ACTA ARITHMETICA
167.3 (2015)

Consecutive primes in tuples

by

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1. Introduction and statement of results. We say that a k-tuple of linear forms in $\mathbb{Z}[x]$, denoted by

$$\mathcal{H}(x) = \{g_j x + h_j\}_{j=1}^{k},$$

is *admissible* if the associated polynomial $f_{\mathcal{H}}(x) = \prod_{1\le j\le k}(g_j x + h_j)$ has no fixed prime divisor, that is, if the inequality

$$\#\{n \bmod p : f_{\mathcal{H}}(n) \equiv 0 \bmod p\} < p$$

holds for every prime number p. In this note we consider only k-tuples for which

$$g_1, \ldots, g_k > 0 \quad \text{and} \quad \prod_{1\le i<j\le k} (g_i h_j - g_j h_i) \neq 0. \tag{1}$$

One form of the *Prime k-Tuple Conjecture* asserts that if $\mathcal{H}(x)$ is admissible and satisfies (1), then $\mathcal{H}(n) = \{g_j n + h_j\}_{j=1}^{k}$ is a k-tuple of primes for infinitely many $n \in \mathbb{N}$. Recently, Maynard [5] and Tao have made great strides towards proving this form of the Prime k-Tuple Conjecture, which rests among the greatest unsolved problems in number theory. The following formulation of their remarkable theorem has been given by Granville [3, Theorem 6.2].

Theorem (Maynard–Tao). *For any $m \in \mathbb{N}$ with $m \ge 2$ there is a number k_m, depending only on m, such that the following holds for every integer $k \ge k_m$: If $\{g_j x + h_j\}_{j=1}^{k}$ is admissible and satisfies* (1)*, then $\{g_j n + h_j\}_{j=1}^{k}$ contains m primes for infinitely many $n \in \mathbb{N}$. In fact, one can take k_m to be any number such that $k_m \log k_m > e^{8m+4}$.*

2010 *Mathematics Subject Classification*: Primary 11N36; Secondary 11A41.
Key words and phrases: consecutive primes in tuples, bounded gaps between primes, Maynard–Tao theorem.

DOI: 10.4064/aa167-3-4

Zhang [10, Theorem 1] was the first to prove that $\liminf_{n\to\infty}(p_{n+1}-p_n)$ is bounded; he showed that for an admissible k-tuple $\mathcal{H}(x)=\{x+b_j\}_{j=1}^k$ there exist infinitely many integers n such that $\mathcal{H}(n)$ contains at least two primes, provided that $k\ge 3.5\cdot 10^6$. Zhang's proof was subsequently refined in a Polymath project [7, Theorem 2.3] to the point where one could take $k_2=632$ (at least in the case of monic linear forms). Maynard [5, Propositions 4.2, 4.3] has shown that one can take $k_2=105$ and $k_m=cm^2e^{4m}$ in the Maynard–Tao theorem, where c is an absolute (and effective) constant. Another Polymath project [8, Theorem 3.2] has since refined Maynard's work so that one can take $k_2=50$ and $k_m=ce^{(4-28/157)m}$. (In [5, 8], only tuples of monic linear forms are treated explicitly, although the results should extend to general linear forms as considered in [3].)

The purpose of the present note is to explain some interesting consequences of the Maynard–Tao theorem. We refer the reader to the expository article [3] of Granville for the recent history and ideas leading up to this breakthrough result, as well as a discussion of its potential impact. Without doubt, this result and its proof will have numerous applications, many of which have already been given in [3]. We are grateful to Granville for pointing out to us that Corollary 2 (below) can now be proved.

The following theorem establishes the existence of m-tuples that infinitely often represent strings of *consecutive* prime numbers.

THEOREM 1. *Let $m,k\in\mathbb{N}$ with $m\ge 2$ and $k\ge k_m$, where k_m is as in the Maynard–Tao theorem. Let $b_1,\dots,b_k$ be distinct integers such that $\{x+b_j\}_{j=1}^k$ is admissible, and let g be any positive integer coprime to $b_1\cdots b_k$. Then, for some subset $\{h_1,\dots,h_m\}\subseteq\{b_1,\dots,b_k\}$, there are infinitely many $n\in\mathbb{N}$ such that $gn+h_1,\dots,gn+h_m$ are consecutive primes.*

A special case of Theorem 1, with $m=2$, $g=1$ (and the weaker bound $k_2\ge 3.5\cdot 10^6$), has already been established in recent work of Pintz [6, Main Theorem], which is based on Zhang's method but uses a different argument to the one presented here.

Theorem 1 (which is proved in §2) has various applications to the study of gaps between consecutive primes. To state our results, let us call a sequence $(\delta_j)_{j=1}^m$ of positive integers a *run of consecutive prime gaps* if

$$\delta_j=d_{r+j}=p_{r+j+1}-p_{r+j}\qquad(1\le j\le m)$$

for some natural number r, where p_n denotes the nth smallest prime. The following corollary of Theorem 1 answers an old question of Erdős and Turán [2] (see also Erdős [1] and Guy [4, A11]).

COROLLARY 2. *For every $m\ge 2$ there are infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps with $\delta_1<\dots<\delta_m$, and infinitely many runs with $\delta_1>\dots>\delta_m$.*

Moreover, in the proof (see §2) we construct infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps with

$$\delta_1 + \cdots + \delta_{j-1} < \delta_j \quad (2 \le j \le m),$$

and infinitely many runs with

$$\delta_j > \delta_{j+1} + \cdots + \delta_m \quad (1 \le j \le m-1).$$

Using a similar argument, we can impose a divisibility requirement amongst gaps between consecutive primes as well.

COROLLARY 3. *For every $m \ge 2$ there are infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps such that $\delta_{j-1} \mid \delta_j$ for $2 \le j \le m$, and infinitely many runs such that $\delta_{j+1} \mid \delta_j$ for $1 \le j \le m-1$.*

In the proof (see §2) we construct infinitely many runs $(\delta_j)_{j=1}^m$ of consecutive prime gaps with $\delta_1 \cdots \delta_{j-1} \mid \delta_j$ for $2 \le j \le m$, and infinitely many runs with $\delta_m \delta_{m-1} \cdots \delta_{j+1} \mid \delta_j$ for $1 \le j \le m-1$.

As another application of Theorem 1, in §2 we prove the following extension of a result of Shiu [9] on consecutive primes in a given congruence class.

COROLLARY 4. *Let a and $D \ge 3$ be coprime integers. For every $m \ge 2$, there are infinitely many $r \in \mathbb{N}$ such that $p_{r+1} \equiv \cdots \equiv p_{r+m} \equiv a \bmod D$ and $p_{r+m} - p_{r+1} \le DC_m$, where C_m is a constant depending only on m.*

Shiu [9] attributes to Chowla the conjecture that there are infinitely many pairs of consecutive primes p_r, p_{r+1} with $p_r \equiv p_{r+1} \equiv a \bmod D$ (see also [4, A4]), and proved the above result without the constraint $p_{r+m} - p_{r+1} \le DC_m$.

2. Proofs

Proof of Theorem 1. Replacing each b_j with $b_j + gN$ for a suitable integer N, we can assume without loss of generality that

$$1 < b_1 < \cdots < b_k.$$

Let $\mathcal{S}$ be the set of integers t such that $1 \le t \le b_k$, $t \notin \{b_1, \ldots, b_k\}$. Let $\{q_t : t \in \mathcal{S}\}$ be distinct primes coprime to g such that $t \not\equiv b_j \bmod q_t$ for all $t \in \mathcal{S}$, $1 \le j \le k$. By the Chinese remainder theorem we can find an integer a such that

$$ga + t \equiv 0 \bmod q_t \quad (t \in \mathcal{S}), \tag{2}$$

and therefore

$$ga + b_j \not\equiv 0 \bmod q_t \quad (t \in \mathcal{S},\ 1 \le j \le k). \tag{3}$$

Consider the k-tuple

$$\mathcal{A}(x) = \{gQx + ga + b_j\}_{j=1}^k \quad \text{where } Q = \textstyle\prod_{t\in\mathcal{S}} q_t.$$

In view of (3) and the equality $\gcd(g, b_1 \cdots b_k) = 1$, we have $\gcd(gQ, ga + b_j) = 1$ for each j, and since $\{x + b_j\}_{j=1}^k$ is admissible, it follows that the k-tuple $\mathcal{A}(x)$ is also admissible. Moreover, $\mathcal{A}(x)$ satisfies (1) (with $g_j = gQ$ and $h_j = ga + b_j$) as the integers $b_1, \ldots, b_k$ are distinct and $gQ \ge 1$.

For every $N \in \mathbb{N}$, the congruences (2) and our choices of Q and a imply that

$$g(QN + a) + t \equiv 0 \bmod q_t \quad (t \in \mathcal{S}).$$

Hence, any prime number in the interval $[g(QN + a) + b_1, g(QN + a) + b_k]$ must lie in $\mathcal{A}(n)$. Let m' be the largest integer for which there exists a subset $\{h_1, \ldots, h_{m'}\} \subseteq \{b_1, \ldots, b_k\}$ with the property that the numbers

$$g(QN + a) + h_i \qquad (1 \le i \le m') \tag{4}$$

are simultaneously prime for infinitely many $N \in \mathbb{N}$. Since $k \ge k_m$, we can apply the Maynard–Tao theorem with $\mathcal{A}(x)$ to deduce that $m' \ge m$.

By the maximal property of m', it must be the case that for all sufficiently large $N \in \mathbb{N}$, if the numbers in (4) are all prime, then $g(QN + a) + b_j$ is composite for every $b_j \in \{b_1, \ldots, b_k\} \setminus \{h_1, \ldots, h_{m'}\}$. Hence, for infinitely many $N \in \mathbb{N}$, the interval $[g(QN+a)+b_1, g(QN+a)+b_k]$ contains precisely m' primes, namely, the numbers $\{gn + h_i\}_{i=1}^{m'}$ with $n = QN + a$. ∎

Proof of Corollary 2. Let $m \ge 2$ and $k \ge k_{m+1}$. Let $\mathcal{A}(x) = \{x + 2^j\}_{j=1}^k$, which is easily seen to be admissible. By Theorem 1, there exists an $(m+1)$-tuple

$$\mathcal{B}(x) = \{x + 2^{\nu_j}\}_{j=1}^{m+1} \subseteq \mathcal{A}(x)$$

such that $\mathcal{B}(n)$ is an $(m+1)$-tuple of consecutive primes for infinitely many n. Here, $1 \le \nu_1 < \cdots < \nu_{m+1} \le k$. For such n, writing

$$\mathcal{B}(n) = \{n + 2^{\nu_j}\}_{j=1}^{m+1} = \{p_{r+1}, \ldots, p_{r+m+1}\}$$

with some integer r, we have

$$\delta_j = d_{r+j} = p_{r+j+1} - p_{r+j} = 2^{\nu_{j+1}} - 2^{\nu_j} \quad (1 \le j \le m).$$

Then

$$\sum_{i=1}^{j-1} \delta_i = \sum_{i=1}^{j-1} (2^{\nu_{i+1}} - 2^{\nu_i}) = 2^{\nu_j} - 2^{\nu_1} < 2^{\nu_{j+1}} - 2^{\nu_j} = \delta_j \quad (2 \le j \le m).$$

Hence, $\delta_{j-1} \le \delta_1 + \cdots + \delta_{j-1} < \delta_j$ for each j, which proves the first statement. To obtain runs of consecutive prime gaps with $\delta_j > \delta_{j+1} + \cdots + \delta_m \ge \delta_{j+1}$, consider instead the admissible k-tuple $\{x - 2^j\}_{j=1}^k$. ∎

Proof of Corollary 3. Let $m \geq 2$, and let $k \geq k_{m+1}$. Put $Q = \prod_{p \leq k} p$, and define the sequence $b_1, \ldots, b_k$ inductively as follows. Let

$$b_1 = 0, \quad b_2 = Q, \quad b_3 = 2Q,$$

and for any $j \geq 3$ let

$$b_j = b_{j-1} + \prod_{1 \leq s < t \leq j-1} (b_t - b_s).$$

Note that

$$(b_{u+1} - b_u) \mid (b_{v+1} - b_v) \quad (v \geq u \geq 1). \tag{5}$$

Now put $\mathcal{A}(x) = \{x + b_j\}_{j=1}^k$, and observe that $\mathcal{A}(x)$ is admissible since Q divides each integer b_j. By Theorem 1, there exists an $(m+1)$-tuple

$$\mathcal{B}(x) = \{x + b_{\nu_j}\}_{j=1}^{m+1} \subseteq \mathcal{A}(x)$$

such that $\mathcal{B}(n)$ is an $(m+1)$-tuple of consecutive primes for infinitely many n. Here, $1 \leq \nu_1 < \cdots < \nu_{m+1} \leq k$. For any such n, writing

$$\mathcal{B}(n) = \{n + b_{\nu_j}\}_{j=1}^{m+1} = \{p_{r+1}, \ldots, p_{r+m+1}\}$$

with some integer r, we have

$$\delta_j = d_{r+j} = p_{r+j+1} - p_{r+j} = b_{\nu_{j+1}} - b_{\nu_j} \quad (1 \leq j \leq m).$$

Then

$$\prod_{i=1}^{j-1} \delta_i = \prod_{i=1}^{j-1} (b_{\nu_{i+1}} - b_{\nu_i}) \Bigm| \prod_{1 \leq s < t \leq \nu_j} (b_t - b_s) = b_{\nu_j+1} - b_{\nu_j}$$

if $2 \leq j \leq m$. On the other hand, using (5) we see that

$$(b_{\nu_j+1} - b_{\nu_j}) \Bigm| \sum_{i=\nu_j}^{\nu_{j+1}-1} (b_{i+1} - b_i) = b_{\nu_{j+1}} - b_{\nu_j} = \delta_j.$$

Hence, $\delta_1 \cdots \delta_{j-1} \mid \delta_j$ for $2 \leq j \leq m$, which proves the first statement. To obtain runs of consecutive prime gaps with $\delta_m \delta_{m-1} \cdots \delta_{j+1} \mid \delta_j$ for $1 \leq j \leq m-1$, consider instead the admissible k-tuple $\{x - b_j\}_{j=1}^k$. ∎

Proof of Corollary 4. Let $m \geq 2$, and let $k \geq k_m$. Let $\{x + a_j\}_{j=1}^k$ be any admissible k-tuple with $a_1 < \cdots < a_k$, and put $b_j = Da_j + a$ for $1 \leq j \leq k$; then $\{x + b_j\}_{j=1}^k$ is also admissible. Since $\gcd(D, b_j) = \gcd(D, a) = 1$ for each j, we can apply Theorem 1 with $g = D$ to conclude that there is a subset $\{h_1, \ldots, h_m\} \subseteq \{b_1, \ldots, b_k\}$ such that $Dn + h_1, \ldots, Dn + h_m$ are consecutive primes for infinitely many $n \in \mathbb{N}$; as such primes lie in the arithmetic progression $a \bmod D$ and are contained in an interval of length $b_k - b_1 = D(a_k - a_1)$, the corollary follows. ∎

Acknowledgements. CLT-B is supported by a GAANN fellowship (grant no. P200A90092). In the first draft of this manuscript, we proved Theorem 1 under the assumption that $k \geq \exp(e^{12m})$. We thank Andrew Granville for showing that k need not be larger than the number k_m in the Maynard–Tao theorem and for simplifying our original proof of Theorem 1. We also thank Gergely Harcos, James Maynard, and the referee for providing helpful comments on our earlier drafts.

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Received on 8.2.2014
and in revised form on 1.12.2014 (7724)