z,x,y) = \sum_{k=0}^{L} p_k(z) x^{L-k} y^k,$$

where $\gcd(p_0(z),\dots,p_L(z))=1$. *If* $L\ge 1$, *then the equality*

$$s(z)P(z,x,y)=P(z^4,y,\ell_1)$$

with a polynomial $s(z)$ *is not possible.*

Proof. A sketch of this proof is given at the end of [3]. Namely, the stated equality is equivalent to

$$s(z)p_k(z)=\sum_{j=0}^{k}(-1)^{L-k}\binom{L-k+j}{L-k}p_{L-k+j}(z^4)z^{L-k}p(z)^j \qquad (k=0,\dots,L), \tag{3.5}$$

as we saw before Lemma 3.2. By using these equalities and our assumption on the coprimality of the p_k's, we get the divisibility relation

$$s(z)\mid z^L$$

– see the *Sketch of the proof of the G-analogue of Theorem 1.11* at the end of [3]. This leads to a contradiction as in the proof of [3, Theorem 1.11] if we just replace equation (5.9) there by our (3.5) and follow the proof there. ■

4. Proof of Theorem 1.1. Assume the contrary of Theorem 1.1, namely that the functions $G(z)$, $H(z)$ $(=G(z^4))$, and $F(z)$ are algebraically dependent over $\mathbb{C}(z)$. We shall deduce a contradiction in four steps.

STEP 1: *basic construction.* By our assumption, there exists an irreducible polynomial $P(z,x,y,w)\in\mathbb{C}[z,x,y,w]$ such that $P(z,G(z),H(z),F(z))=0$. Let

$$P(z,x,y,w)=\sum_{k=0}^{U}P_k(z,x,y)w^k \quad \text{ with } P_U(z,x,y)\ne 0.$$

Since $G(z)$ and $H(z)$ are algebraically independent over $\mathbb{C}(z)$, we have $U\ge 1$. Moreover, it follows from the transcendence of $F(z)$ that $\deg_{x,y}P_k(z,x,y)\ge 1$ for at least one k. We now use (3.1) to obtain

$$\begin{aligned}
&P(z^4,G(z^4),H(z^4),F(z^4))\\
&\ = P\Big(z^4,H(z),-zG(z)+p(z)H(z),\frac{H(z)F(z)-1}{zG(z)}\Big)\\
&\ = (zG(z))^{-U}\\
&\quad \times\sum_{k=0}^{U}P_k\big(z^4,H(z),-zG(z)+p(z)H(z)\big)(zG(z))^{U-k}(H(z)F(z)-1)^k=0.
\end{aligned}$$

This yields another polynomial $Q(z,x,y,w)$ with $Q(z,G(z),H(z),F(z))=0$, namely

$$\begin{aligned}Q(z,x,y,w) &:= \sum_{k=0}^{U} P_k\big(z^4,y,-zx+p(z)y\big)(zx)^{U-k}(yw-1)^k\\ &= \sum_{k=0}^{U}\Bigg(\sum_{j=0}^{U-k}(-1)^{U-k-j}\binom{U-j}{k}y^k(zx)^jP_{U-j}\big(z^4,y,-zx+p(z)y\big)\Bigg)w^k\\ &=: \sum_{k=0}^{U} Q_k(z,x,y)w^k.\end{aligned}$$

Note here that $Q_U(z,x,y) = y^U P_U(z^4,y,\ell_1) \neq 0$ by the remark before Lemma 3.2. Thus, there exists a polynomial $S(z,x,y) \neq 0$ such that

$$S(z,x,y)P(z,x,y,w) = Q(z,x,y,w);$$

in particular,

$$S(z,x,y)P_U(z,x,y) = y^U P_U(z^4,y,\ell_1) \tag{4.1}$$

and

$$S(z,x,y)P_{U-1}(z,x,y) = -Uy^{U-1}P_U(z^4,y,\ell_1) + zxy^{U-1}P_{U-1}(z^4,y,\ell_1). \tag{4.2}$$

By these two equations, we see that $\deg_{x,y} S(z,x,y) = U$ and

$$S(z,x,y)\big(yP_{U-1}(z,x,y) + UP_U(z,x,y)\big) = zxy^U P_{U-1}(z^4,y,\ell_1). \tag{4.3}$$

Step 2: *equation (4.1).* To study (4.1) in more detail, we separate the homogeneous terms in $S(z,x,y)$ and $P_U(z,x,y)$. So let

$$S(z,x,y) = \sum_{j=0}^{U} S_j(z,x,y), \qquad P_U(z,x,y) = \sum_{j=0}^{V} P_{U,j}(z,x,y),$$

where $S_j(z,x,y)$ and $P_{U,j}(z,x,y)$ are homogeneous of degree j in x,y (or vanish), and $S_U(z,x,y) \neq 0$, $P_{U,V}(z,x,y) \neq 0$. Then (4.1) gives

$$S_U(z,x,y)P_{U,V}(z,x,y) = y^U P_{U,V}(z^4,y,\ell_1). \tag{4.4}$$

We may now write

$$S_U(z,x,y) = s(z)y^{j_0}\sum_{j=0}^{U-j_0} s_j(z)x^j y^{U-j_0-j} =: s(z)y^{j_0}S_{U,0}(z,x,y),$$

$$P_{U,V}(z,x,y) = \tilde{p}(z)y^{k_0}\sum_{j=0}^{V-k_0} p_j(z)x^j y^{V-k_0-j} =: \tilde{p}(z)y^{k_0}P_{U,V,0}(z,x,y),$$

where j_0, k_0 are non-negative integers with $j_0 + k_0 = U$, and $s(z)$, $\tilde{p}(z), s_j(z), p_j(z)$ are polynomials with $\gcd(s_0(z), \ldots, s_{U-j_0}(z)) = 1$ and $\gcd(p_0(z), \ldots, p_{V-k_0}(z)) = 1$. Then (4.4) implies

$$s(z)S_{U,0}(z,x,y)\tilde{p}(z)P_{U,V,0}(z,x,y) = \tilde{p}(z^4)\ell_1^{k_0}P_{U,V,0}(z^4,y,\ell_1). \tag{4.5}$$

If $k_0 = 0$, then $j_0 = U$ and $S_{U,0}(z,x,y) = 1$, hence (4.5) has the form

$$s(z)\tilde{p}(z)P_{U,V,0}(z,x,y) = \tilde{p}(z^4)P_{U,V,0}(z^4,y,\ell_1). \tag{4.6}$$

Since $\gcd(p_0(z), \ldots, p_{V-k_0}(z)) = 1$, this gives $\tilde{p}(z^4) \mid s(z)\tilde{p}(z)$, say $s(z)\tilde{p}(z) = S(z)\tilde{p}(z^4)$ with $S(z) \in \mathbb{C}[z] \setminus \{0\}$, whence

$$S(z)P_{U,V,0}(z,x,y) = P_{U,V,0}(z^4,y,\ell_1).$$

By Lemma 3.3, this is possible only if $V = 0$ and $P_{U,V,0}(z,x,y) = 1$. Thus, in the case $k_0 = 0$, we have

$$S_U(z,x,y) = s(z)y^U, \quad P_{U,V}(z,x,y) = \tilde{p}(z), \quad s(z)\tilde{p}(z) = \tilde{p}(z^4). \tag{4.7}$$

In the case $k_0 \geq 1$, from (4.5) we obtain

$$S_{U,0}(z,x,y) = \ell_1^{j_1}S_{U,1}(z,x,y), \quad P_{U,V,0}(z,x,y) = \ell_1^{k_1}P_{U,V,1}(z,x,y),$$

where $j_1 + k_1 = k_0$ ($k_1 = 0$ if $V = k_0$); note here that $\gcd(-z, p(z)) = 1$ for the $p(z)$ from (2.3). By (4.5),

$$s(z)S_{U,1}(z,x,y)\tilde{p}(z)P_{U,V,1}(z,x,y) = \tilde{p}(z^4)\ell_2^{k_1}P_{U,V,1}(z^4,y,\ell_1). \tag{4.8}$$

If $k_1 = 0$, then $j_1 = k_0 = U - j_0$ and $S_{U,1}(z,x,y) = 1$. If $k_1 \geq 1$, we may continue in the same way to get a sequence of positive integers $k_1 \geq \cdots \geq k_{m-1}$ such that if $j_2 = k_1 - k_2, \ldots, j_{m-1} = k_{m-2} - k_{m-1}$, $j_m = k_{m-1}$ ($= k_{m-1} - k_m$, where $k_m = 0$), then

$$S_{U,1}(z,x,y) = \ell_2^{j_2}\cdots\ell_m^{j_m}, \quad P_{U,V,1}(z,x,y) = \ell_2^{k_2}\cdots\ell_{m-1}^{k_{m-1}}P_{U,V,m-1}(z,x,y).$$

This holds since $U = j_0 + k_0 = j_0 + j_1 + k_1 = \cdots = j_0 + j_1 + \cdots + j_{m-1} + j_m$ and $\gcd(a_m(z), b_m(z)) = 1$ for all m, by Lemma 3.1. Now (4.8) implies

$$s(z)\tilde{p}(z)P_{U,V,m-1}(z,x,y) = \tilde{p}(z^4)P_{U,V,m-1}(z^4,y,\ell_1),$$

but this is analogous to (4.6), and therefore Lemma 3.3 gives a contradiction unless $V - k_0 - \cdots - k_{m-1} = 0$, and in this case $P_{U,V,m-1}(z,x,y) = 1$. Thus, we must have

(4.9)
$$S_U(z,x,y) = s(z)y^{j_0}\prod_{i=1}^{m}\ell_i^{j_i}, \ P_{U,V}(z,x,y) = \tilde{p}(z)y^{k_0}\prod_{i=1}^{m-1}\ell_i^{k_i}, \ s(z)\tilde{p}(z) = \tilde{p}(z^4),$$

where $j_0 + \cdots + j_m = U$, $k_0 + \cdots + k_{m-1} = V$. Next we shall use these representations with (4.2). From (4.2) the relation $P_{U-1}(z,x,y) \neq 0$ is obvious. Let us denote $\deg_{x,y} P_{U-1}(z,x,y) =: N$. If $N < V - 1$, then (4.2) is impossible, whence $N \geq V - 1$ holds, and we first deal with the case $N \geq V$.

STEP 3: *equation (4.2), case $N \geq V$.* Note that in (4.7), $V = 0$, and therefore the above condition $N \geq V$ holds in this case. Let

$$\widetilde{Q}_N(z,x,y) = q(z)\sum_{j=0}^{N} q_j(z)x^j y^{N-j} =: q(z)Q_{N,-2}(z,x,y)$$

be the highest homogeneous term in $P_{U-1}(z,x,y)$, where $q(z)$ and the $q_j(z)$ are polynomials with $\gcd(q_0(z),\dots,q_N(z)) = 1$. Then (4.2) gives

$$S_U(z,x,y)\widetilde{Q}_N(z,x,y) = zxy^{U-1}\widetilde{Q}_N(z^4,y,\ell_1).$$

The above equation and (4.9) mean that x is a factor of $Q_{N,-2}(z,x,y)$, and so we may write $Q_{N,-2}(z,x,y) = xQ_{N,-1}(z,x,y)$ with a polynomial $Q_{N,-1}(z,x,y)$, whence

$$s(z)y^{j_0}\Big(\prod_{i=1}^{m}\ell_i^{j_i}\Big)q(z)Q_{N,-1}(z,x,y) = zq(z^4)y^U Q_{N,-1}(z^4,y,\ell_1).$$

Since $U - j_0 = k_0$, we get $Q_{N,-1}(z,x,y) = y^{k_0}Q_{N,0}(z,x,y)$, and therefore

$$s(z)\Big(\prod_{i=1}^{m}\ell_i^{j_i}\Big)q(z)Q_{N,0}(z,x,y) = zq(z^4)\ell_1^{k_0}Q_{N,0}(z^4,y,\ell_1).$$

By using the fact $j_1+k_1=k_0$, we may now write $Q_{N,0}(z,x,y)=\ell_1^{k_1}Q_{N,1}(z,x,y)$ and

$$s(z)\Big(\prod_{i=2}^{m}\ell_i^{j_i}\Big)q(z)Q_{N,1}(z,x,y) = zq(z^4)\ell_2^{k_1}Q_{N,1}(z^4,y,\ell_1).$$

Repeating this (and recalling $j_m = k_{m-1}$) we come to the equation

$$s(z)q(z)Q_{N,m-1}(z,x,y) = zq(z^4)Q_{N,m-1}(z^4,y,\ell_1).$$

As in the case of (4.6), we now apply Lemma 3.3 to obtain $N = 1 + k_0 + \cdots + k_{m-1} = V + 1$ and $Q_{N,m-1}(z,x,y) = 1$. Therefore $s(z)q(z) = zq(z^4)$ and, by (4.7) or (4.9), we have $s(z)\tilde{p}(z) = \tilde{p}(z^4)$. If we denote, for a moment, $\deg s(z) = d$, $\deg \tilde{p}(z) = \tilde{p}$, and $\deg q(z) = q$, then $d + \tilde{p} = 4\tilde{p}$ and $d + q = 1 + 4q$. This implies $3(\tilde{p} - q) = 1$, a contradiction.

STEP 4: *case $N = V - 1$.* In this case, by (4.2) and (4.9),

$$s(z)y^{j_0}\Big(\prod_{i=1}^{m}\ell_i^{j_i}\Big)\widetilde{Q}_N(z,x,y) + Uy^{U-1}\tilde{p}(z^4)\prod_{i=1}^{m}\ell_i^{k_{i-1}} = zxy^{U-1}\widetilde{Q}_N(z^4,y,\ell_1).$$

Therefore $\widetilde{Q}_N(z,x,y) = y^{n_0}Q^*_{N,0}(z,x,y)$, where $n_0 \geq U - 1 - j_0 = k_0 - 1$ and y is not a factor of $Q^*_{N,0}(z,x,y)$. Thus

(4.10)
$$s(z)y^{n_0-k_0+1}\Big(\prod_{i=1}^{m}\ell_i^{j_i}\Big)Q^*_{N,0}(z,x,y) + U\tilde{p}(z^4)\prod_{i=1}^{m}\ell_i^{k_{i-1}} = zx\ell_1^{n_0}Q^*_{N,0}(z^4,y,\ell_1).$$

Let us assume first that $n_0 = k_0 + J$, $J \geq 0$. Then $\deg_{x,y} Q^*_{N,0}(z,x,y) = N - k_0 - J = V - k_0 - 1 - J = k_1 + \cdots + k_{m-1} - 1 - J$. By $j_1 + k_1 = k_0$, (4.10) implies that $Q^*_{N,0}(z,x,y) = \ell_1^{k_1} Q^*_{N,1}(z,x,y)$, which is possible only if $k_2 + \cdots + k_{m-1} - 1 - J \geq 0$, otherwise we have a contradiction. Substituting this to (4.10) we obtain

$$s(z)y^{1+J}\Big(\prod_{i=2}^{m} \ell_i^{j_i}\Big) Q^*_{N,1}(z,x,y) + U\tilde{p}(z^4)\prod_{i=2}^{m} \ell_i^{k_{i-1}} = zx\ell_1^J \ell_2^{k_1} Q^*_{N,1}(z^4,y,\ell_1).$$

Let t be the greatest index such that $k_t + \cdots + k_{m-1} - 1 - J \geq 0$ (then certainly $t \leq m-1$). By continuing as above, we get

$$Q^*_{N,1}(z,x,y) = \Big(\prod_{i=2}^{t-1} \ell_i^{k_i}\Big) Q^*_{N,t-1}(z,x,y),$$
$$\deg_{x,y} Q^*_{N,t-1}(z,x,y) = k_t + \cdots + k_{m-1} - 1 - J < k_t,$$

and

$$s(z)y^{1+J}\Big(\prod_{i=t}^{m} \ell_i^{j_i}\Big) Q^*_{N,t-1}(z,x,y) + U\tilde{p}(z^4)\prod_{i=t}^{m} \ell_i^{k_{i-1}} = zx\ell_1^J \ell_t^{k_{t-1}} Q^*_{N,t-1}(z^4,y,\ell_1).$$

Thus $\ell_t^{k_t}$ must divide $Q^*_{N,t-1}(z,x,y)$. But this is impossible since we have $\deg_{x,y} Q^*_{N,1}(z,x,y) < k_t$. Thus, the assumption $n_0 \geq k_0$ gives a contradiction.

Our final task is to prove that also the case $n_0 = k_0 - 1$ leads to a contradiction. In this case, (4.10) has the form

$$s(z)\Big(\prod_{i=1}^{m} \ell_i^{j_i}\Big) Q^*_{N,0}(z,x,y) + U\tilde{p}(z^4)\prod_{i=1}^{m} \ell_i^{k_{i-1}} = zx\ell_1^{k_0-1} Q^*_{N,0}(z^4,y,\ell_1), \tag{4.11}$$

where y is not a factor of $Q^*_{N,0}(z,x,y)$. By Lemma 3.2, we know that ℓ_1 is not a factor of $Q^*_{N,0}(z^4,y,\ell_1)$. Thus $Q^*_{N,0}(z,x,y) = \ell_1^{k_1-1}\overline{Q}_{N,1}(z,x,y)$, where $\overline{Q}_{N,1}(z,x,y)$ is not divisible by ℓ_1. By (4.11),

$$s(z)\Big(\prod_{i=2}^{m} \ell_i^{j_i}\Big) \overline{Q}_{N,1}(z,x,y) + U\tilde{p}(z^4)\ell_1\prod_{i=2}^{m} \ell_i^{k_{i-1}} = zx\ell_2^{k_1-1} \overline{Q}_{N,1}(z^4,y,\ell_1),$$

where ℓ_2 is not a factor of $\overline{Q}_{N,1}(z^4,y,\ell_1)$ by Lemma 3.2. Repeating this argument we obtain

$$\overline{Q}_{N,1}(z,x,y) = \Big(\prod_{i=2}^{m-1} \ell_i^{k_i-1}\Big) \overline{Q}_{N,m-1}(z,x,y),$$

where the polynomial $\overline{Q}_{N,m-1}(z,x,y)$ is not divisible by ℓ_{m-1}. By substituting this into the above equation we have

$$s(z)\ell_m^{j_m}\overline{Q}_{N,m-1}(z,x,y) + U\tilde{p}(z^4)\ell_1\cdots\ell_{m-1}\ell_m^{k_{m-1}} = zx\ell_m^{k_{m-1}-1}\overline{Q}_{N,m-1}(z^4,y,\ell_1).$$

This is a contradiction since $j_m = k_{m-1}$, and Lemma 3.2 says that ℓ_m is not a factor of $\overline{Q}_{N,m-1}(z^4,y,\ell_1)$. Thus, Theorem 1.1 is proved. ∎

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