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Approximate formulae for $L(1,\chi)$, II

by

Olivier Ramaré (Lille)

1. Introduction and results. Upper bounds of $|L(1,\chi)|$ are mainly useful in number theory to study class numbers of algebraic extensions. In [1]–[3] Louboutin establishes bounds for $|L(1,\chi)|$ that take into account the behavior of χ at small primes. His method uses special representations of $L(1,\chi)$ and does not extend to odd characters. For instance in [2] he uses $L(1,\chi)=2\sum_n\sum_{l\le n}\chi(l)/(n(n+1)(n+2))$ which comes from an integration by parts; such a formula fails in the odd case. But the effect of this integration by parts is in fact similar to the introduction of a smoothing, something we did in [5], the only difficulty being to handle properly the Fourier transform of functions behaving like $1/t$ near ∞. This method gives good numerical results in a uniform way.

In this note we improve on the results given in [2] and [3] and extend them to the odd character case. Let us mention that we take this opportunity to correct several typos occurring in [5].

We first state a general formula.

Theorem. *Let χ be a primitive Dirichlet character modulo q and let h be an integer prime to q. Let $F:\mathbb{R}\to\mathbb{R}$ be such that $f(t)=F(t)/t$ is in $C^2(\mathbb{R})$ (also at 0), vanishes at $\pm\infty$ and f' and f'' are in $\mathcal{L}^1(\mathbb{R})$. Assume also that F is even if χ is odd, and odd if χ is even. Then, for every $\delta>0$, we have*

$$\prod_{p\mid h}\Big(1-\frac{\chi(p)}{p}\Big)L(1,\chi)=\sum_{\substack{n\ge1\\(n,h)=1}}\chi(n)\,\frac{1-F(\delta n)}{n} + \frac{\chi(-h)\tau(\chi)}{qh}\sum_{m\ge1}c_h(m)\overline{\chi}(m)\int_{-\infty}^{\infty}\frac{F(t)}{t}\,e(mt/(\delta qh))\,dt.$$

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Here the Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e(a/q) \tag{1}$$

and the Ramanujan sums $c_h(m)$ by

$$c_h(m) = \sum_{a \bmod^* h} e(ma/q). \tag{2}$$

Of course $e(\cdot) = e^{2i\pi\cdot}$, and $a \bmod^* h$ denotes summation over all invertible residue classes modulo h. We further restrict our attention to square-free h.

Here are two interesting choices for F which we take directly from Proposition 2 of [5]. Set

$$F_3(t) = \left(\frac{\sin \pi t}{\pi}\right)^2 \left(\frac{2}{t} + \sum_{m \in \mathbb{Z}} \frac{\operatorname{sgn}(m)}{(t-m)^2}\right), \tag{3}$$

$$j(u) = \int_{-\infty}^{\infty} \frac{F_3(t)}{t} e(ut)\, dt = \mathbb{1}_{[-1,1]}(u) \int_{|u|}^{1} (\pi(1-t)\cot \pi t + 1)\, dt, \tag{4}$$

$$F_4(t) = 1 - \left(\frac{\sin \pi t}{\pi t}\right)^2 \tag{5}$$

which satisfies

$$\int_{-\infty}^{\infty} \frac{F_4(t)}{t} e(ut)\, dt = -i\pi(1-|u|)^2 \mathbb{1}_{[-1,1]}(u). \tag{6}$$

Notice furthermore that F_3 and F_4 take their values in $[0,1]$.

In order to compute efficiently the resulting sums we select several levels of hypotheses, starting by the most general ones. We use the Euler ϕ-function and the number $\omega(t)$ of distinct prime factors of t.

Corollary 1. *Let χ be a primitive Dirichlet character modulo q and h an integer prime to q. Assume q is divisible by a square-free k and set $\kappa_\chi = 0$ if χ is even, and $\kappa_\chi = 5 - 2\log 6 = 1.41648\ldots$ if χ is odd. Then*

$$\left|\prod_{p|h}\left(1 - \frac{\chi(p)}{p}\right) L(1,\chi)\right| - \frac{\phi(hk)}{2hk}\left[\log q + 2\sum_{p|hk} \frac{\log p}{p-1} + \omega(h)\log 4 + \kappa_\chi\right]$$

is bounded from above if χ is even and $q \ge k^2 4^{\omega(h)}$ by

$$\frac{\phi(h) 2^{\omega(k)-1}}{h\sqrt{q}} \times \begin{cases} \log(q 4^{-\omega(h)+1}) & \text{if } q \ge k^2 4^{\omega(h)}, \\ 1.81 + \omega(h)\log 4 - \log q & \text{if } k = 1, \end{cases}$$

and if χ is odd by

$$\frac{3\pi\phi(hk)}{2hkq}\prod_{p|hk}\frac{p^2-1}{4p^2}+\begin{cases}\dfrac{\pi\phi(h)2^{\omega(k)}}{2h\sqrt{q}} & \text{if } q>k^2\max\Big(\dfrac{11}{10}\cdot 4^{\omega(h)},\dfrac{4h^2}{4^{\omega(h)}}\Big),\\ 0 & \text{if } k=1.\end{cases}$$

This improves on Theorems 1, 4 and 5 of [3] in the quality of the bounds and in their range, and also by the fact that it covers the case of odd characters. For instance in Theorem 5 of [3], where Louboutin studies separately the cases $h=3$ and $k=2$, he gets the upper bound $\frac{1}{6}(\log q+4.83\ldots+o(1))$ for even characters, while we get $\frac{1}{6}(\log q+3.87\ldots+3(\log q)/\sqrt{q})$. Recently in [4], by generalizing his method introduced in [2], Louboutin has reached a similar result for the case of even characters, albeit with a slightly larger constant $\kappa_\chi=2+\gamma-\log(4\pi)=0.046\ldots$ instead of $\kappa_\chi=0$. This enabled him to replace $\frac{1}{6}(\log q+4.83\ldots+o(1))$ by $\frac{1}{6}(\log q+3.91\ldots)$.

Notice that the upper bound in the case of even characters is non-positive when $k=1$ as soon as $q\ge 6.2\cdot 4^{\omega(h)}$.

When $h=2$ we can get slightly more precise results:

Corollary 2. *Let χ be a primitive Dirichlet character modulo odd q. Then*

$$|(1-\chi(2)/2)L(1,\chi)|\le \tfrac{1}{4}(\log q+\kappa(\chi))$$

where $\kappa(\chi)=4\log 2$ if χ is even, and $\kappa(\chi)=5-2\log(3/2)$ otherwise.

In [2], the value $\kappa(\chi)\simeq 2.818\ldots$ is proved to hold true for even characters while $4\log 2=2.772\ldots$

We introduce the character ψ induced by χ modulo qh. Furthermore (m,t) denotes the gcd of m and t.

As for the typos in [5], first, Proposition 2 gives a wrong formula for $L(1,\chi)$ if χ is even: the sign preceding $\tau(\chi)$ should be $+$ and not $-$. Then Lemma 8 gives a fancy value for ϱ_4. In fact $\varrho_4(t)=-i\pi(1-|t|)^2\mathbb{1}_{[-1,1]}(t)$, which is what is proved and used throughout the paper! Finally, in the 6th line of page 264, it is written, "and this last summand is non-negative", while this summand is without any doubt non-positive.

We thank the referee for his careful reading and for improving Lemma 11.

2. Lemmas. We essentially combine Louboutin's proof [2] and ours [5], while generalizing both situations.

First here is a generalization of the new part in Louboutin's paper [2]:

Lemma 1. *For every m in $\mathbb{Z}$, we have*

$$\sum_{a \bmod qh}\psi(a)e(am/(qh))=c_h(m)\chi(h)\overline{\chi}(m)\tau(\chi).$$

Proof. By the Chinese remainder theorem,

$$\begin{aligned}\sum_{a \bmod hq} \psi(a)e(am/(hq)) &= \sum_{x \bmod h}\sum_{y \bmod q} \psi(xq+yh)e((xq+yh)m/(hq))\\ &= \sum_{x \bmod^* h} e(xm/h) \sum_{y \bmod q} \chi(yh)e(ym/q)\\ &= c_h(m)\chi(h)\overline{\chi}(m)\tau(\chi),\end{aligned}$$

where $c_h(m)$ is the Ramanujan sum defined by (2).

Now, Lemma 3 of [5] can be extended to

LEMMA 2. *The sum* $\sum_n^w f(\delta n)\chi(n)$ *exists in the restricted sense given in* [5] *and*

$$\sum_{n\in\mathbb{Z}}^{w} f(\delta n)\psi(n) = \frac{\chi(-h)\tau(\chi)}{qh}\sum_{m\in\mathbb{Z}\setminus\{0\}} c_h(m)\overline{\chi}(m)\int_{-\infty}^{\infty} f(\delta t)e(mt/(qh))\,dt.$$

Note: $\int_{-\infty}^{\infty} g(t)e(ut)\,dt = \lim_{T\to\infty}\int_{-T}^{T} g(t)e(ut)\,dt$ *for* $u \neq 0$.

Now we state and prove lemmas that give approximations of the relevant quantities.

LEMMA 3. *For* $\delta > 0$ *and* $hk \geq 2$ *we have*

$$\frac{hk}{\phi(hk)}\sum_{\substack{n\geq 1\\ (n,hk)=1}} \frac{1-F_3(\delta n)}{n} = -\log\delta - 1 + \sum_{p\mid hk}\frac{\log p}{p-1}.$$

Proof. We have

$$\begin{aligned}\sum_{\substack{n\geq 1\\ (n,hk)=1}} \frac{1-F_3(\delta n)}{n} &= \sum_{d\mid hk}\mu(d)\sum_{\substack{n\geq 1\\ d\mid n}}\frac{1-F_3(\delta n)}{n}\\ &= \sum_{d\mid hk}\frac{\mu(d)}{d}\sum_{n\geq 1}\frac{1-F_3(d\delta n)}{n}.\end{aligned}$$

Lemma 16 of [5] gives the value of the above if $hk = 1$, which is $-\log\delta - 1 + \delta$. This equality is stated only for $\delta \leq 1$ but since only analytic functions are involved, it naturally extends to $\delta > 0$. We infer that

$$\begin{aligned}\sum_{\substack{n\geq 1\\ (n,hk)=1}} \frac{1-F_3(\delta n)}{n} &= \sum_{d\mid hk}\frac{\mu(d)}{d}(-\log(d\delta) - 1 + d\delta)\\ &= -\frac{\phi(hk)}{hk}\log\delta - \frac{\phi(hk)}{hk} + \frac{\phi(hk)}{hk}\sum_{p\mid hk}\frac{\log p}{p-1}\end{aligned}$$

provided $hk \geq 2$.

LEMMA 4. *For $\delta uq \ge 1$ we have*

$$\delta uq - 2\log(e\delta uq) \le \sum_{1\le m\le \delta uq} j(m/(\delta uq)) \le \delta uq - \log(2\pi\delta uq/e).$$

The upper bound is proved between (6.3) and (6.4) in [5]. There also the restriction $\delta \le 1$ can be dispensed with. The lower bound comes simply from a comparison to an integral since j is non-increasing and since $j(t) \le -2\log|t|$ for $t \le 1$ (shown to be true in Lemma 7 of [5]),

$$\int_0^r j(t)\,dt \le -2(r\log r - r) \qquad (r\in[0,1]). \tag{7}$$

LEMMA 5. *For $\delta > 0$ and $h' = h/(2,h)$ we have*

$$\sum_{1\le m\le \delta q} \frac{\phi((m,h))}{\phi(h)}\, j(m/(\delta hq)) \le 2^{\omega(h)}\delta q + 1 - \log(2\pi\delta q) + \frac{H(h')}{\phi(h)}\sum_{p\mid h'}\frac{\log p}{p-2}.$$

Proof. Let us introduce the non-negative multiplicative function $H = \mu\star\phi$. We have $H(p) = p-2$. We get

$$\begin{aligned}\sum_{1\le m\le \delta q}\phi((m,h))j(m/(\delta q)) &= \sum_{d\mid h} H(d)\sum_{1\le m\le \delta q/d} j(dm/(\delta q))\\ &\le \sum_{d\mid h}\frac{hH(d)}{d}\,\delta q + \phi(h)(1-\log(2\pi\delta hq)) + \sum_{d\mid h} H(d)\log d.\end{aligned}$$

Now and since h is square-free we see that $\sum_{d\mid h} hH(d)/d = 2^{\omega(h)}\phi(h)$.

LEMMA 6. *For $\delta \ge k/q$ we have*

$$\sum_{\substack{1\le m\le \delta q\\ (m,k)=1}} \frac{\phi((m,h))}{\phi(h)}\, j(m/(\delta hq)) \le 2^{\omega(h)}\,\frac{\phi(k)}{k}\,\delta q + 2^{\omega(k)}\log(e\delta q/2).$$

Proof. Following the proof of Lemma 5, our sum equals

$$\begin{aligned}&\sum_{d\mid h} H(d)\sum_{l\mid k}\mu(l)\sum_{1\le m\le \delta q/(dl)} j(dlm/(\delta hq))\\ &\qquad\le \delta q 2^{\omega(h)}\phi(h)\,\frac{\phi(k)}{k} + \sum_{d\mid h} H(d)\sum_{\substack{l\mid k\\ \mu(l)=-1}} 2\log(e\delta q/(dl))\\ &\qquad\le \delta q 2^{\omega(h)}\phi(h)\,\frac{\phi(k)}{k} + \phi(h)2^{\omega(k)}\log(e\delta q/2)\end{aligned}$$

provided that $\delta q/k \ge 1$.

LEMMA 7. *For $\delta > 0$ and $hk \geq 2$ we have*

$$\frac{hk}{\phi(hk)} \sum_{\substack{n\geq 1\\ (n,hk)=1}} \frac{1-F_4(\delta n)}{n} = \log\delta + \frac{3}{2} - \log(2\pi) + \sum_{p|hk}\frac{\log p}{p-1}$$
$$+\frac{2\phi(hk)}{hk}\sum_{d|hk}\mu(d)\int_0^1 (1-t)\log\left|\frac{\pi d\delta t}{\sin(\pi d\delta t)}\right|\frac{dt}{d}.$$

When $hk = 2$ the last summand is non-positive, and in general if $\delta \leq 1/(2hk)$, it is not more than $\frac{\pi^3}{6}\delta^2\prod_{p|hk}(p^2-1)/p^2$.

Proof. Lemma 17 of [5] gives us

$$\sum_{n\geq 1}\frac{1-F_4(\delta n)}{n} = -\log\delta + \frac{3}{2} - \log(2\pi) + 2\int_0^1(1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt$$

and we use the same technique as in the previous lemma. The error term is non-positive if $hk = 2$ as shown in [5] between (7.2) and (7.3). Furthermore the integral is shown there (in Lemma 18) to be not more than $\pi^3\delta^2/12$ as soon as $\delta \leq 1/2$.

A simple comparison to an integral yields:

LEMMA 8. *For $\delta uq \geq 1$ we have*

$$\frac{\delta uq}{3} - 1 \leq \sum_{1\leq m\leq \delta uq}\left(1-\frac{m}{\delta uq}\right)^2 \leq \frac{\delta uq}{3}.$$

LEMMA 9. *For $\delta \geq k/q$ we have*

$$\sum_{\substack{1\leq m\leq \delta hq\\ (m,k)=1}} \frac{\phi((m,h))}{\phi(h)}\left(1-\frac{m}{\delta hq}\right)^2 \leq \frac{\phi(k)}{k}\,\frac{\delta q}{3}\,2^{\omega(h)} + 2^{\omega(k)-1}$$

where the last summand can be omitted if $k = 1$.

Proof. We proceed as in Lemma 6 to deduce that our sum is

$$\sum_{d|h} H(d)\sum_{l|k}\mu(l)\sum_{1\leq m\leq \delta q/(dl)}\left(1-\frac{dlm}{\delta q}\right)^2$$

and the conclusion follows readily.

From [6, (3.22), (2.11) and (3.26)], we get

LEMMA 10. *We have*

$$\sum_{1<p\le X}\frac{\log p}{p}\le \log X-1.332+\frac{1}{2\log X}\qquad (X\ge 319),$$

$$\prod_{2<p\le X}\frac{p-1}{p}\le\frac{2e^{-\gamma}}{\log X}\Big(1+\frac{1}{2\log^2 X}\Big)\qquad (X>1),$$

where γ is Euler's constant.

LEMMA 11. *For $h>1$, we have*

$$\prod_{2<p|h}\frac{p-2}{p-1}\sum_{2<p|h}\frac{\log p}{p-2}\le 0.7414.$$

Proof. First writing $h=h_1p_1$ where p_1 is a prime factor, the reader readily checks that our quantity is a non-increasing function of p_1. We thus find that its maximum is obtained when $h=\prod_{2<p\le X}p$. As a function of X, it numerically seems increasing and GP/PARI needs at most 10 seconds to prove it is ≤ 0.72 if the product is taken over primes $\le 10^6$. Using Lemma 10, we get

$$\begin{aligned}S(X)=\sum_{2<p\le X}\frac{\log p}{p-2}&=\sum_{2<p\le X}\frac{2\log p}{p(p-2)}+\sum_{1<p\le X}\frac{\log p}{p}-\frac{\log 2}{2}\\&\le 1.27+\log X-1.332+\frac{1}{2\log X}-0.346\\&\le \log X-0.4\end{aligned}$$

for $X\ge 10^6$. Furthermore, still invoking Lemma 10, we have

$$\begin{aligned}\Pi(X)&=\prod_{2<p\le X}\frac{p-2}{p-1}\\&\le\prod_{2<p\le X}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{2<p\le X}\frac{p-1}{p}\\&\le\prod_{2<p\le 10^6}\Big(1-\frac{1}{(p-1)^2}\Big)\frac{2e^{-\gamma}}{\log X}\Big(1+\frac{1}{2\log^2 X}\Big)\end{aligned}$$

also for $X\ge 10^6$. Since $(1-0.4y)(1+0.5y^2)\le 1$ if $0\le y\le 0.4$, our function is not more than

$$2e^{-\gamma}\prod_{2<p\le 10^6}\Big(1-\frac{1}{(p-1)^2}\Big)\le 0.7414. \tag{8}$$

3. Proof of the Theorem. Let us start with

$$L(1,\psi)=\sum_{n\geq 1}\psi(n)\,\frac{1-F(\delta n)}{n}+\sum_{n\geq 1}\psi(n)\,\frac{F(\delta n)}{n}. \tag{9}$$

Thanks to the hypothesis concerning the respective parities of F and χ, we get

$$\sum_{n\geq 1}\psi(n)\,\frac{F(\delta n)}{n}=\frac{1}{2}\sum_{n\in\mathbb{Z}}\psi(n)\delta f(\delta n), \tag{10}$$

to which we apply Lemma 2, and the Theorem follows readily.

4. Proofs of the corollaries. For even characters we take $F=F_3$. Combining the Theorem with Lemmas 3 and 6, and noticing that $|c_h(m)|\leq\phi((h,m))$, we get

$$\begin{aligned}\Big|\prod_{p|h}\Big(1-\frac{\chi(p)}{p}\Big)L(1,\chi)\Big|\frac{hk}{\phi(hk)}\\ \leq -\log\delta-1+\sum_{p|hk}\frac{\log p}{p-1}+\frac{1}{\sqrt{q}}\Big(2^{\omega(h)}\delta q+\frac{k2^{\omega(k)}}{\phi(k)}\log(e\delta q/2)\Big)\end{aligned} \tag{11}$$

provided $\delta\geq k/q$. We simply have to choose $\delta=1/(2^{\omega(h)}\sqrt{q})$ and the claimed formula follows readily.

For odd characters we use $F=F_4$ and Lemmas 7 and 9 to get

$$\begin{aligned}\Big|\prod_{p|h}\Big(1-\frac{\chi(p)}{p}\Big)L(1,\chi)\Big|\frac{hk}{\phi(hk)}\leq -\log\delta+\frac{3}{2}-\log(2\pi)\\ +\sum_{p|hk}\frac{\log p}{p-1}+\frac{\pi^3}{6}\,\delta^2\prod_{p|hk}\frac{p^2-1}{p^2}+\frac{\pi}{\sqrt{q}}\Big(\frac{\delta 2^{\omega(h)}q}{3}+2^{\omega(k)-1}\,\frac{k}{\phi(k)}\Big)\end{aligned} \tag{12}$$

provided $\delta\in[k/q,1/(2hk)]$. We take $\delta=3/(2^{\omega(h)}\pi\sqrt{q})$ and the claimed formula follows readily.

To prove the second corollary (i.e. with $k=1$), we simply adapt the above proof, but we can simplify the bound in the even case. We first obtain

$$\frac{1}{\sqrt{q}}\Big(1-\log((2\pi/e)\sqrt{q}\,2^{-\omega(h)})+\prod_{2<p|h}\frac{p-2}{p-1}\sum_{2<p|h}\frac{\log p}{p-2}\Big). \tag{13}$$

The last factor is bounded in Lemma 11 by 0.7414, so the above term is not more than $(1.81+\omega(h)\log 4-\log q)/(2\sqrt{q})$ as announced.

When $h=2$, the claimed upper bounds are proved if $q\geq 39$, in part because the term in δ^2 appearing in (12) disappears by Lemma 7. We complete the verification by appealing to GP/PARI as indicated in [5]. The maximum of $\kappa(\chi)$ for even characters of module ≤ 1000 is ≤ 1.705, attained for $q = 109$, while the maximum of $\kappa(\chi)$ for odd characters of module ≤ 1000 is ≤ 3.360, attained for $q = 131$.

References

[1] S. Louboutin, *Majorations au point 1 des fonctions L associées aux caractères de Dirichlet primitifs, ou au caractère d'une extension quadratique d'un corps quadratique imaginaire principal*, J. Reine Angew. Math. 419 (1991), 213–219.

[2] —, *Majorations explicites de $|L(1,\chi)|$ (quatrième partie)*, C. R. Acad. Sci. Paris Sér. I Math. 334 (2002), 625–628.

[3] —, *Explicit upper bounds for $|L(1,\chi)|$ for primitive even Dirichlet characters*, Acta Arith. 101 (2002), 1–18.

[4] —, *Explicit upper bounds for values at $s = 1$ of Dirichlet L-series associated with primitive even characters*, J. Number Theory, to appear.

[5] O. Ramaré, *Approximate formulae for $L(1,\chi)$*, Acta Arith. 100 (2001), 245–266.

[6] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.

UMR 8524
Université Lille I
59 655 Villeneuve d'Ascq Cedex, France
E-mail: ramare@agat.univ-lille1.fr

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