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The number of k-sums of abelian groups of order k

by

Hong Bing Yu (Suzhou)

1. Introduction. Let G be an abelian group of order k. Given a sequence of elements $a_1, \ldots, a_n$ in G (possibly with repetitions), a *t-sum* is a sum of the form $a_{i_1} + \ldots + a_{i_t}$ $(i_1 < \ldots < i_t)$. In [6] Erdős, Ginzburg and Ziv proved an important result in Combinatorial Number Theory, which states that if $n = 2k - 1$ then some k-sum is 0. Since then, numerous other proofs and generalizations of this result have been given (see for example [2] and the survey paper [4]). More recently, Bollobás and Leader [3] proved the following interesting result: for $n = k + r$ $(1 \le r \le k - 1)$, if 0 is not a k-sum then there are at least $r + 1$ k-sums. This clearly implies the Erdős–Ginzburg–Ziv theorem, by taking $r = k - 1$.

In this paper we shall prove several results concerning k-sums for abelian groups of order k. Our first result here is the following theorem, which settles a conjecture of Bollobás and Leader (see [3, Section 2]).

Theorem 1. *Let G be an abelian group of order k, and let $r \ge 1$. Then the minimum number of k-sums for a sequence $a_1, \ldots, a_{k+r}$ of elements of G that does not have* 0 *as a k-sum is attained at the sequence $b_1, \ldots, b_{r+1}, 0, \ldots, 0$, where $b_1, \ldots, b_{r+1}$ is chosen to minimize the number of* (*non-empty*) *sums without* 0 *being a* (*non-empty*) *sum.*

Our second result gives a characterization of the extremal cases in Bollobás–Leader's theorem mentioned above.

Theorem 2. *Let G be an abelian group of order k, and let $d(G)$ be the maximal order of an element in G. Let $a_1, \ldots, a_{k+r} \in G$. Then if* 0 *is not a k-sum then the number of k-sums is at least $r + 1$, and the bound is attained if and only if $1 \le r \le d(G) - 2$ and the sequence is of the form $a, \ldots, a, b, \ldots, b$ with the order of $a - b$ being at least $r + 2$.*

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From Theorems 1 and 2 we see that to estimate the minimum number of k-sums for a sequence of elements of G with length $k+r$ that does not have 0 as a k-sum, it suffices to consider the problem in the case of G non-cyclic and $d(G)-1 \le r \le D(G)-2$, where $D(G)$ is the *Davenport constant* of G, i.e., the minimal n such that, whenever $a_1, \dots, a_n \in G$, some non-empty sum of the a_i is 0. We remark here that Eggleton and Erdős [5] have proved that $D(G) \le k/2+1$ for any abelian non-cyclic group G of order k.

The following result can be easily deduced from Theorem 1 and the theorem of Olson and White [7], so its proof is omitted.

THEOREM 3. *Let G be an abelian non-cyclic group of order k, and let $d(G)-1 \le r \le D(G)-2$. Let $a_1, \dots, a_{k+r} \in G$. Then if 0 is not a k-sum then the number of k-sums is at least $2r+1$.*

We do not know whether the bound of Theorem 3 is sharp in general. It should be mentioned here that in the case of $G = \mathbb{Z}_n^2$ Bollobás and Leader have conjectured that the bound in question is $n(r-n+3)-1$ for $n-1 \le r \le 2n-3$ $(= D(\mathbb{Z}_n^2)-2)$ (see [3, Section 2]).

2. Preliminary lemmas. In the proof of Theorems 1 and 2 we need the following two well known results. The first follows from Corollary 2.3 of Alon [1], and the second is Lemma 1 of Olson and White [7].

LEMMA 1. *Let G be an abelian group of order k, and let $a_1, \dots, a_n$ be a sequence of elements of G in which no value is repeated $l+1$ times. If $n \ge k$ then the sequence has a t-sum equal to 0 for some $1 \le t \le l$.*

LEMMA 2. *Let $c_1, \dots, c_{r+1}$ be a sequence of elements of an abelian group without 0 being a non-empty sum. Then there are at least $r+1$ non-empty sums, and the bound is attained only when $c_1 = \dots = c_{r+1}$.*

3. Proof of Theorems 1 and 2. Let $N_{k+r}(A)$ be the number of k-sums for a sequence $A = \{a_1, \dots, a_{k+r}\}$ that does not have 0 as a k-sum. We observe that Theorem 1 together with Lemma 2 implies immediately that $N_{k+r}(A) \ge r+1$, and equality holds only if $1 \le r \le d(G)-2$. Therefore, to prove the theorems it suffices to prove the following assertions:

(i) Let $b_1, \dots, b_{r+1}$ be as in Theorem 1, and let N_{r+1} be the number of (non-empty) sums for this sequence. Then $N_{k+r}(A) \ge N_{r+1}$.

(ii) If $N_{k+r}(A) = r+1$ then A must be of the form stated in Theorem 2.

Translating (which does not affect k-sums), we may assume that 0 is the most often repeated value in A. Let L be the subsequence of all 0 in A, and write $l = |L|$ (here and below $|X|$ denotes the length of a sequence X). Clearly $l \le k-1$. We distinguish two cases.

Case 1: $l > r$. Then $|A\backslash L| < k$. Let H be a subsequence of maximal cardinality of $A\backslash L$ summing to 0 (H may be empty), and let $h = |H|$. Clearly $0 \leq h \leq k-1$, which implies that

$$l + h \leq k - 1, \tag{1}$$

for otherwise, H with $k-h$ zeros of L added would be a subsequence of A with length k summing to 0. Hence $|A\backslash L \cup H| \geq r+1$. Furthermore, $A\backslash L \cup H$ has no non-empty sum equal to 0 by the maximality of H. Take a subsequence $C \subseteq A\backslash L \cup H$ with $|C| = r+1$; then C has at least N_{r+1} non-empty sums (by the definition of N_{r+1}). It follows that $L \cup C$ has at least N_{r+1} $l+1$-sums (recall that $l > r$). Adding the sum of all elements of $A\backslash L \cup C$ to each $l+1$-sum of $L \cup C$, we obtain at least N_{r+1} k-sums of A (noting that $|A\backslash L \cup C| = k-l-1$). This proves (i) in Case 1.

Suppose now that $N_{r+1}(A) = r+1$. By Lemma 2 and the argument above, it follows easily that the elements in $A\backslash L \cup H$ must be all equal to some $c \in G$, and hence $r+1 \leq d_1 - 1$, where d_1 is the order of c.

If $H \neq \emptyset$, we claim that all elements in H are also equal to c. Suppose that there exists a $x \in H$ with $x \neq c$ (note that $x \neq 0$). Removing x and $r-1$ zeros from A we obtain a sequence of length k. Since $N_{k+r}(A) = r+1$, the sum of all elements of this sequence must be equal to some k-sum obtained in the above. It follows that there exists an integer t $(1 < t \leq r)$ such that $x = tc$. Then, replacing x in H by t elements c of $A\backslash L \cup H$, we obtain a subsequence H' of $A\backslash L$ summing to 0; but $|H'| > |H|$, contradicting the maximality of H. Hence the elements of $A\backslash L$ are all equal. This completes the proof of (ii) in Case 1.

Case 2: $l \leq r$. Then $|A\backslash L| \geq k$. By repeatedly applying Lemma 1 we can find a system of subsequences $S_1, \dots, S_q$ of $A\backslash L$ with the following properties:

(2) The S_j are disjoint.
(3) Each S_j sums to 0 and $2 \leq |S_j| \leq l$ $(j = 1, \dots, q)$.
(4) $|L \cup S_1 \cup \ldots \cup S_{q-1}| \leq r < |L \cup S_1 \cup \ldots \cup S_{q-1} \cup S_q|$

(where S_{q-1} is interpreted to be $\emptyset$ when $q = 1$).

Write

$$S = S_1 \cup \ldots \cup S_q, \qquad s = |S|. \tag{5}$$

Then by (4), $|A\backslash L \cup S| < k$. Let H be a subsequence of maximal cardinality of $A\backslash L \cup S$ summing to 0, and let $h = |H|$. Then $0 \leq h \leq k-1$; and, in analogy to (1), $l + h \leq k-1$. We claim that

$$|H \cup S| \leq k - 1. \tag{6}$$

To see this, we first note that $|H \cup S_1| = h + |S_1| \leq h + l \leq k-1$. Suppose

(6) is false. Then there exists some u $(1 \le u \le q-1)$ such that

$$|H \cup S_1 \cup \ldots \cup S_u| \le k-1 < |H \cup S_1 \cup \ldots \cup S_u \cup S_{u+1}|.$$

Since $|S_{u+1}| \le l$, it follows that $1 \le k - |H \cup S_1 \cup \ldots \cup S_u| \le l$. Then, by (2), (3) and the definition of H, $H \cup S_1 \cup \ldots \cup S_u$ with $k - |H \cup S_1 \cup \ldots \cup S_u|$ zeros of L added would be a subsequence of A with length k summing to 0, a contradiction. Hence (6) holds. Further, in analogy to (1), from (6) we deduce that $|L \cup H \cup S| \le k-1$. Hence $|A \backslash L \cup H \cup S| \ge r+1$. Take a subsequence $C \subseteq A \backslash L \cup H \cup S$ with $|C| = r+1$. Then C has no non-empty sum equal to 0 (by the maximality of H). Hence C has at least N_{r+1} non-empty sums.

We shall prove that $L \cup S \cup C$ has at least N_{r+1} $l+s+1$-sums. To do this it suffices to show that for each i-sum σ_i of C $(1 \le i \le r+1)$, $L \cup S \cup C$ has an $l+s+1$-sum equal to σ_i. We first note that $s \le r < l+s$ by using (4), (5) and $|S_q| \le l$. If $1 \le i \le l+1$, then $0 \le l+1-i \le l$. It is easily seen that $S \cup C$ with $l+1-i$ zeros from L appended has an $l+s+1$-sum equal to σ_i. If $s+1 \le i \le r+1$, then $0 \le l+s+1-i \le l$. It follows that C with $l+s+1-i$ zeros from L appended has an $l+s+1$-sum equal to σ_i. Thus we are done unless $s > l+1$. In the latter case, for $l+1 < i < s+1$, we have $i + |S_1| \le i + l < l+s+1 < i+s$. It follows that there exists a v $(1 \le v \le q-1)$ such that

$$i + |S_1 \cup \ldots \cup S_v| < l+s+1 \le i + |S_1 \cup \ldots \cup S_v \cup S_{v+1}|. \tag{7}$$

Recalling that $|S_{v+1}| \le l$, by (7) we have $1 \le l+s+1-i-|S_1 \cup \ldots \cup S_v| \le l$. Hence $C \cup S_1 \cup \ldots \cup S_v$ with $l+s+1-i-|S_1 \cup \ldots \cup S_v|$ zeros from L appended has an $l+s+1$-sum equal to σ_i. The desired result is thus proved.

Now, adding the sum of all elements of $A \backslash L \cup S \cup C$ to each of the $l+s+1$-sums of $L \cup S \cup C$, we obtain at least N_{r+1} k-sums of A (noting that $|A \backslash L \cup S \cup C| = k-l-s-1$). This completes the proof of (i) in Case 2.

Finally, since $l \le r$ and $|C| = r+1$, the elements in C cannot be all equal (recalling the definition of l). Hence, by Lemma 2, C has at least $r+2$ non-empty sums and thus we must have $N_{k+r}(A) > r+1$ in Case 2.

The proof of Theorems 1 and 2 is now complete.

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Department of Mathematics
Suzhou University
Suzhou 215006, Jiangsu, P.R. China
E-mail: yuhb@suda.edu.cn

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