Approximate formulae for $L(1, \chi)$, II

by

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1. Introduction and results. Upper bounds of $|L(1, \chi)|$ are mainly useful in number theory to study class numbers of algebraic extensions. In [1]–[3] Louboutin establishes bounds for $|L(1, \chi)|$ that take into account the behavior of $\chi$ at small primes. His method uses special representations of $L(1, \chi)$ and does not extend to odd characters. For instance in [2] he uses $L(1, \chi) = 2 \sum_n \sum_{l \leq n} \chi(l)/(n(n+1)(n+2))$ which comes from an integration by parts; such a formula fails in the odd case. But the effect of this integration by parts is in fact similar to the introduction of a smoothing, something we did in [5], the only difficulty being to handle properly the Fourier transform of functions behaving like $1/t$ near $1$. This method gives good numerical results in a uniform way.

In this note we improve on the results given in [2] and [3] and extend them to the odd character case. Let us mention that we take this opportunity to correct several typos occurring in [5].

We first state a general formula.

Theorem. Let $\chi$ be a primitive Dirichlet character modulo $q$ and let $h$ be an integer prime to $q$. Let $F : \mathbb{R} \to \mathbb{R}$ be such that $f(t) = F(t)/t$ is in $C^2(\mathbb{R})$ (also at 0), vanishes at $\pm \infty$ and $f'$ and $f''$ are in $L^1(\mathbb{R})$. Assume also that $F$ is even if $\chi$ is odd, and odd if $\chi$ is even. Then, for every $\delta > 0$, we have

$$
\prod_{p|h} \left(1 - \frac{\chi(p)}{p}\right) L(1, \chi) = \sum_{n \geq 1} \chi(n) \frac{1 - F(\delta n)}{n} \\
+ \frac{\chi(-h)\tau(\chi)}{qh} \sum_{m \geq 1} c_h(m) \overline{\chi}(m) \int_{-\infty}^{\infty} \frac{F(t)}{t} e(mt/(\delta qh)) dt.
$$

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Here the Gauss sum $\tau(\chi)$ is defined by
\begin{equation}
\tau(\chi) = \sum_{a \mod q} \chi(a)e(a/q)
\end{equation}
and the Ramanujan sums $c_h(m)$ by
\begin{equation}
c_h(m) = \sum_{a \mod^* h} e(ma/q).
\end{equation}
Of course $e(\cdot) = e^{2i\pi \cdot}$, and $a \mod^* h$ denotes summation over all invertible residue classes modulo $h$. We further restrict our attention to square-free $h$.

Here are two interesting choices for $F$ which we take directly from Proposition 2 of [5]. Set
\begin{equation}
F_3(t) = \left(\frac{\sin \pi t}{\pi}\right)^2 \left(\frac{2}{t} + \sum_{m \in \mathbb{Z}} \frac{\text{sgn}(m)}{(t-m)^2}\right),
\end{equation}
\begin{equation}
j(u) = \int_{-\infty}^{\infty} \frac{F_3(t)}{t} e(ut) dt = 1_{[-1,1]}(u) \frac{1}{|u|} \int_{|u|}^{\infty} (\pi(1-t) \cot \pi t + 1) dt,
\end{equation}
\begin{equation}
F_4(t) = 1 - \left(\frac{\sin \pi t}{\pi t}\right)^2
\end{equation}
which satisfies
\begin{equation}
\int_{-\infty}^{\infty} \frac{F_4(t)}{t} e(ut) dt = -i\pi (1 - |u|)^2 \text{Im}_{[-1,1]}(u).
\end{equation}
Notice furthermore that $F_3$ and $F_4$ take their values in $[0,1]$.

In order to compute efficiently the resulting sums we select several levels of hypotheses, starting by the most general ones. We use the Euler $\phi$-function and the number $\omega(t)$ of distinct prime factors of $t$.

**Corollary 1.** Let $\chi$ be a primitive Dirichlet character modulo $q$ and $h$ an integer prime to $q$. Assume $q$ is divisible by a square-free $k$ and set $\kappa_\chi = 0$ if $\chi$ is even, and $\kappa_\chi = 5 - 2 \log 6 = 1.41648 \ldots$ if $\chi$ is odd. Then
\[
\prod_{p|h} \left(1 - \frac{\chi(p)}{p}\right)L(1, \chi) = -\phi(hk) \left[ \frac{\log q + 2}{2hk} \sum_{p|hk} \frac{\log p}{p-1} + \omega(h) \log 4 + \kappa_\chi \right]
\]
is bounded from above if $\chi$ is even and $q \geq k^2 4^{\omega(h)}$ by
\[
\frac{\phi(h)2^{\omega(k)-1}}{h \sqrt{q}} \times \begin{cases} 
\log(q4^{-\omega(h)+1}) & \text{if } q \geq k^2 4^{\omega(h)}, \\
1.81 + \omega(h) \log 4 - \log q & \text{if } k = 1,
\end{cases}
\]
and if $\chi$ is odd by
\[
\frac{3\pi \phi(hk)}{2hkq} \prod_{p|h} \frac{p^2 - 1}{4p^2} + \left\{ \begin{array}{ll}
\frac{\pi \phi(h)2^{\omega(k)}}{2h\sqrt{q}} & \text{if } q > k^2 \max\left(\frac{11}{10} \cdot 4^{\omega(h)}, \frac{4h^2}{4^{\omega(h)}}\right), \\
0 & \text{if } k = 1.
\end{array} \right.
\]

This improves on Theorems 1, 4 and 5 of [3] in the quality of the bounds and in their range, and also by the fact that it covers the case of odd characters. For instance in Theorem 5 of [3], where Louboutin studies separately the cases $h = 3$ and $k = 2$, he gets the upper bound $\frac{1}{6}(\log q + 4.83\ldots + o(1))$ for even characters, while we get $\frac{1}{6}(\log q + 3.87\ldots + 3(\log q)/\sqrt{q})$. Recently in [4], by generalizing his method introduced in [2], Louboutin has reached a similar result for the case of even characters, albeit with a slightly larger constant $\kappa_\chi = 2 + \gamma - \log(4\pi) = 0.046\ldots$ instead of $\kappa_\chi = 0$. This enabled him to replace $\frac{1}{6}(\log q + 4.83\ldots + o(1))$ by $\frac{1}{6}(\log q + 3.91\ldots)$.

Notice that the upper bound in the case of even characters is non-positive when $k = 1$ as soon as $q \geq 6.2 \cdot 4^{\omega(h)}$.

When $h = 2$ we can get slightly more precise results:

**Corollary 2.** Let $\chi$ be a primitive Dirichlet character modulo odd $q$. Then
\[
|1 - \chi^2(2)/2)_{L(1, \chi)}| \leq \frac{1}{4}(\log q + \kappa(\chi))
\]
where $\kappa(\chi) = 4\log 2$ if $\chi$ is even, and $\kappa(\chi) = 5 - 2\log(3/2)$ otherwise.

In [2], the value $\kappa(\chi) \simeq 2.818\ldots$ is proved to hold true for even characters while $4\log 2 = 2.772\ldots$.

We introduce the character $\psi$ induced by $\chi$ modulo $qh$. Furthermore $(m, t)$ denotes the gcd of $m$ and $t$.

As for the typos in [5], first, Proposition 2 gives a wrong formula for $L(1, \chi)$ if $\chi$ is even: the sign preceding $\tau(\chi)$ should be $+$ and not $-$. Then Lemma 8 gives a fancy value for $g_4$. In fact $g_4(t) = -i\pi(1 - |t|)^2\mathbb{1}_{[-1,1]}(t)$, which is what is proved and used throughout the paper! Finally, in the 6th line of page 264, it is written, “and this last summand is non-negative”, while this summand is without any doubt non-positive.

We thank the referee for his careful reading and for improving Lemma 11.

2. **Lemmas.** We essentially combine Louboutin’s proof [2] and ours [5], while generalizing both situations.

First here is a generalization of the new part in Louboutin’s paper [2]:

**Lemma 1.** For every $m$ in $\mathbb{Z}$, we have
\[
\sum_{a \pmod{qh}} \psi(a)e(am/(qh)) = c_h(m)\chi(h)\overline{\chi}(m)\tau(\chi).
\]
Proof. By the Chinese remainder theorem,

\[
\sum_{a \mod hq} \psi(a)e(am/(hq)) = \sum_{x \mod h} \sum_{y \mod q} \psi(x+y)e((x+y)m/(hq))
\]

\[
= \sum_{x \mod* h} e(xm/h) \sum_{y \mod q} \chi(yh)e(ym/q)
\]

\[
= c_h(m)\chi(h)\chi(m)\tau(\chi),
\]

where \(c_h(m)\) is the Ramanujan sum defined by (2).

Now, Lemma 3 of [5] can be extended to

**Lemma 2.** The sum \(\sum_{n} f(\delta n)\chi(n)\) exists in the restricted sense given in [5] and

\[
\sum_{n \in \mathbb{Z}} f(\delta n)\psi(n) = \frac{\chi(-h)\tau(\chi)}{qh} \sum_{m \in \mathbb{Z}\setminus\{0\}} c_h(m)\chi(m) \int_{-\infty}^{\infty} f(\delta t)e(mt/(qh)) \, dt.
\]

Note: \(\int_{-\infty}^{\infty} g(t)e(ut) \, dt = \lim_{T \to \infty} \int_{-T}^{T} g(t)e(ut) \, dt\) for \(u \neq 0\).

Now we state and prove lemmas that give approximations of the relevant quantities.

**Lemma 3.** For \(\delta > 0\) and \(hk \geq 2\) we have

\[
\frac{hk}{\phi(hk)} \sum_{\substack{n \geq 1 \\ (n,hk)=1}} \frac{1-F_3(\delta n)}{n} = -\log \delta - 1 + \sum_{p|h} \frac{\log p}{p-1}.
\]

**Proof.** We have

\[
\sum_{\substack{n \geq 1 \\ (n,hk)=1}} \frac{1-F_3(\delta n)}{n} = \sum_{d|hk} \mu(d) \sum_{\substack{n \geq 1 \\ d|n}} \frac{1-F_3(\delta n)}{n}
\]

\[
= \sum_{d|hk} \mu(d) \sum_{n \geq 1} \frac{1-F_3(d\delta n)}{n}.
\]

Lemma 16 of [5] gives the value of the above if \(hk = 1\), which is \(-\log \delta - 1 + \delta\). This equality is stated only for \(\delta \leq 1\) but since only analytic functions are involved, it naturally extends to \(\delta > 0\). We infer that

\[
\sum_{\substack{n \geq 1 \\ (n,hk)=1}} \frac{1-F_3(\delta n)}{n} = \sum_{d|hk} \frac{\mu(d)}{d} (-\log(d\delta) - 1 + d\delta)
\]

\[
= -\frac{\phi(hk)}{hk} \log \delta - \frac{\phi(hk)}{hk} + \frac{\phi(hk)}{hk} \sum_{p|hk} \frac{\log p}{p-1}
\]

provided \(hk \geq 2\).
Lemma 4. For $\delta uq \geq 1$ we have

$$
\delta uq - 2\log(e\delta uq) \leq \sum_{1 \leq m \leq \delta uq} j(m/(\delta uq)) \leq \delta uq - \log(2\pi \delta uq/e).
$$

The upper bound is proved between (6.3) and (6.4) in [5]. There also the restriction $\delta \leq 1$ can be dispensed with. The lower bound comes simply from a comparison to an integral since $j$ is non-increasing and since $j(t) \leq -2\log|t|$ for $t \leq 1$ (shown to be true in Lemma 7 of [5]),

$$
(7) \quad \int_0^r j(t) \, dt \leq -2(r \log r - r) \quad (r \in [0, 1]).
$$

Lemma 5. For $\delta > 0$ and $h' = h/(2, h)$ we have

$$
\sum_{1 \leq m \leq \delta q} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta hq)) \leq 2^{\omega(h)} \delta q + 1 - \log(2\pi \delta q) + \frac{H(h')}{\phi(h)} \sum_{p|h'} \log p \frac{p}{p - 2}.
$$

Proof. Let us introduce the non-negative multiplicative function $H = \mu \ast \phi$. We have $H(p) = p - 2$. We get

$$
\sum_{1 \leq m \leq \delta q} \phi((m, h)) j(m/(\delta q)) = \sum_{d|h} H(d) \sum_{1 \leq m \leq \delta q/d} j(dm/(\delta q))
$$

$$
\leq \sum_{d|h} \frac{hH(d)}{d} \delta q + \phi(h)(1 - \log(2\pi \delta hq)) + \sum_{d|h} H(d) \log d.
$$

Now and since $h$ is square-free we see that $\sum_{d|h} hH(d)/d = 2^{\omega(h)} \phi(h)$.

Lemma 6. For $\delta \geq k/q$ we have

$$
\sum_{1 \leq m \leq \delta q \atop (m, k) = 1} \frac{\phi((m, h))}{\phi(h)} j(m/(\delta hq)) \leq 2^{\omega(h)} \phi(k) \frac{k}{k} \delta q + 2^{\omega(k)} \log(e\delta q/2).
$$

Proof. Following the proof of Lemma 5, our sum equals

$$
\sum_{d|h} H(d) \sum_{l|k} \mu(l) \sum_{1 \leq m \leq \delta q/(dl)} j(dm/(\delta hq))
$$

$$
\leq \delta q2^{\omega(h)} \phi(h) \frac{\phi(k)}{k} + \sum_{d|h} H(d) \sum_{l|k \atop \mu(l) = -1} 2\log(e\delta q/(dl))
$$

$$
\leq \delta q2^{\omega(h)} \phi(h) \frac{\phi(k)}{k} + \phi(h)2^{\omega(k)} \log(e\delta q/2)
$$

provided that $\delta q/k \geq 1$. 

Lemma 7. For \( \delta > 0 \) and \( hk \geq 2 \) we have

\[
\frac{hk}{\phi(hk)} \sum_{n \geq 1, (n, hk) = 1} \frac{1 - F_4(\delta n)}{n} = \log \delta + \frac{3}{2} - \log(2\pi) + \sum_{p \mid hk} \frac{\log p}{p - 1} + \frac{2\phi(hk)}{hk} \sum_{d \mid hk} \mu(d) \int_0^1 (1 - t) \log \left| \frac{\pi d\delta t}{\sin(\pi d\delta t)} \right| \, dt.
\]

When \( hk = 2 \) the last summand is non-positive, and in general if \( \delta \leq 1/(2hk) \), it is not more than \( \frac{\pi^3}{6} \delta^2 \prod_{p \mid hk} (p^2 - 1)/p^2 \).


\[
\sum_{n \geq 1} \frac{1 - F_4(\delta n)}{n} = - \log \delta + \frac{3}{2} - \log(2\pi) + 2 \int_0^1 (1 - t) \log \left| \frac{\pi \delta t}{\sin(\pi \delta t)} \right| \, dt
\]

and we use the same technique as in the previous lemma. The error term is non-positive if \( hk = 2 \) as shown in [5] between (7.2) and (7.3). Furthermore the integral is shown there (in Lemma 18) to be not more than \( \frac{\pi^3}{6} \delta^2/12 \) as soon as \( \delta \leq 1/2 \).

A simple comparison to an integral yields:

Lemma 8. For \( \delta uq \geq 1 \) we have

\[
\frac{\delta uq}{3} - 1 \leq \sum_{1 \leq m \leq \delta uq} \left( 1 - \frac{m}{\delta uq} \right)^2 \leq \frac{\delta uq}{3}.
\]

Lemma 9. For \( \delta \geq k/q \) we have

\[
\sum_{1 \leq m \leq \delta hq, (m, k) = 1} \frac{\phi((m, h))}{\phi(h)} \left( 1 - \frac{m}{\delta hq} \right)^2 \leq \frac{\phi(k)}{k} \frac{\delta q}{3} 2^{\omega(h)} + 2^{\omega(k)-1}
\]

where the last summand can be omitted if \( k = 1 \).

Proof. We proceed as in Lemma 6 to deduce that our sum is

\[
\sum_{d \mid h} H(d) \sum_{l \mid k} \mu(l) \sum_{1 \leq m \leq \delta q/(dl)} \left( 1 - \frac{dlm}{\delta q} \right)^2
\]

and the conclusion follows readily.

From [6, (3.22), (2.11) and (3.26)], we get
**Lemma 10.** We have

\[ \sum_{1 < p \leq X} \frac{\log p}{p} \leq \log X - 1.332 + \frac{1}{2 \log X} \quad (X \geq 319), \]

\[ \prod_{2 < p \leq X} \frac{p - 1}{p} \leq \frac{2e^{-\gamma}}{\log X} \left( 1 + \frac{1}{2 \log^2 X} \right) \quad (X > 1), \]

where \( \gamma \) is Euler’s constant.

**Lemma 11.** For \( h > 1 \), we have

\[ \prod_{2 < p | h} \frac{p - 2}{p - 1} \sum_{2 < p \leq X} \frac{\log p}{p - 2} \leq 0.7414. \]

**Proof.** First writing \( h = h_1 p_1 \) where \( p_1 \) is a prime factor, the reader readily checks that our quantity is a non-increasing function of \( p_1 \). We thus find that its maximum is obtained when \( h = \prod_{2 < p \leq X} p \). As a function of \( X \), it numerically seems increasing and GP/PARI needs at most 10 seconds to prove it is \( \leq 0.72 \) if the product is taken over primes \( \leq 10^6 \). Using Lemma 10, we get

\[ S(X) = \sum_{2 < p \leq X} \frac{\log p}{p - 2} = \sum_{2 < p \leq X} \frac{2 \log p}{p(p - 2)} + \sum_{1 < p \leq X} \frac{\log p}{p} - \frac{\log 2}{2} \]

\[ \leq 1.27 + \log X - 1.332 + \frac{1}{2 \log X} - 0.346 \]

\[ \leq \log X - 0.4 \]

for \( X \geq 10^6 \). Furthermore, still invoking Lemma 10, we have

\[ II(X) = \prod_{2 < p \leq X} \frac{p - 2}{p - 1} \]

\[ \leq \prod_{2 < p \leq 10^6} \left( 1 - \frac{1}{(p - 1)^2} \right)^{\prod_{2 < p \leq X} \frac{p - 1}{p}} \]

\[ \leq \prod_{2 < p \leq 10^6} \left( 1 - \frac{1}{(p - 1)^2} \right) \frac{2e^{-\gamma}}{\log X} \left( 1 + \frac{1}{2 \log^2 X} \right) \]

also for \( X \geq 10^6 \). Since \((1 - 0.4y)(1 + 0.5y^2) \leq 1\) if \( 0 \leq y \leq 0.4 \), our function is not more than

\[ 2e^{-\gamma} \prod_{2 < p \leq 10^6} \left( 1 - \frac{1}{(p - 1)^2} \right) \leq 0.7414. \]
3. Proof of the Theorem. Let us start with

\[ L(1, \psi) = \sum_{n \geq 1} \psi(n) \frac{1 - F(\delta n)}{n} + \sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n}. \]

Thanks to the hypothesis concerning the respective parities of \( F \) and \( \chi \), we get

\[ \sum_{n \geq 1} \psi(n) \frac{F(\delta n)}{n} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \psi(n) \delta f(\delta n), \]

to which we apply Lemma 2, and the Theorem follows readily.

4. Proofs of the corollaries. For even characters we take \( F = F_3 \). Combining the Theorem with Lemmas 3 and 6, and noticing that \( |\chi_h(m)| \leq \phi((h, m)) \), we get

\[ \prod_{p \mid h} \left( 1 - \frac{\chi(p)}{p} \right) L(1, \chi) \frac{hk}{\phi(hk)} \leq -\log \delta - 1 + \sum_{p \mid h} \frac{\log p}{p - 1} + \frac{1}{\sqrt{q}} \left( 2^{\omega(h)} \delta q + \frac{k2^{\omega(k)}}{\phi(k)} \log(e\delta q/2) \right) \]

provided \( \delta \geq k/q \). We simply have to choose \( \delta = 1/(2^{\omega(h)} \sqrt{q}) \) and the claimed formula follows readily.

For odd characters we use \( F = F_4 \) and Lemmas 7 and 9 to get

\[ \prod_{p \mid h} \left( 1 - \frac{\chi(p)}{p} \right) L(1, \chi) \frac{hk}{\phi(hk)} \leq -\log \delta + \frac{3}{2} - \log(2\pi) \]

\[ + \sum_{p \nmid h} \frac{\log p}{p - 1} + \frac{\pi^3}{6} \delta^2 \prod_{p \nmid h} \frac{p^2 - 1}{p^2} + \frac{\pi}{\sqrt{q}} \left( \frac{\delta 2^{\omega(h)} q}{3} + 2^{\omega(k)-1} \frac{k}{\phi(k)} \right) \]

provided \( \delta \in [k/q, 1/(2hk)] \). We take \( \delta = 3/(2^{\omega(h)} \pi \sqrt{q}) \) and the claimed formula follows readily.

To prove the second corollary (i.e. with \( k = 1 \)), we simply adapt the above proof, but we can simplify the bound in the even case. We first obtain

\[ \frac{1}{\sqrt{q}} \left( 1 - \log((2\pi/e)\sqrt{q}2^{-\omega(h)}) + \prod_{2 < p \nmid h} \frac{p - 2}{p - 1} \sum_{2 < p \nmid h} \frac{\log p}{p - 2} \right). \]

The last factor is bounded in Lemma 11 by 0.7414, so the above term is not more than \((1.81 + \omega(h) \log 4 - \log q)/(2\sqrt{q})\) as announced.

When \( h = 2 \), the claimed upper bounds are proved if \( q \geq 39 \), in part because the term in \( \delta^2 \) appearing in (12) disappears by Lemma 7. We complete the verification by appealing to GP/PARI as indicated in [5]. The maximum of \( \kappa(\chi) \) for even characters of module \( \leq 1000 \) is \( \leq 1.705 \), attained
for \( q = 109 \), while the maximum of \( \kappa(\chi) \) for odd characters of module \( \leq 1000 \) is \( \leq 3.360 \), attained for \( q = 131 \).

References


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