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The moduli space of totally marked degree two rational maps

by

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1. Introduction. A rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree two over a field k is given by a pair of homogeneous polynomials

$$\phi = [\phi_0, \phi_1] = [aX^2 + bXY + cY^2, dX^2 + eXY + fY^2]$$

such that ϕ_0, ϕ_1 have no common roots. In non-homogeneous form, ϕ may be expressed as

$$\phi(z) = \frac{az^2 + bz + c}{dz^2 + ez + f}.$$

Let $\phi_0(z) = az^2 + bz + c$ and $\phi_1(z) = dz^2 + ez + f$. We define $\operatorname{Res}(\phi)$, the *resultant* of ϕ, as the product

$$\prod_{(\alpha,\beta):\, \phi_0(\alpha)=\phi_1(\beta)=0} (\alpha - \beta).$$

The condition that ϕ_0, ϕ_1 have no common roots is equivalent to $\operatorname{Res}(\phi) \neq 0$.

Let Rat_2 denote the space of degree two rational maps $\mathbb{P}^1 \to \mathbb{P}^1$. The special linear group SL_2 acts via conjugation on Rat_2: for $f \in \mathrm{SL}_2$ and $\phi \in \mathrm{Rat}_2$, $f \cdot \phi = f \circ \phi \circ f^{-1}$. The moduli space $\mathrm{Rat}_2/\mathrm{SL}_2$, denoted M_2, arises naturally in the study of dynamical systems on $\mathbb{P}^1$. Over the complex numbers Milnor [2] proved that $\mathrm{Rat}_2(\mathbb{C})/\mathrm{SL}_2(\mathbb{C})$ is biholomorphic to $\mathbb{C}^2$. This fact was generalized by Silverman [6], who showed that M_2 is an affine integral scheme over $\mathbb{Z}$ and is isomorphic to $\mathbb{A}^2_{\mathbb{Z}}$.

Inspired by Milnor [3] we consider a rational map along with an ordered list of its fixed and critical points. Since a rational map of degree two is completely determined by its fixed and critical points, we dispose of the map and focus on the ordered lists of fixed and critical points. We refer to this as the space of *totally marked degree two rational maps*, $\mathrm{Rat}_2^{\mathrm{tm}}$. It can be viewed as an affine open subvariety of $(\mathbb{P}^1)^5$.

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Let p_1, p_2, p_3, q_1, q_2 be an ordered list of fixed points and critical points of some degree two rational map. The natural action of the special linear group SL_2 on $(\mathbb{P}^1)^5$ induces an action on $\mathrm{Rat}_2^{\mathrm{tm}}$. In this article we analyze the quotient $\mathrm{Rat}_2^{\mathrm{tm}}/\mathrm{SL}_2$ and prove:

THEOREM 1.1. *Let* $\mathrm{Rat}_2^{\mathrm{tm}}$ *denote the space of totally marked degree two rational maps. Consider the following action of* SL_2 *on* $\mathrm{Rat}_2^{\mathrm{tm}}$*:*

$$f \cdot (p_1, p_2, p_3, q_1, q_2) = (f(p_1), f(p_2), f(p_3), f(q_1), f(q_2)).$$

Then the moduli space $\mathrm{Rat}_2^{\mathrm{tm}}/\mathrm{SL}_2$ *is isomorphic to a Del Pezzo surface and the isomorphism is defined over* $\mathbb{Z}[1/2]$.

Recall that a cubic in $\mathbb{P}^3$ is a Del Pezzo surface. We give the explicit equation of the surface in §5. The above theorem generalizes a similar result by Milnor [3] over $\mathbb{C}$. The two most significant facts which allow us to prove the theorem above are:

(a) The fixed points and critical points of a degree two rational map determine the map completely.
(b) The three cross ratios formed by selecting both critical points and selecting two out of the three fixed points at a time (see Definition 3.1) are SL_2-invariant functions on $\mathrm{Rat}_2^{\mathrm{tm}}$.

Observe that for $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^2$, each point of $\mathbb{P}^1$ is a critical point in characteristic two. Thus the notion of a totally marked rational map is not well defined in characteristic two, so the isomorphism in the theorem above cannot be defined over $\mathbb{Z}$.

The moduli space of totally marked degree two rational maps, $\mathrm{M}_2^{\mathrm{tm}}$, is a 12-to-1 cover of M_2. Indeed, the map $\mathrm{M}_2^{\mathrm{tm}} \to \mathrm{M}_2$ factors through the moduli space of fixed point marked degree two rational maps, $\mathrm{M}_2^{\mathrm{fm}}$. The latter is a 6-to-1 cover of M_2, and $\mathrm{M}_2^{\mathrm{tm}}$ is a double cover of $\mathrm{M}_2^{\mathrm{fm}}$.

It is natural to ask about the structure of the quotient $\mathrm{M}_d^{\mathrm{tm}} := \mathrm{Rat}_d^{\mathrm{tm}}/\mathrm{SL}_2$. To answer this, we need analogs of (a) and (b) for $d > 2$. As in the degree two case, $\mathrm{M}_d^{\mathrm{tm}}$ will be a finite cover of M_d, and studying $\mathrm{M}_d^{\mathrm{tm}}$ is useful for finding equations defining M_d.

In §2 we prove some basic facts about degree two rational maps. In §3 we decribe the moduli scheme $\mathrm{M}_2^{\mathrm{tm}}$ of totally marked degree two rational maps, followed by the moduli functor for totally marked degree two rational maps, $\underline{\mathrm{M}}_2^{\mathrm{tm}}$ in §4. We prove that the moduli scheme $\mathrm{M}_2^{\mathrm{tm}}$ is a coarse moduli scheme for the functor $\underline{\mathrm{M}}_2^{\mathrm{tm}}$. Finally in §5 we prove our main result.

Notation/Conventions. Throughout this article we fix k to be a field of characteristic different from two. We denote the fixed points by p_1, p_2, p_3 and critical points by q_1, q_2.

2. Preliminaries

LEMMA 2.1. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree two defined over $\mathbb{Z}[1/2]$ such that the resultant* $\mathrm{Res}(\phi)$ *is nonzero. Then ϕ has two distinct critical points.*

Proof. Let

$$\phi(z) = \frac{az^2 + bz + c}{dz^2 + ez + f},$$

and denote the fixed points of ϕ by p_1, p_2, p_3. We split the proof into two cases.

CASE 1. Suppose there is a fixed point of multiplicity three. Without loss of generality we may assume p_1 has multplicity three. Then

$$\phi(z) = \frac{az^2 + bz - p_1^3}{z^2 + (a - 3p_1)z + (b + 3p_1^2)}$$

with an appropriate change of coordinates if $p_1 = \infty$.

CASE 2. Suppose there is no fixed point with multiplicity three. Without loss of generality we may assume that $p_2 \neq p_3$. Applying a change of coordinates we let $p_2 = 0$, $p_3 = \infty$. Then

$$\phi(z) = \frac{az^2 + bz}{ez + f}.$$

In both cases it can be easily verified that the critical points are distinct. ■

LEMMA 2.2. *Let $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree two defined over $\mathbb{Z}[1/2]$. Then ϕ is uniquely determined by its fixed points and critical points.*

Proof. Let

$$\phi(z) = \frac{az^2 + bz + c}{dz^2 + ez + f},$$

and denote its fixed and critical points by p_1, p_2, p_3 and q_1, q_2 respectively. By the previous lemma we know that $q_1 \neq q_2$, so we may assume $q_1 = 0$ and $q_2 = \infty$. Observe that

$$\begin{aligned} q_1 = 0 \text{ and } q_2 = \infty \ &\Leftrightarrow\ ae - bd = 0 \text{ and } bf - ce = 0 \\ &\Leftrightarrow\ \phi(z) = \phi(-z) \text{ for all } z \in \mathbb{P}^1 \\ &\Leftrightarrow\ b = e = 0. \end{aligned}$$

Therefore,

$$\phi(z) = \frac{az^2 + c}{dz^2 + f}.$$

There is a fixed point at infinity if and only if $d = 0$. The fixed points of ϕ are the roots of the equation $dz^3 - az^2 + fz - c = 0$, and they uniquely determine the point $(d : a : f : c)$ in $\mathbb{P}^3$. Thus the coefficients a, c, d, f and hence the rational map ϕ are uniquely determined by its fixed points and critical points. ∎

3. The moduli scheme $\mathrm{M}_2^{\mathrm{tm}}$. For any vector $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ we define a line bundle on $(\mathbb{P}^1)^n$ by

$$L_v = \bigotimes_{i=1}^{n} \pi_i^*(\mathcal{O}_{\mathbb{P}^1}(1)^{\otimes v_i})$$

where $\pi_i : (\mathbb{P}^1)^n \to \mathbb{P}^1$ is the projection on the ith factor.

DEFINITION 3.1. Let $(\omega_1, \omega_2, \omega_3, \xi_1, \xi_2)$ be nonhomogeneous coordinates on $(\mathbb{P}^1)^5$. Fix the linearization $m = (1, 1, 1, 2, 2)$ on $(\mathbb{P}^1)^5$ and denote it by $(\mathbb{P}^1)^5(L_m)$. Let

$$C := \{(\omega_1, \omega_2, \omega_3, \xi_1, \xi_2) \in (\mathbb{P}^1)^5(L_m) \mid \xi_1 = \xi_2\}$$

and

$$R_i := \left\{ (\omega_1, \omega_2, \omega_3, \xi_1, \xi_2) \in (\mathbb{P}^1)^5(L_m) \,\middle|\, r_i := \frac{(\omega_j - \xi_1)(\omega_k - \xi_2)}{(\omega_j - \xi_2)(\omega_k - \xi_1)} = -1 \right\}$$

where (i, j, k) is any permutation of $(1, 2, 3)$. We define the *space of totally marked degree two rational maps* as

$$\mathrm{Rat}_2^{\mathrm{tm}} := (\mathbb{P}^1)^5(L_m) \setminus \{C \cup R_1 \cup R_2 \cup R_3\}.$$

A generic element of $\mathrm{Rat}_2^{\mathrm{tm}}$ is an ordered set of fixed points and critical points of a degree two rational map. Observe that if $\xi_1 = 0$ and $\xi_2 = \infty$, then $\omega_i \neq -\omega_j$ for $i \neq j$. The automorphism group of $\mathbb{P}^1$, PGL_2, acts on each coordinate of $\mathrm{Rat}_2^{\mathrm{tm}}$. For technical reasons we consider the action of SL_2 instead of PGL_2.

DEFINITION 3.2. Two elements $\{p_1, p_2, p_3, q_1, q_2\}$ and $\{p_1', p_2', p_3', q_1', q_2'\}$ of $\mathrm{Rat}_2^{\mathrm{tm}}$ are said to be SL_2-*equivalent* if there exists $f \in \mathrm{SL}_2$ such that $f(p_i) = p_i'$ and $f(q_i) = q_i'$. The quotient $\mathrm{Rat}_2^{\mathrm{tm}}/\mathrm{SL}_2$ is called the *moduli space of totally marked degree two rational maps* and is denoted by $\mathrm{M}_2^{\mathrm{tm}}$.

A priori, for an algebraically closed field k the quotient $\mathrm{M}_2^{\mathrm{tm}}(k)$ exists as a set. We shall show that this set is isomorphic to a Del Pezzo surface whenever $\mathrm{char}(k) \neq 2$. We now describe the sets of stable and of semistable points of projective space. This is well known; we recall it here for the reader's convenience. The sets of stable and of semistable points of a scheme (say V) are denoted by V^{s} and V^{ss} respectively.

THEOREM 3.3. *Let $P = (x_1, \ldots, x_m) \in (\mathbb{P}^r)^m$ and let $v = (v_1, \ldots, v_m) \in \mathbb{Z}^m$. Then*

$$P \in ((\mathbb{P}^r)^m)^{\mathrm{ss}}(L_v) \quad (\textit{resp. } P \in ((\mathbb{P}^r)^m)^{\mathrm{s}}(L_v))$$

if and only if for every proper linear subspace W of $\mathbb{P}^r$,

$$\sum_{i,\,x_i\in W} v_i \le \frac{\dim W+1}{n+1}\sum_{i=1}^{m} v_i$$

(*resp. the strict inequality holds*).

Proof. See [1, p. 172]. ∎

COROLLARY 3.4.

$$((\mathbb{P}^r)^m)^{\mathrm{ss}}(L_v)\neq\emptyset \ \Leftrightarrow\ \forall i=1,\dots,m,\ (r+1)v_i\le\sum_{i=1}^{m} v_i,$$
$$((\mathbb{P}^r)^m)^{\mathrm{s}}(L_v)\neq\emptyset \ \Leftrightarrow\ \forall i=1,\dots,m,\ (r+1)v_i<\sum_{i=1}^{m} v_i.$$

Proof. See [1, p. 172]. ∎

Using Corollary 3.4 with the linearization $m=(1,1,1,2,2)$ we have

$$\mathrm{Rat}_2^{\mathrm{tm}}\subset((\mathbb{P}^1)^5)^{\mathrm{s}}(L_m)=((\mathbb{P}^1)^5)^{\mathrm{ss}}(L_m). \tag{1}$$

The equality in (1) follows from Corollary 3.4, and the (strict) inclusion follows by observing that $(1,-1,2,0,\infty)\in((\mathbb{P}^1)^5)^{\mathrm{s}}(L_m)$ but $(1,-1,2,0,\infty)\notin\mathrm{Rat}_2^{\mathrm{tm}}$ since $\omega_1=-\omega_2$. The choice of linearization $(1,1,1,2,2)$ is not arbitrary. If we use $(1,1,1,1,1)$, then by Corollary 3.4 it can be verified that $(1,1,1,0,\infty)\notin((\mathbb{P}^1)^5)^{\mathrm{ss}}(L_m)$. The rational map $(3z^2+1)/(z^2+3)$ has a triple fixed point at 1 and critical points at 0 and ∞.

THEOREM 3.5. *Using the notation above and the linearization $m=(1,1,1,2,2)$ we have:*

- (a) *The space $\mathrm{Rat}_2^{\mathrm{tm}}$ of totally marked degree two rational maps is an SL_2-invariant open subset of the stable locus $((\mathbb{P}^1)^5)^{\mathrm{s}}(L_m)$ in $(\mathbb{P}^1)^5(L_m)$. Hence, the geometric quotient $\mathrm{M}_2^{\mathrm{tm}}=\mathrm{Rat}_2^{\mathrm{tm}}/\mathrm{SL}_2$ exists as a scheme over $\mathbb{Z}[1/2]$.*
- (b) *The geometric quotient $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{s}}=((\mathbb{P}^1)^5)^{\mathrm{s}}(L_m)/\mathrm{SL}_2$ and the categorical quotient $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{ss}}=((\mathbb{P}^1)^5)^{\mathrm{ss}}(L_m)/\mathrm{SL}_2$ exist as schemes over $\mathbb{Z}[1/2]$ and are the same for the linearization $(1,1,1,2,2)$.*
- (c) *The schemes $\mathrm{M}_2^{\mathrm{tm}}$, $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{s}}$ and $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{ss}}$ are connected, integral, normal and of finite type over $\mathbb{Z}[1/2]$. Moreover, $\mathrm{M}_2^{\mathrm{tm}}$ is affine over $\mathbb{Z}[1/2]$.*

Proof. The assertions follow from standard invariant-theoretic results in [4] and [5].

(a) The inclusion $\mathrm{Rat}_2^{\mathrm{tm}}\subset((\mathbb{P}^1)^5)(L_m)$ follows from (1). The action of SL_2 on $(\mathbb{P}^1)^5(L_m)$ fixes the sets R_i and C defined in Definition 3.1. Hence, $\mathrm{Rat}_2^{\mathrm{tm}}$ is an SL_2-stable and SL_2-invariant scheme, so the geometric quotient

$\mathrm{M}_2^{\mathrm{tm}} = \mathrm{Rat}_2^{\mathrm{tm}}/\mathrm{SL}_2$ exists. Over a field this a consequence of Mumford's construction of quotients [4, Chapter 1], and over $\mathbb{Z}[1/2]$ it follows by essentially the same methods, using Seshadri's theorem that a reductive group scheme is geometrically reductive (see [4] and [5]).

(b) The existence of quotients follows from Mumford [4] and Seshadri [5], and the equality $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{s}} = (\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{ss}}$ follows from Corollary 3.4.

(c) The schemes $\mathrm{Rat}_2^{\mathrm{tm}}, ((\mathbb{P}^1)^5)^{\mathrm{s}}$ and $((\mathbb{P}^1)^5)^{\mathrm{ss}}$ are open subschemes of $(\mathbb{P}^1)^5$, so they are connected, integral and normal. By [4, Section 2, Remark 2], we conclude that the respective quotients $\mathrm{M}_2^{\mathrm{tm}}$, $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{s}}$ and $(\mathrm{M}_2^{\mathrm{tm}})^{\mathrm{ss}}$ are connected, integral and normal. The fact that $\mathrm{M}_2^{\mathrm{tm}}$ is affine over $\mathbb{Z}[1/2]$ follows from [4, Theorem 1.1]. ■

4. The moduli functor $\underline{\mathrm{M}}_2^{\mathrm{tm}}$

Definition 4.1. The functor $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$ of totally marked degree two rational maps is the functor

$$\underline{\mathrm{Rat}}_2^{\mathrm{tm}} : (\mathrm{Sch}/\mathbb{Z}[1/2]) \to (\mathrm{Sets})$$

defined by

$$\underline{\mathrm{Rat}}_2^{\mathrm{tm}}(S) = \begin{cases} \text{separable } S\text{-morphisms } \phi : \mathbb{P}^1_S \to \mathbb{P}^1_S \text{ with } \phi^*\mathcal{O}(1) \cong \mathcal{O}(2), \\ \text{sections } r_i \text{ of } \mathbb{P}^1_S \to S,\ i=1,2,3, \text{ with } \phi \circ r_i = r_i, \text{ and} \\ \text{sections } s_j \text{ of } \mathbb{P}^1_S \to S,\ j=1,2, \text{ with } \operatorname{div}(s_1) + \operatorname{div}(s_2) = R_\phi, \end{cases}$$

where R_ϕ is the ramification divisor of ϕ.

Observe that the sections r_i correspond to the fixed points and the sections s_j correspond to the critical points. An S-point of $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$ consists of a degree two rational map and five sections satisfying the above conditions.

Definition 4.2. We say two S-points of $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$, say $(\phi, r_1, r_2, r_3, s_1, s_2)$ and $(\phi', r'_1, r'_2, r'_3, s'_1, s'_2)$, are *equivalent* if there exists $f \in \mathrm{Aut}(\mathbb{P}^1_S)$ such that $\phi \circ f = f \circ \phi'$, $f(r_i) = r'_i$ and $f(s_j) = s'_j$. We define the moduli functor $\underline{\mathrm{M}}_2^{\mathrm{tm}}$ to be the quotient of $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$ under the above equivalence relation:

$$\underline{\mathrm{M}}_2^{\mathrm{tm}} : (\mathrm{Sch}\,/\mathbb{Z}[1/2]) \to (\mathrm{Sets}), \qquad S \mapsto \underline{\mathrm{Rat}}_2^{\mathrm{tm}}(S)/\!\sim.$$

We now prove that the functor $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$ is representable.

Theorem 4.3. *The scheme* $\mathrm{Rat}_2^{\mathrm{tm}}$ *defined in Definition* 3.1 *represents the functor* $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$. *In particular,* $\mathrm{Rat}_2^{\mathrm{tm}}$ *is a fine moduli space for* $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$.

Proof. Let S be an arbitary scheme and $(p_1, p_2, p_3, q_1, q_2) \in \mathrm{Rat}_2^{\mathrm{tm}}(S)$. By Lemma 2.2 there exists a unique rational map $\phi : \mathbb{P}^1_S \to \mathbb{P}^1_S$ with fixed points p_1, p_2, p_3 and critical points q_1, q_2. Let r_i, s_j be the sections of $\mathbb{P}^1_S \to S$

corresponding to the fixed and critical points. This gives a well defined map

$$\begin{aligned} (2) \qquad & \mathrm{Rat}_2^{\mathrm{tm}}(S) \to \underline{\mathrm{Rat}}_2^{\mathrm{tm}}(S), \\ & (p_1, p_2, p_3, q_1, q_2) \mapsto (\phi, r_1, r_2 r_3, s_1, s_2). \end{aligned}$$

The inverse

$$\begin{aligned} (3) \qquad & \underline{\mathrm{Rat}}_2^{\mathrm{tm}}(S) \to \mathrm{Rat}_2^{\mathrm{tm}}(S), \\ & (\phi, r_1, r_2, r_3, s_1, s_2) \mapsto (p_1, p_2, p_3, q_1, q_2), \end{aligned}$$

maps the sections r_i, s_j to the corresponding fixed and critical points and forgets ϕ. Thus the scheme $\mathrm{Rat}_2^{\mathrm{tm}}$ represents the functor $\underline{\mathrm{Rat}}_2^{\mathrm{tm}}$. ∎

We now show that $\mathrm{M}_2^{\mathrm{tm}}$ is a coarse moduli scheme for the functor $\underline{\mathrm{M}}_2^{\mathrm{tm}}$.

THEOREM 4.4. *There is a natural map of functors*

$$\underline{\mathrm{M}}_2^{\mathrm{tm}} \to \mathrm{Hom}(-, \mathrm{M}_2^{\mathrm{tm}})$$

with the property that $\underline{\mathrm{M}}_2^{\mathrm{tm}}(k) \cong \mathrm{M}_2^{\mathrm{tm}}(k)$ *for every algebraically closed field* k *of characteristic* $\neq 2$.

Proof. Let S be an arbitrary scheme over $\mathbb{Z}[1/2]$, let $[\eta] \in \underline{\mathrm{M}}_2^{\mathrm{tm}}(S)$ and let $(\phi, r_1, r_2, r_3, s_1, s_2) \in \underline{\mathrm{Rat}}_2^{\mathrm{tm}}(S)$ be a representative of $[\eta]$. Let $(p_1, p_2, p_3, q_1, q_2) \in \mathrm{Rat}_2^{\mathrm{tm}}$ be the image of $(\phi, r_1, r_2, r_3, s_1, s_2)$ along the map defined in (3). Taking the quotient of $(p_1, p_2, p_3, q_1, q_2)$ by SL_2 we get the image of $[\eta]$ in $\mathrm{Hom}(-, \mathrm{M}_2^{\mathrm{tm}})$. The image is independent of the choice of the lifting of $[\eta]$ since we are quotienting by SL_2.

For any algebraically closed field k of characteristic different than two,

$$\underline{\mathrm{M}}_2^{\mathrm{tm}}(k) \cong \mathrm{Rat}_2^{\mathrm{tm}}(k)/\mathrm{PGL}_2(k), \qquad \mathrm{M}_2^{\mathrm{tm}}(k) \cong \mathrm{Rat}_2^{\mathrm{tm}}(k)/\mathrm{SL}_2(k).$$

The map $\mathrm{SL}_2 \to \mathrm{PGL}_2$ is surjective, hence so are the quotients. ∎

5. Main theorem. Recall that $\mathrm{Rat}_2^{\mathrm{tm}} := (\mathbb{P}^1)^5(L_m) \setminus \{C \cup R_1 \cup R_2 \cup R_3\}$ where

$$C := \{(\omega_1, \omega_2, \omega_3, \xi_1, \xi_2) \in (\mathbb{P}^1)^5(L_m) \mid \xi_1 = \xi_2\},$$

and

$$R_i := \left\{ (\omega_1, \omega_2, \omega_3, \xi_1, \xi_2) \in (\mathbb{P}^1)^5(L_m) \,\middle|\, r_i := \frac{(\omega_j - \xi_1)(\omega_k - \xi_2)}{(\omega_j - \xi_2)(\omega_k - \xi_1)} = -1 \right\}$$

where (i, j, k) is any permutation of $(1, 2, 3)$. We shall show that the cross ratios r_i are SL_2-invariant functions on the quotient space $\mathrm{M}_2^{\mathrm{tm}}$, and they uniquely determine the conjugacy class.

PROPOSITION 5.1. *The cross ratios* r_i *are rational functions on* $\mathrm{Rat}_2^{\mathrm{tm}}$. *Moreover, they are invariant under the action of* SL_2 *on* $\mathrm{Rat}_2^{\mathrm{tm}}$. *Thus they descend to give rational functions on* $\mathrm{M}_2^{\mathrm{tm}}$.

Proof. Two elements $(p_1, p_2, p_3, q_1, q_2), (p'_1, p'_2, p'_3, q'_1, q'_2) \in \mathrm{Rat}_2^{\mathrm{tm}}$ are SL_2-equivalent if there exists $f \in \mathrm{SL}_2$ such that $f(p_i) = p'_i$ and $f(q_i) = q'_i$, where p_i denote the fixed points and q_i denote the critical points. Note that each cross ratio is determined by selecting two of the three fixed points and both critical points. If

$$r_1 = \frac{(p_2 - q_1)(p_3 - q_2)}{(p_2 - q_2)(p_3 - q_1)},$$

then the cross ratio determined by $f(p_2), f(p_3), f(q_1), f(q_2)$ is

$$r'_1 = \frac{(f(p_2) - f(q_1))(f(p_3) - f(q_2))}{(f(p_2) - f(q_2))(f(p_3) - f(q_1))} = \frac{(p'_2 - q'_1)(p'_3 - q'_2)}{(p'_2 - q'_2)(p'_3 - q'_1)}.. \tag{4}$$

Claim. $r_1 = r'_1$.

Proof of Claim. By Lemma 2.1 we may assume $q_1 = 0$, $q_2 = \infty$, hence $r_1 = p_2/p_3$. Write $p_i = [p_i : 1]$, $\infty = [1 : 0]$ and $0 = [0 : 1]$. For

$$f = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2,$$

we have $f(p_i) = \left[\frac{ap_i+b}{cp_i+d} : 1\right]$, $f(\infty) = [a/c : 1]$ and $f(0) = [b/d : 1]$. Writing these in nonhomogeneous form and substituting in (4), we get $r'_1 = p_2/p_3$. Similarly for r_2, r_3. Since the cross ratios are invariant under the SL_2 action, they descend to give rational functions on $\mathrm{M}_2^{\mathrm{tm}}$. ∎

Let $V = \mathrm{Spec}\big(\mathbb{Z}[1/2][x_1, x_2, x_3]/(x_1 + x_2 + x_3 + x_1x_2x_3)\big)$.

Proposition 5.2. *The cross ratios form a complete conjugacy invariant, i.e. they determine the conjugacy class in* $\mathrm{M}_2^{\mathrm{tm}}$ *uniquely.*

Proof. We begin by defining a map from the scheme V to the fixed point marked moduli space $\mathrm{M}_2^{\mathrm{fm}}$ and then extending it to $\mathrm{M}_2^{\mathrm{tm}}$. The fixed point marked moduli space is determined by the multipliers at the three fixed points which we denote by μ_1, μ_2, μ_3. Define a map from V to $\mathrm{M}_2^{\mathrm{fm}}$ by setting

$$\mu_i = 1 + x_j x_k.$$

Observe that $\mu_1 + \mu_2 + \mu_3 = \mu_1\mu_2\mu_3 + 2$. If $\omega_j \neq \omega_k$, then we can put $\omega_j = 0$ and $\omega_k = \infty$, and write the map as

$$\phi(z) = \frac{z^2 + \mu_j z}{\mu_k z + 1}.$$

The critical points of ϕ are

$$\xi_1 = \frac{-1 + x_i}{\mu_k}, \qquad \xi_2 = \frac{-1 - x_i}{\mu_k},$$

and the cross ratio r_i is given by

$$r_i = \frac{\xi_1}{\xi_2} = \frac{1 - x_i}{1 + x_i}.$$

Conversely, given r_i we can solve for x_i, obtaining $x_i = (1 - r_i)/(1 + r_i)$. This shows that the cross ratios and hence the conjugacy class in $\mathrm{M}_2^{\mathrm{tm}}$ are completely determined by the coordinates x_1, x_2, x_3, yielding a smooth map from $\mathrm{M}_2^{\mathrm{tm}}$ to V. ∎

Let r_1, r_2, r_3 be nonhomogeneous coordinates on $(\mathbb{P}^1 \setminus \{-1\})^3$, and let W be the subvariety cut out by the equation $r_1 r_2 r_3 - 1$. We now prove that $\mathrm{M}_2^{\mathrm{tm}}$ is isomorphic to W, where the isomorphism is defined over $\mathbb{Z}[1/2]$.

REMARK 5.3. The schemes V and W are isomorphic to each other. Using x_1, x_2, x_3 and r_1, r_2, r_3 as coordinates on V and W respectively define $\sigma : V \to W$, $x_i \mapsto r_i = (1 - x_i)/(1 + x_i)$. It can be easily verified that $\sigma = \sigma^{-1}$.

THEOREM 5.4. *The map $\mathrm{M}_2^{\mathrm{tm}} \to W$ is an isomorphism of schemes defined over $\mathbb{Z}[1/2]$.*

Proof. Let $(\omega_1, \omega_2, \omega_3, \xi_1, \xi_2)$ be any element of $\mathrm{M}_2^{\mathrm{tm}}$. The map $\mathrm{M}_2^{\mathrm{tm}} \to W$ is given by

$$r_i = \frac{(\omega_j - \xi_1)(\omega_k - \xi_2)}{(\omega_j - \xi_2)(\omega_k - \xi_1)},$$

where (i, j, k) is any permutation of $(1, 2, 3)$. We now construct the inverse. Without loss of generality we may assume that one of $\omega_1, \omega_2, \omega_3$ is finite and nonzero, and $\omega_1 = 1$, $\omega_2 = 1/r_3$, $\omega_3 = r_2$, $\xi_1 = 0$, $\xi_2 = \infty$. Since $\xi_1 = 0$, $\xi_2 = \infty$, we have $\phi(z) = \frac{az^2+b}{cz^2+d}$. We shall determine the coefficients of ϕ explicitly. The image for these values of $\omega_1, \omega_2, \omega_3, \xi_1, \xi_2$ in $(\mathbb{P}^1 \setminus \{-1\})^3$ is the complement of the curves $(r_2 = \infty, r_3 = 0)$ and $(r_2 = 0, r_3 = \infty)$. We denote the image in $(\mathbb{P}^1 \setminus \{-1\})^3$ by U. For $\omega_1, \omega_2, \omega_3, \xi_1, \xi_2$ as above,

$$a/c = -(1 + r_2 + 1/r_3), \quad b/c = r_2/r_3, \quad d/c = r_2 + 1/r_3 + r_2/r_3.$$

We now break U into four subsets based on values of r_2 and r_3.

CASE 1: If $r_2 \neq \infty$, $r_3 \neq 0$, then let $c = 1$ and we are done.
CASE 2: If $r_2 \neq \infty$, $r_3 \neq \infty$, then let $c = r_3$, so $b = -r_2$, $a = -(1+r_3+r_2r_3)$, $d = 1 + r_2 + r_2 r_3$.
CASE 3: If $r_2 \neq 0$, $r_3 \neq 0$, then let $b = -1/r_3$, so $a = -(1 + 1/r_2 + 1/r_2 r_3)$, $c = 1/r_2$, $d = 1 + 1/r_3 + 1/r_2 r_3$.
CASE 4: If $r_2 \neq 0$, $r_3 \neq \infty$, then let $b = -1$, so $a = -(r_3 + 1/r_2 + r_3/r_2)$, $c = r_3/r_2$, $d = 1 + r_3 + 1/r_2$.

In each of the four cases $\phi(z) \in \mathrm{M}_2^{\mathrm{tm}}$. On the intersections the maps agree not only in $\mathrm{M}_2^{\mathrm{tm}}$ but also in $\mathrm{Rat}_2^{\mathrm{tm}}$. So we can glue the four affine pieces together to obtain a map $U \to \mathrm{M}_2^{\mathrm{tm}}$. By symmetry we can assume that ω_2, ω_3 are finite and nonzero as well. We get morphisms from three affine open pieces to $\mathrm{M}_2^{\mathrm{tm}}$. The union of these three affine pieces is W, so we have three morphisms from W to $\mathrm{M}_2^{\mathrm{tm}}$. It remains to show that the morphisms agree on the intersections.

On the first affine piece (i.e. $\omega_1 \neq 0, \infty$) we have $\omega_1 = 1$, $\omega_2 = 1/r_3$, $\omega_3 = r_2$, so $a/c = -(1 + r_2 + 1/r_3)$, $b/c = -r_2/r_3$, $d/c = r_2 + 1/r_3 + r_2/r_3$. On the second affine piece (i.e. $\omega_2 \neq 0, \infty$) we have $\omega_2 = 1$, $\omega_1 = 1/r_3$, $\omega_3 = r_2 r_3$, so $a/c = -(1 + r_3 + r_2 r_3)$, $b/c = -r_2 r_3^2$, $d/c = r_3(1 + r_2 + r_2 r_3)$. Applying the transformation $z \mapsto r_3 \cdot z$ to the equation $\phi(z) = z$ we see that the expressions for a/c, b/c, d/c are the same. On the third affine piece (i.e. $\omega_3 \neq 0, \infty$) if $r_2 \neq \infty$, then let $c = 1$, and if $r_2 \neq 0$, then let $c = 1/r_2$. In either case r_1, r_2, r_3 determine the same quadruple (a, b, c, d) defining the same point in $\mathrm{M}_2^{\mathrm{tm}}$, though not the same point in $\mathrm{Rat}_2^{\mathrm{tm}}$. ∎

A cubic in $\mathbb{P}^3$ is a Del Pezzo surface. In homogeneous coordinates the surface W is cut out by the equation $r_1 r_2 r_3 - r_4^3$. Thus the moduli space of totally marked degree two rational maps is isomorphic to a Del Pezzo surface and the isomorphism is defined over $\mathbb{Z}[1/2]$.

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