ACTA ARITHMETICA
112.4 (2004)

2-extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel

by

Mikaël Lescop (Limoges and Dublin)

Introduction. Let F be a number field with ring of integers o_F. Let F_v denote the local field at a finite or real infinite prime v. For K a number field or a local field, let $\mu(K)$ be the group of roots of unity of K and, for a finite group A, denote by $A(2)$ its 2-primary part. Furthermore, let K_2 be the functor of Milnor.

The Hilbert kernel or wild kernel $\mathrm{WK}_2(F)$ of F is defined to be the kernel of the map $K_2(F) \to \bigoplus_v \mu(F_v)$, given by the $|\mu(F_v)|$th power norm residue symbol at all finite or real infinite primes v. Moore's exact sequence states that

$$0 \to \mathrm{WK}_2(F) \to K_2(F) \to \bigoplus_v \mu(F_v) \to \mu(F) \to 0,$$

where v runs through all the finite and real infinite primes of F.

H. Garland proved in [Ga] that the Hilbert kernel is a finite abelian group. Moreover, the 2-primary part $\mathrm{WK}_2(F)(2)$ of the Hilbert kernel of F fits into the exact sequence

$$0 \to \mathrm{WK}_2(F)(2) \to K_2(o_F)(2) \to \bigoplus_{v|2} \mu(F_v)(2) \bigoplus_{v\,\mathrm{real}} \mu_2 \to \mu(F)(2) \to 0.$$

J. Browkin and A. Schinzel computed in [BS] the 2-rank of the Hilbert kernel of quadratic fields. It is an easy consequence of their results that the fields $F = \mathbb{Q}(\sqrt{d})$, which have trivial 2-primary Hilbert kernel, are given by the following values of the squarefree integer d:

- $-1, \pm 2,$
- $\pm p, \pm 2p$ $\quad$ p a prime with $p \equiv \pm 3 \bmod 8$,
- $-p$ $\quad$ p a prime with $p \equiv 7 \bmod 8$,

2000 *Mathematics Subject Classification*: 11R70, 19F15.
Key words and phrases: Hilbert (or wild) and tame kernels, genus formula.

- p — p a prime with $p \equiv 1 \bmod 8$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$,
- pq — p, q primes with $p \equiv q \equiv 3 \bmod 8$,
- $-pq$ — p, q primes with $p \equiv -q \equiv 3 \bmod 8$.

More recently, M. Kolster and A. Movahhedi established in [KM] a genus formula for Hilbert kernels of a relative quadratic extension and consequently classified all bi-quadratic extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel. We recall that the term bi-quadratic means here the compositum of two quadratic fields. Finally, R. Griffiths determined, with the previous genus formula, all multi-quadratic extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel (cf. [Gr]; see also on this subject [C]).

In this paper, we give a complete list of cyclic 2-extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel. Because of the above list, we concentrate on extensions of degree at least 4. This work is based on the list of all quadratic extensions with trivial 2-primary Hilbert kernel and on the genus formula for Hilbert kernels of a relative quadratic extension stated in [KM]. More precisely, the main theorem we will prove, and which is contained in Proposition 2.4 and Theorem 3.4, is the following:

Theorem. *For $n \geq 2$, let μ_n denote the group of nth roots of unity and $\mathbb{Q}(\mu_{2^\infty})$ be the union of all the fields $\mathbb{Q}(\mu_{2^s})$ $(s \geq 1)$. Then all cyclic 2-extensions of $\mathbb{Q}$ of degree at least 4 with trivial 2-primary Hilbert kernel are the following*:

(1) *the cyclic subfields of degree at least 4 of the composite*

$$\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{-p}) \quad \text{with } p \equiv 3 \bmod 8,$$

(2) *the cyclic subfields of degree at least 4 of the composite*

$$\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}\big(\sqrt{2\sqrt{p}(a - \sqrt{p})}\,\big)$$

with $p \equiv 5 \bmod 8$ (where $p = a^2 + b^2$ in $\mathbb{Z}$, with a odd),

(3) *the unique subfield of degree 4 of $\mathbb{Q}(\mu_p)$, where p is a prime with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$*,

(4) *the unique subfield of degree 8 of $\mathbb{Q}(\mu_p)$, where p is a prime with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$ and $2^{(p-1)/8} \equiv 1 \bmod p$.*

Thus, we note that the extensions appearing in cases (3) and (4) are the only cyclic 2-extensions of $\mathbb{Q}$ of degree at least 4 with trivial 2-primary Hilbert kernel, but with non-trivial 2-primary positive tame kernel (compare with results of Gras in [G]). Furthermore, we determine in the final section all *totally real* abelian 2-extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel (see Theorem 4.3). In particular, we show that, for totally real abelian 2-extensions of degree ≥ 8, the triviality of the 2-primary Hilbert kernel is equivalent to that of the 2-primary positive tame kernel.

Acknowledgements. The author would like to thank Manfred Kolster and his supervisor Abbas Movahhedi for many helpful discussions and relevant suggestions.

1. Genus formula. This section is devoted to finding information on the 2-primary Hilbert kernel of a cyclic 2-extension of $\mathbb{Q}$. The main tool used here is the genus formula for relative quadratic extensions (see [KM] for the general description).

Let $E/\mathbb{Q}$ be a cyclic 2-extension of $\mathbb{Q}$ of degree at least 4. We denote by F the unique subfield of E such that $[E:F]=2$ and by $G=\mathrm{Gal}(E/F)$ the Galois group of E/F. Moreover, let $K=\mathbb{Q}(\sqrt{d})$ be the quadratic subfield of E; note that d is a positive squarefree integer which is a sum of two squares, since K is embedded in a cyclic extension (see [S3] for a proof).

If $K\neq\mathbb{Q}(\sqrt{2})$, an odd prime ramifies in $K/\mathbb{Q}$ and the relative quadratic extension E/F is then ramified at a non-dyadic prime (since in a cyclic 2-extension, a rational prime must first decompose, then stay inert and finally ramify). Consequently, by Proposition 2.2 in [KM], the transfer map $\mathrm{WK}_2(E)(2)\to\mathrm{WK}_2(F)(2)$ is surjective. Hence, if $\mathrm{WK}_2(E)(2)$ is trivial, so is $\mathrm{WK}_2(F)(2)$. By a similar argument, we can even say that, in the case where $K\neq\mathbb{Q}(\sqrt{2})$, if $\mathrm{WK}_2(E)(2)$ is trivial, so is $\mathrm{WK}_2(L)(2)$ for all subfields L of E. Now, using the fact that, if p is an odd prime dividing d, then p is congruent to 1 modulo 4 (since d is squarefree and a sum of two squares), the above list of all quadratic extensions with trivial 2-primary Hilbert kernel implies the following result: if $\mathrm{WK}_2(E)(2)$ is trivial, then $K=\mathbb{Q}(\sqrt{d})$ where the only possible values of d are:

- 2,
- p or $2p$, $\quad p$ a prime with $p\equiv 5 \bmod 8$,
- p $\quad p$ a prime with $p\equiv 1 \bmod 8$,
  $p\neq x^2-32y^2$, $x>0$, $x\equiv 1 \bmod 4$.

In fact, for reasons which will become clear later, we will firstly concentrate, in Section 2, on the case where $d=2$, p or $2p$ ($p\equiv 5 \bmod 8$) and, secondly, in Section 3, on the case where $p\equiv 1 \bmod 8$, $p\neq x^2-32y^2$, $x>0$, $x\equiv 1 \bmod 4$.

Let $T_{E/F}$ be the finite set of the primes of F which are tamely ramified in E, and the dyadic primes v of F, undecomposed in E, for which either $\mu(E_w)(2) = \mu(F_v)(2)$, or E_w is not contained in the cyclotomic $\mathbb{Z}_2$-extension of F_v, where w is the prime above v in E.

LEMMA 1.1. *Let $E/\mathbb{Q}$ be a cyclic 2-extension of $\mathbb{Q}$ and denote by F the unique subfield of E such that $[E : F] = 2$. A dyadic prime v of F which is undecomposed in E always belongs to $T_{E/F}$.*

Proof. Let w be the prime above v in E. Assume that $v \notin T_{E/F}$. We then have $|\mu(E_w)(2)| > |\mu(F_v)(2)|$ and E_w is contained in the cyclotomic $\mathbb{Z}_2$-extension of F_v, and therefore i and $\sqrt{2}$ belong to E_w; hence, the cyclic extension E_w of $\mathbb{Q}_2$ contains the two quadratic extensions $\mathbb{Q}_2(i)$ and $\mathbb{Q}_2(\sqrt{2})$, which is impossible. ∎

Let $\mathrm{N}_{E/F}$ denote the norm map of the extension E/F and denote by D_F the Tate kernel of F: by definition, $D_F = \{x \in F^* \mid \{-1, x\} = 1 \in K_2(F)\}$. Since in our case F is totally real, the Genus Formula 2.8 in [KM] becomes

$$\frac{|\mathrm{WK}_2(E)(2)_G|}{|\mathrm{WK}_2(F)(2)|} = \frac{2^{|T_{E/F}|-\varrho}}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]},$$

where $\varrho \in \{0, 1\}$.

Moreover, if $\mathrm{WK}_2(E)(2) = 0$, then $|T_{E/F}| \le 2$. Indeed, we know (cf. Theorem 6.3 in [T]) that the index of F^{*2} in the Tate kernel D_F is equal to 2 in the case where F is totally real and consequently the index appearing in the genus formula is equal to 1 or 2 depending on whether the generator of D_F/F^{*2} is an element of $\mathrm{N}_{E/F}(E^*)$ or not. So the result follows from the genus formula.

We have to be more precise about ϱ. We know that if either a real infinite prime of F ramifies in E, or $\mu(F_v)(2) = \mu(F)(2)$ for some prime $v \in T_{E/F}$, then $\varrho = 1$. Otherwise, the precise value of ϱ can be computed as follows: In the remaining cases no real infinite prime of F ramifies in E and for all primes $v \in T_{E/F}$, $|\mu(F_v)(2)| > |\mu(F)(2)|$. Now, to begin with, we denote by S the set of all dyadic primes of F of all finite primes of F which ramify in E, and of all real infinite primes of F, and by S_E the set of all non-complex extensions of primes in S to E. The integral closure of the ring of S-integers o_F^S of F in E is simply denoted by o_E^S. Moreover, for a local field M, let D_M denote the kernel of the map $M^* \to \mu_2$ given by $x \mapsto (-1, x)_m$ where $m = |\mu(M)(2)|$ and $(\cdot, \cdot)_m$ denotes the Hilbert symbol with values in $\mu(M)(2)$. Now, [KM] gives the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{WK}_2(F)(2) & \longrightarrow & K_2(o_F^S)(2) & \longrightarrow & \bigoplus_{v\in S}\mu(F_v)(2) \\
 & & \downarrow & & \downarrow & & \downarrow j_S \\
0 & \longrightarrow & \mathrm{WK}_2(E)(2)^G & \xrightarrow{\gamma} & K_2(o_E^S)(2)^G & \xrightarrow{\alpha} & \Big(\bigoplus_{w\in S_E}\mu(E_w)(2)\Big)^G \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{WK}_2(E)(2)^G/\mathrm{Im}\,\mathrm{WK}_2(F)(2) & \xrightarrow{\gamma'} & D_F/F^{*2}\,\mathrm{N}_{E/F}(D_E) & \xrightarrow{\alpha'} & \bigoplus_{v\in T_{E/F}} D_{F_v}/F_v^{*2}\,\mathrm{N}_{E_w/F_v}(D_{E_w}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where the first two lines are exact. The proof of the genus formula in [KM] now states that

$$2^\varrho = |\mathrm{Ker}\,\alpha'/\mathrm{Im}\,\gamma'| \in \{1,2\}.$$

So we have to decide when $\mathrm{Ker}\,\alpha' = \mathrm{Im}\,\gamma'$ where γ' is the map from the cokernel $\mathrm{WK}_2(E)(2)^G/\mathrm{Im}\,\mathrm{WK}_2(F)(2)$ to the cokernel $K_2(o_E^S)^G/\mathrm{Im}\,K_2(o_F^S)$.

Let us recall some facts stated in [KM]: first of all, notice that here $\mathrm{Ker}\,\alpha' \cong D_F \cap \mathrm{N}_{E/F}(E^*)/F^{*2}\,\mathrm{N}_{E/F}(D_E)$. Thus let us take $[\varepsilon]$ in $\mathrm{Ker}\,\alpha'$ where $[\varepsilon]$ denotes the class of $\varepsilon \in D_F \cap \mathrm{N}_{E/F}(E^*)$ in $\mathrm{Ker}\,\alpha'$. So $\varepsilon \in D_F$ and we write $\varepsilon = \mathrm{N}_{E/F}(\eta)$ for some $\eta \in E^*$. The class of ε is represented by $\{\sqrt{\delta},\varepsilon\} \in K_2(o_E^S)$, where δ satisfies $E = F(\sqrt{\delta})$. Now, if φ generates $G = \mathrm{Gal}(E/F)$, then

$$\{\sqrt{\delta},\varepsilon\} = \{\sqrt{\delta},\eta\}\{\sqrt{\delta},\eta^\varphi\} = \{\sqrt{\delta},\eta\}\{\sqrt{\delta},\eta\}^\varphi\{-1,\eta\}$$

in $K_2(E)^G$, hence the class of ε is represented in $K_2(E)^G$ by $\{-1,\eta\}$. Now

$$K_2(o_E^S)^G/\mathrm{Im}\,K_2(o_F^S) \cong K_2(E)^G/\mathrm{Im}\,K_2(F),$$

and we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{WK}_2(F)(2) & \longrightarrow & K_2(F)(2) & \longrightarrow & \bigoplus_{v}\mu(F_v)(2) \\
 & & \downarrow & & \downarrow & & \downarrow j_S \\
0 & \longrightarrow & \mathrm{WK}_2(E)(2)^G & \xrightarrow{\gamma} & K_2(E)(2)^G & \xrightarrow{\alpha} & \Big(\bigoplus_{w}\mu(E_w)(2)\Big)^G \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{WK}_2(E)(2)^G/\mathrm{Im}\,\mathrm{WK}_2(F)(2) & \xrightarrow{\gamma'} & D_F/F^{*2}\,\mathrm{N}_{E/F}(D_E) & \xrightarrow{\alpha'} & \bigoplus_{v\in T_{E/F}} D_{F_v}/F_v^{*2}\,\mathrm{N}_{E_w/F_v}(D_{E_w}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

where the top rows are exact. Set $n = |\mu(F)(2)| = 2$, $n_v = |\mu(F_v)(2)|$ and $m_w = |\mu(E_w)(2)|$. The local symbols $(-1,\eta)_{m_w}$ are trivial for all $w\,|\,v$ with $v \in T_{E/F}$. For $v \notin T_{E/F}$ we have an isomorphism $\mu(F_v)(2) \cong$

$(\bigoplus_{w|v}\mu(E_w)(2))^G$ and hence for those v there exist $x_v \in F_v$ such that $(-1,x_v)_{n_v} = (-1,\eta)_{m_w}$ for $w \mid v$. Now we have

$$\begin{aligned}\varrho = 0 &\Leftrightarrow \operatorname{Ker}\alpha' = \operatorname{Im}\gamma'\\ &\Leftrightarrow [\varepsilon] \in \operatorname{Im}\gamma' \text{ where } [\varepsilon] \text{ generates } D_F \cap \mathrm{N}_{E/F}(E^*)/F^{*2}\,\mathrm{N}_{E/F}(D_E).\end{aligned}$$

But the right part of the previous diagram is

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \bigoplus_{v\in T_{E/F}} \mu_2 \ \bigoplus_{v\in S_\infty^r} \mu_2 & & & & \\ & & \downarrow & & & & \\ K_2(F)(2) & \longrightarrow & \bigoplus_v \mu(F_v)(2) & \xrightarrow{\pi} & \mu(F)(2) & \longrightarrow & 0 \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_S} & & \downarrow & & \\ K_2(E)(2)^G & \xrightarrow{\alpha} & \big(\bigoplus_w \mu(E_w)(2)\big)^G & \longrightarrow & (\mu(E)(2))^G & & \end{array}$$

where S_∞^r consists of the real infinite primes in F which ramify in E, and π is defined by $\pi((\zeta_v)_v) = \prod_v \zeta_v^{n_v/n}$; however, in our case, S_∞^r is empty. Moreover, define $\varrho_\varepsilon = 0$ or 1 depending on whether the product $\prod_v (-1,x_v)_{n_v}^{n_v/n}$ is equal to 1 or -1. Now, since $n_v > n$ for all v in $T_{E/F}$, we have shown the following proposition:

Proposition 1.2 (Genus formula). *Let E be a cyclic 2-extension of $\mathbb{Q}$, F its subfield satisfying $[E:F]=2$ and $G = \mathrm{Gal}(E/F)$. Then*

$$\frac{|\mathrm{WK}_2(E)(2)_G|}{|\mathrm{WK}_2(F)(2)|} = \frac{2^{|T_{E/F}|-\varrho}}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]},$$

where $\varrho \in \{0,1\}$. More precisely,

(i) *if either a real infinite prime of F ramifies in E, or $\mu(F_v)(2) = \mu(F)(2)$ for a certain prime $v \in T_{E/F}$, then $\varrho = 1$;*

(ii) *in all other cases,*

$$\varrho = 0 \ \Leftrightarrow\ \varrho_\varepsilon = 0 \text{ where } [\varepsilon] \text{ generates } D_F \cap \mathrm{N}_{E/F}(E^*)/F^{*2}\,\mathrm{N}_{E/F}(D_E).$$

2. The first case. We now have efficient tools at our disposal to determine at least some cyclic 2-extensions of $\mathbb{Q}$ which have trivial 2-primary Hilbert kernel. In order to compute the index appearing in the genus formula, the next lemma will be useful in the following.

For $n \geq 0$, let $\mathbb{Q}_n$ be the maximal real subfield of $\mathbb{Q}(\mu_{2^{n+2}})$ and let $\mathbb{Q}_\infty = \bigcup_{n\geq 0}\mathbb{Q}_n$ denote the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$. Then, if we introduce the sequence $(\alpha_k)_{k\in\mathbb{N}}$ defined by $\alpha_0 = 0$ and $\alpha_{k+1} = \sqrt{2+\alpha_k}$, it is well known that $\mathbb{Q}_n = \mathbb{Q}(\alpha_n)$.

LEMMA 2.1. *Let p and ℓ be two distinct odd primes. For $n \geq 2$, let M_n be a cyclic extension of $\mathbb{Q}$ of degree 2^n satisfying one of the following two conditions*:

(i) *$M_n/\mathbb{Q}$ is unramified outside 2 and p,*
   2 and p are undecomposed in $M_n/\mathbb{Q}$;

(ii) *$M_n/\mathbb{Q}$ is unramified outside 2, p and ℓ,*
   2 is totally decomposed in $M_n/\mathbb{Q}$,
   p and ℓ are undecomposed in $M_n/\mathbb{Q}$.

For $i \in \{0,\ldots,n\}$, let M_i be the unique subfield of degree 2^i of M_n, and let r be defined by $\mathbb{Q}_r = M_n \cap \mathbb{Q}_\infty$ $(= M_r)$. For $n \geq r+1$, we now have

$$2+\alpha_r \in \mathrm{N}_{M_n/M_{n-1}}(M_n^*) \;\Leftrightarrow\; 2+\alpha_r \in \mathrm{N}_{M_{r+1}/M_r}(M_{r+1}^*).$$

Note that this lemma will be useful to compute the index $[D_{M_{n-1}} : D_{M_{n-1}} \cap \mathrm{N}_{M_n/M_{n-1}}(M_n^*)]$, since we know that $D_{M_{n-1}}/(M_{n-1}^*)^2$ is generated by the class of $2+\alpha_r$.

Proof. The result uses induction. We can assume that $n-r \geq 2$ and so for $i \in \{r,\ldots,n-2\}$, set $L = M_i$, $F = M_{i+1}$ and $E = M_{i+2}$; it is sufficient to show that

$$2+\alpha_r \in \mathrm{N}_{E/F}(E^*) \;\Leftrightarrow\; 2+\alpha_r \in \mathrm{N}_{F/L}(F^*).$$

Now, according to our assumptions on ramification, it is sufficient to look at this equivalence locally at the place $\mathfrak{p}$ of F above p (indeed, in the case (ii), $2+\alpha_r$ is locally a norm at the dyadic place, since 2 is totally decomposed in $M_n/\mathbb{Q}$). But E/L is a cyclic degree 4 extension and so we can write $F = L(\sqrt{a})$ and $E = F(\sqrt{b})$ where $a \in L$, $b \in F$ and $\mathrm{N}_{F/L}(b) = ac^2$ with $c \in L^*$ (see [S3] for a proof). We then deduce that the Hilbert symbols $(2+\alpha_r, b)_{F_\mathfrak{p}}$ and $(2+\alpha_r, a)_{L_\mathfrak{p}}$ are equal, since the corestriction

$$\mathrm{cor}_{F_\mathfrak{p}/L_\mathfrak{p}}((2+\alpha_r, b)_{F_\mathfrak{p}}) = (2+\alpha_r, \mathrm{N}_{F/L}(b))_{L_\mathfrak{p}} = (2+\alpha_r, a)_{L_\mathfrak{p}},$$

and the corestriction map ${}_2\mathrm{Br}(F_\mathfrak{p}) \to {}_2\mathrm{Br}(L_\mathfrak{p})$ is injective, where ${}_2\mathrm{Br}(K)$ denotes all the elements which are killed by 2 in the Brauer group of K (note that the surjectivity of the corestriction map for the Brauer groups of local fields is proved in [K, Théorème 7.1], and the injectivity follows immediately). Hence the result. ■

In the following, we will keep the notations introduced in the first section: $E/\mathbb{Q}$ will denote a cyclic 2-extension of degree at least 4, F the unique subfield of E such that $[E:F]=2$ and $K=\mathbb{Q}(\sqrt{d})$ the quadratic subfield of E.

2.1. *The case* $K=\mathbb{Q}(\sqrt{2})$. Assume that E contains $K=\mathbb{Q}(\sqrt{2})$ and that $\mathrm{WK}_2(E)(2)$ is trivial.

The prime 2 is ramified in $K/\mathbb{Q}$, and even in $E/\mathbb{Q}$ (since $E/\mathbb{Q}$ is a cyclic extension of degree a prime power). Lemma 1.1 shows that the dyadic place of F is in $T_{E/F}$. As we mentioned earlier, $|T_{E/F}|\le 2$ and so at most one non-dyadic prime $\mathcal{L}$ of F is ramified in E. Moreover, if $\mathcal{L}$ lies above a rational prime ℓ, then necessarily ℓ must be inert in $K/\mathbb{Q}$, which means that $\ell\equiv\pm 3 \bmod 8$.

But since E is unramified outside 2 and ℓ, we deduce that E is contained in the field $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_\ell)$ (recall that E is an extension of degree a power of 2).

Consequently, we have shown the following result: if $\mathrm{WK}_2(E)(2)$ is trivial and if E contains $\mathbb{Q}(\sqrt{2})$, then E is contained in a field $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_\ell)$ where ℓ is a prime with $\ell\equiv\pm 3 \bmod 8$.

Conversely, we want to know if such extensions E have in fact a trivial 2-primary Hilbert kernel. To do this, we will use induction and so we will assume that $\mathrm{WK}_2(F)(2)$ is trivial. It is easy to see that the dyadic place of F belongs to the set $T_{E/F}$ and

$$|T_{E/F}|=\begin{cases}1 & \text{if } E\subset\mathbb{Q}(\mu_{2^\infty}),\\ 2 & \text{otherwise.}\end{cases}$$

Since 2 is totally ramified in the cyclic extension $E/\mathbb{Q}$ and since $\mu(\mathbb{Q}_2(\sqrt{2}))(2)=\mu_2$, we obtain $\mu(E_w)(2)=\mu_2$, where w is the dyadic place of E; thus we have $\varrho=1$ according to part (i) of the genus formula, which means that

$$|\mathrm{WK}_2(E)(2)_G|=\frac{2^{|T_{E/F}|-1}}{[D_F:D_F\cap \mathrm{N}_{E/F}(E^*)]}.$$

Hence, in the case where $E\subset\mathbb{Q}(\mu_{2^\infty})$, we directly get

$$|\mathrm{WK}_2(E)(2)_G|=\frac{1}{[D_F:D_F\cap \mathrm{N}_{E/F}(E^*)]}=1.$$

Otherwise, we have to compute the index appearing in the genus formula. But this index is equal to 1 or 2 and, keeping in mind Lemma 2.1 and its notations (applied to $M_n:=E$), we obtain:

$$\begin{aligned}[D_F:D_F\cap \mathrm{N}_{E/F}(E^*)]=1 &\Leftrightarrow 2+\alpha_r\in \mathrm{N}_{E/F}(E^*)\\ &\Leftrightarrow 2+\alpha_r\in \mathrm{N}_{M_{r+1}/M_r}(M_{r+1}^*).\end{aligned}$$

Moreover, we note that M_r is the $(r+1)$th layer $\mathbb{Q}(\sqrt{2+\alpha_r})$ of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$ and that M_{r+1}/M_r is necessarily ramified at the prime $\mathcal{L}$ of M_r lying above ℓ. As a result, since M_{r+1} is a cyclic extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{\pm\ell})$, we can show that M_{r+1} is of the form $\mathbb{Q}(\sqrt{(-1)^\alpha\ell(2+\alpha_r)})$, where $\alpha \in \{0,1\}$. However, the Hilbert symbol $(2+\alpha_r, (-1)^\alpha\ell(2+\alpha_r))_{(M_r)_\mathcal{L}} = (2+\alpha_r, -(-1)^\alpha\ell)_{(M_r)_\mathcal{L}}$ is non-trivial, since the corestriction

$$\begin{aligned}\mathrm{cor}_{(M_r)_\mathcal{L}/\mathbb{Q}_\ell}((2+\alpha_r, -(-1)^\alpha\ell)_{(M_r)_\mathcal{L}}) &= (\mathrm{N}_{M_r/\mathbb{Q}}(2+\alpha_r), -(-1)^\alpha\ell)_\ell \\ &= (2, -(-1)^\alpha\ell)_\ell \\ &= (2,\ell)_\ell = -1,\end{aligned}$$

for $\ell \equiv \pm 3 \bmod 8$. We thus showed that $2+\alpha_r$ is not a norm locally at the place of M_r lying above ℓ, and so neither is it globally. We then obtain $[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)] = 2$ and finally

$$|\mathrm{WK}_2(E)(2)_G| = \frac{2^{2-1}}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]} = \frac{2}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]} = \frac{2}{2} = 1.$$

To conclude, we have proved the triviality of $\mathrm{WK}_2(E)(2)_G$, and hence of $\mathrm{WK}_2(E)(2)$. We thus have shown

PROPOSITION 2.2. *The only cyclic 2-extensions of $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{2})$ with trivial 2-primary Hilbert kernel are exactly those contained in the composite $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_\ell)$ where ℓ is a prime with $\ell \equiv \pm 3 \bmod 8$.*

2.2. *The case $K = \mathbb{Q}(\sqrt{d})$ where $d = p$ or $2p$, p a prime with $p \equiv 5 \bmod 8$.* Here, the main ideas are the same as in the previous case. We assume that E contains $K = \mathbb{Q}(\sqrt{d})$ and that $\mathrm{WK}_2(E)(2)$ is trivial.

On the one hand, p is ramified in $K/\mathbb{Q}$, and so in E/F. Therefore, the place of F above p is in $T_{E/F}$. On the other hand, since $p \equiv 5 \bmod 8$, the prime 2 is undecomposed in $K/\mathbb{Q}$ and consequently the dyadic place of F is also undecomposed in E/F (keep in mind that $E/\mathbb{Q}$ is cyclic of degree a prime power). This place is, by Lemma 1.1, in $T_{E/F}$. Hence, since $\mathrm{WK}_2(E)(2)$ is trivial, we necessarily have $|T_{E/F}| = 2$.

Thus, E is unramified outside 2 and p and so we have the following result: if $\mathrm{WK}_2(E)(2)$ is trivial and if E contains $\mathbb{Q}(\sqrt{d})$ where $d = p$ or $2p$, p being a prime with $p \equiv 5 \bmod 8$, then E is contained in the composite $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$.

Once again, we want to show that the converse is true. Let $E \supset K$ be a cyclic subfield of degree a power of 2 of the composite $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$. We want to show that $\mathrm{WK}_2(E)(2)$ is trivial. As in Section 2.1 we will use induction on the degree and we may assume that $\mathrm{WK}_2(F)(2)$ is trivial.

The arguments given previously to obtain necessary conditions for the triviality of $\mathrm{WK}_2(E)(2)$ remain true, and so the set $T_{E/F}$ consists of the dyadic place of F and the place of F lying above p. Hence $|T_{E/F}| = 2$ and, by the same argument as before, we can apply part (i) of the genus formula to the dyadic place of F. Therefore the genus formula becomes

$$|\mathrm{WK}_2(E)(2)_G| = \frac{2^{2-1}}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]} = \frac{2}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]}.$$

Since $2 \notin F^{*2}$, we have $[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)] = 1$ if and only if $2 \in \mathrm{N}_{E/F}(E^*)$, which is the case, by Lemma 2.1, if and only if $2 \in \mathrm{N}_{K/\mathbb{Q}}(K^*)$. But $p \equiv 5 \bmod 8$, and so $2 \notin \mathrm{N}_{K/\mathbb{Q}}(K^*)$. We then see that $[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)] = 2$ and finally

$$|\mathrm{WK}_2(E)(2)_G| = \frac{2}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]} = \frac{2}{2} = 1,$$

and the triviality of $\mathrm{WK}_2(E)(2)$ follows. We obtain

Proposition 2.3. *Let $d = p$ or $2p$, where p is a prime with $p \equiv 5 \bmod 8$. The only cyclic 2-extensions of $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{d})$ with trivial 2-primary Hilbert kernel are exactly those contained in the composite $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$.*

As mentioned in [G], if ℓ is a prime such that $\ell \equiv 3 \bmod 8$, then the maximal cyclic 2-extension contained in $\mathbb{Q}(\mu_\ell)$ is $\mathbb{Q}(\sqrt{-\ell})$ and has degree 2 over $\mathbb{Q}$, and if ℓ is a prime such that $\ell \equiv 5 \bmod 8$, the maximal cyclic 2-extension contained in $\mathbb{Q}(\mu_\ell)$ is $\mathbb{Q}\big(\sqrt{2\sqrt{\ell}(a-\sqrt{\ell})}\,\big)$ (where $l = a^2 + b^2$ in $\mathbb{Z}$, with a odd) and has degree 4 over $\mathbb{Q}$. Thus, we can make the results of Propositions 2.2 and 2.3 more explicit:

Proposition 2.4. *All cyclic 2-extensions of $\mathbb{Q}$ of degree at least 4, containing $\mathbb{Q}(\sqrt{d})$ (d being 2, p or $2p$, where p is a prime with $p \equiv 5 \bmod 8$), with trivial 2-primary Hilbert kernel are*

- *the cyclic subfields of degree at least 4 of the composite*

$$\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\sqrt{-p}) \quad \text{if } p \equiv 3 \bmod 8,$$

- *the cyclic subfields of degree at least 4 of the composite*

$$\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}\big(\sqrt{2\sqrt{p}(a-\sqrt{p})}\,\big) \quad \text{if } p \equiv 5 \bmod 8$$

(where $p = a^2 + b^2$ in $\mathbb{Z}$, with a odd).

Remark. We note that this set is exactly the set of all cyclic 2-extensions of $\mathbb{Q}$ with trivial 2-primary positive tame kernel. This result is due to G. Gras (cf. [G]), who has actually determined all abelian 2-extensions of $\mathbb{Q}$ with trivial 2-primary positive tame kernel. Recall that, if E denotes a number field, the positive tame kernel $K_2(o_E)^+$ (or $H_2^0(E)$ in [G]) of E is the kernel of the surjective homomorphism

$$K_2(o_E) \to \bigoplus_{v \text{ real}} \mu_2,$$

and its 2-primary part fits into the exact sequence

$$0 \to \mathrm{WK}_2(E)(2) \to K_2(o_E)^+(2) \to \bigoplus_{v|2} \mu(E_v)(2) \to \mu(E)(2) \to 0.$$

We can then show that $\mathrm{WK}_2(E)(2) = K_2(o_E)^+(2)$ if and only if 2 is undecomposed in $E/\mathbb{Q}$ and if, for the unique place $v \mid 2$, we have $\mu(E_v)(2) = \mu(E)(2)$.

Now, as we will see in the next section, only one case remains, which is more complicated since part (i) of the genus formula may not apply.

3. The second case. In this section, K will denote $\mathbb{Q}(\sqrt{p})$ where p is a prime with $p \equiv 1 \bmod 8$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$.

3.1. *Necessary conditions for the triviality of the* 2*-primary Hilbert kernel.* E still denotes a cyclic 2-extension of $\mathbb{Q}$ of degree at least 4 and F is its subfield satisfying $[E : F] = 2$. Assume that E contains K and that $\mathrm{WK}_2(E)(2)$ is trivial.

The prime p is ramified in $K/\mathbb{Q}$, and even in E/F. The place $\mathfrak{p}$ above p in F then belongs to the set $T_{E/F}$.

Moreover, 2 decomposes in $K/\mathbb{Q}$. Let $\mathfrak{p}_2$ denote a dyadic place of F; if $\mathfrak{p}_2$ remains undecomposed in E/F, then Lemma 1.1 would imply that $\mathfrak{p}_2 \in T_{E/F}$. Thus at least two dyadic places would be in $T_{E/F}$ and so $|T_{E/F}| \geq 3$, which is impossible (see Section 1). Consequently, 2 must be totally decomposed in $E/\mathbb{Q}$. In particular, no dyadic place can belong to $T_{E/F}$.

Therefore, at most one additional non-dyadic prime of F, different from $\mathfrak{p}$, can ramify in E. Furthermore, the condition $|T_{E/F}| \leq 2$ implies that this prime of F is undecomposed in $F/\mathbb{Q}$. Since 2 is totally decomposed in $E/\mathbb{Q}$, the assumptions of the case (ii) of Lemma 2.1 are satisfied and we obtain $2 \in \mathrm{N}_{E/F}(E^*)$ if and only if $2 \in \mathrm{N}_{K/\mathbb{Q}}(K^*)$, which is true since $p \equiv 1 \bmod 8$. Hence, $[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)] = 1$ and the genus formula gives

$$|\mathrm{WK}_2(E)(2)_G| = \frac{2^{|T_{E/F}|-\varrho}}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]} = 2^{|T_{E/F}|-\varrho},$$

and so the triviality of $\mathrm{WK}_2(E)(2)$ implies that $|T_{E/F}| \leq 1$. Necessarily, we have $T_{E/F} = \{\mathfrak{p}\}$.

Consequently, we have shown that if $\mathrm{WK}_2(E)(2)$ is trivial, then E is contained in the cyclotomic field $\mathbb{Q}(\mu_p)$ and 2 must be totally decomposed in $E/\mathbb{Q}$. However, we know that if $\mathrm{WK}_2(E)(2)$ is trivial, then so is $\mathrm{WK}_2(L)(2)$ for all subfields L of E. Thus, first of all, let us focus our attention on the subfield of $\mathbb{Q}(\mu_p)$ of degree 4 over $\mathbb{Q}$.

Hence, we now assume that E is the unique subfield of degree 4 in the cyclotomic field $\mathbb{Q}(\mu_p)$ and we will study the possible triviality of $\mathrm{WK}_2(E)(2)$. If ε denotes the fundamental unit of $F = K = \mathbb{Q}(\sqrt{p})$, we know (cf. Corollary 2, Part 1, Chap. V in [FT]) that ε has norm -1 in $F/\mathbb{Q}$. Thus, as $p \equiv 1 \bmod 8$, we can write $\varepsilon = a + b\sqrt{p}$ where a and b are integers satisfying $a^2 - pb^2 = -1$. Hence, $\mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big)$ is really a $\mathbb{C}_4$-extension of $\mathbb{Q}$, namely a cyclic degree 4 extension of $\mathbb{Q}$. This extension is also unramified outside p. Therefore,

$$E = \mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big).$$

Before proceeding further, let us recall some simple facts about the fundamental unit ε of $F = \mathbb{Q}(\sqrt{p})$ when p is a prime with $p \equiv 1 \bmod 8$. It is easy to see that $a \equiv 0 \bmod 4$ and b is odd. Moreover, we note that, p being congruent to 1 modulo 8, we can obviously write $p = 16m^2 + n^2$ with n odd. The following result is due to M. Stinner ([St]), but since the sources are not easily accessible, we include a proof.

LEMMA 3.1. *Let $p = 16m^2 + n^2 \equiv 1 \bmod 8$ be a prime. Denote by $\varepsilon = a + b\sqrt{p}$ the fundamental unit of $\mathbb{Q}(\sqrt{p})$. Then*

$$4m \equiv na + (-1)^{(n-1)/2}n - 1 \bmod 8.$$

Proof. We have $a^2 + 1 = pb^2$. So in $\mathbb{Z}[i]$, we obtain $(a+i)(a-i) = pb^2$ (where $i^2 = -1$). Now by unique factorization in $\mathbb{Z}[i]$, we can write

$$a + i = (x + iy)(c + id)^2, \tag{1}$$

where $x, y, c, d \in \mathbb{Z}$ and $x + iy$ is squarefree.

Since $a + i$ and $a - i$ are relatively prime, so are $x + iy$ and $x - iy$. Thus the relation $(x^2 + y^2)(c^2 + d^2)^2 = a^2 + 1 = pb^2$ implies that

$$p = x^2 + y^2, \tag{2}$$

$$b = c^2 + d^2. \tag{3}$$

Furthermore from (1) we have

$$a = x(c^2 - d^2) - 2ycd, \tag{4}$$

$$1 = y(c^2 - d^2) + 2xcd. \tag{5}$$

Since b is odd and a is even, we deduce by (3) that $c^2 - d^2$ is odd, by (4) that x is even and by (2) that y is odd. Now, by (5), we have $y(c^2 - d^2) \equiv 1 \bmod 8$ and consequently, by (4), $ya = xy(c^2 - d^2) - 2y^2cd \equiv x - 2cd \bmod 8$. Hence

$$x \equiv ay + 2cd \bmod 8. \tag{6}$$

Furthermore, by (5), $y = y^2(c^2 - d^2) + 2xycd \equiv c^2 - d^2 \bmod 8$. Writing $c^2 - d^2 = (c - d)^2 + 2cd - 2d^2$, we obtain

$$y \equiv 1 + 2cd - 2d^2 \bmod 8. \tag{7}$$

Now, if $y \equiv 1 \bmod 4$, then reduction modulo 4 in (7) implies that d is even and (7) then gives $2cd \equiv y-1 \bmod 8$. If $y \equiv 3 \bmod 4$, then reduction modulo 4 in (7) implies that d is odd and replacing in (7), we finally obtain $2cd \equiv -2cd \equiv -y-1 \bmod 8$. Hence, in both cases, $2cd \equiv (-1)^{(y-1)/2}y - 1 \bmod 8$ and, by (6), we have

$$x \equiv ya + (-1)^{(y-1)/2}y - 1 \bmod 8.$$

Hence the lemma, since the representation of p as a sum of two squares is unique up to signs. ∎

Notice that we have in particular shown that $b = c^2 + d^2$, and consequently $b \equiv 1 \bmod 4$.

We then have the following equivalent conditions:

(i) $p = x^2 - 32y^2, x > 0, x \equiv 1 \bmod 4$,
(ii) $(-1)^{(n-1)/2}n - 1 \equiv 4m \bmod 8$,
(iii) $a \equiv 0 \bmod 8$.

The first equivalence between (i) and (ii) is part of the Main Theorem in [BC] and the second one between (ii) and (iii) immediately follows from Lemma 3.1.

Let us return to the case concerning the C_4-extension E of $\mathbb{Q}$. According to our assumptions, we have $a \equiv 4 \bmod 8$ and we are now ready to simplify the condition which states that 2 must be totally decomposed in $E/\mathbb{Q}$. Actually, 2 being totally decomposed in $E/\mathbb{Q}$ is equivalent to the fact that $\varepsilon\sqrt{p}$ is a square in $\mathbb{Q}_2^*$. If we write $p = u^2$ with $u \in \mathbb{Q}_2^*$, and note that $\varepsilon = a + b\sqrt{p} = a + bu \in \mathbb{Q}_2^*$, we obtain

$$\begin{aligned}
\varepsilon\sqrt{p} \in (\mathbb{Q}_2^*)^2 &\Leftrightarrow (a+bu)u \equiv 1 \bmod 8 \text{ in } \mathbb{Q}_2^* \\
&\Leftrightarrow au + bp \equiv 1 \bmod 8 \\
&\Leftrightarrow au + b \equiv 1 \bmod 8 \\
&\Leftrightarrow 4 + b \equiv 1 \bmod 8 \qquad (\text{since } a \equiv 4 \bmod 8) \\
&\Leftrightarrow b \equiv 5 \bmod 8, \\
&\Leftrightarrow p \equiv 9 \bmod 16.
\end{aligned}$$

To sum up, we have shown

PROPOSITION 3.2. *Let E denote a C_4-extension of $\mathbb{Q}$ containing $F = \mathbb{Q}(\sqrt{p})$ where p is a prime with $p \equiv 1 \bmod 8$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$. Then, if $\mathrm{WK}_2(E)(2)$ is trivial,*

$$E = \mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big),$$

where $p \equiv 9 \bmod 16$.

Therefore, we are now able to handle our initial problem: E denotes any cyclic 2-extension of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel. So we have shown that E is contained in the cyclotomic field $\mathbb{Q}(\mu_p)$ and 2 must be totally decomposed in $E/\mathbb{Q}$. The previous argument about its subfield of degree 4 shows that we necessarily have $p \equiv 9 \bmod 16$. Hence, since $[\mathbb{Q}(\mu_p) : \mathbb{Q}] = p - 1 \equiv 8 \bmod 16$, E must have degree 4 or 8 over $\mathbb{Q}$. So it remains to determine in the next part if the unique subfields of $\mathbb{Q}(\mu_p)$ of degree 4 and 8 with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$ have trivial 2-primary Hilbert kernel.

3.2. *Sufficient conditions to have trivial* 2*-primary Hilbert kernel*

First step. Assume that $E = \mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big)$, where p is a prime with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$. From what we said in the previous subsection we know that 2 is totally decomposed in $E/\mathbb{Q}$ and that the set $T_{E/F}$ is exactly the set $\{\mathfrak{p}\}$. Moreover, $2 \in \mathrm{N}_{E/F}(E^*)$ and the genus formula becomes:

$$|\mathrm{WK}_2(E)(2)_G| = \frac{2^{|T_{E/F}|-\varrho}}{[D_F : D_F \cap \mathrm{N}_{E/F}(E^*)]} = 2^{1-\varrho},$$

where $\varrho \in \{0, 1\}$.

The triviality of $\mathrm{WK}_2(E)(2)$ happens if and only if $\varrho = 1$ and to compute the precise value of ϱ we have to apply part (ii) of the genus formula. First of all, F being totally real, we have $\operatorname{Ker}\alpha' \cong D_F \cap \mathrm{N}_{E/F}(E^*)/F^{*2}\,\mathrm{N}_{E/F}(D_E) \cong D_F/F^{*2}$, which is generated by the class of $\varepsilon = 2$ (since $2 \notin F^{*2}$). Hence, part (ii) states that $\mathrm{WK}_2(E)(2)$ is trivial if and only if

$$\prod_v (-1, \eta)_{m_w}^{n_v/n} = -1,$$

where v runs through all finite and real infinite primes of F which are not in $T_{E/F}$ and the notations are:

- w denotes any place of E lying above v,
- $n = |\mu(F)(2)| = 2$, $n_v = |\mu(F_v)(2)|$ and $m_w = |\mu(E_w)(2)|$,
- η is some element of E^* satisfying $\mathrm{N}_{E/F}(\eta) = 2$.

Now under our assumptions 2 is actually a norm in the C_4-extension $E/\mathbb{Q}$. Indeed, since the primes different from 2 and p are unramified in E, 2 is locally a norm everywhere except possibly at one place, the ramified place p, and hence is a global norm.

Therefore, let $\xi \in E^*$ denote an element satisfying $2 = \mathrm{N}_{E/\mathbb{Q}}(\xi)$ and σ a generator of $\mathrm{Gal}(E/\mathbb{Q})$. Set $\eta = \xi\xi^\sigma$ so that $\mathrm{N}_{E/F}(\eta) = 2$. We are now interested in determining the product

$$\phi = \prod_v (-1, \eta)_{m_w}^{n_v/n} = \prod_{v \in V} (-1, \eta)_{m_w},$$

where $V := \{\text{places } v \text{ of } F \text{ such that } n_v = n = 2\}$.

Let us notice that, if v is a non-dyadic place of V lying above a rational prime number q, then $q \equiv 3 \bmod 4$ (since $\mathbb{Q}_q \hookrightarrow F_v$) and v decomposes in $F/\mathbb{Q}$ (otherwise the residue field of F_v would be the finite field with q^2 elements and $q^2 - 1 \equiv 0 \bmod 8$).

Let U be the set of all rational prime numbers q lying below some place of V. In fact, U consists of the prime 2 and the odd prime numbers congruent to 3 modulo 4. Then

$$\phi = \prod_{q \in U} (-1, \eta\eta^{\sigma})_{m_w},$$

where w is any prime of E above q. But $\eta\eta^{\sigma} = \xi\xi^{\sigma^2}(\xi^{\sigma})^2$ and consequently

$$\phi = \prod_{q \in U} (-1, \mathrm{N}_{E/F}(\xi))_{m_w}.$$

Since $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$, we can write $p = r^2 - 8s^2$ with $r > 0, r \equiv 1 \bmod 8$ and s odd. Thus $2 = \mathrm{N}_{F/\mathbb{Q}}(x)$ where $x = (r + \sqrt{p})/2s$ and we have $2 = \mathrm{N}_{F/\mathbb{Q}}(x) = \mathrm{N}_{F/\mathbb{Q}}(\mathrm{N}_{E/F}(\xi))$. In other words, $\mathrm{N}_{E/F}(\xi) = xy$ where $y \in F^*$ with $\mathrm{N}_{F/\mathbb{Q}}(y) = 1$. This means by Hilbert's theorem 90 that there exists $z \in F^*$ such that $y = z/z^{\sigma}$. Hence

$$\phi = \phi_1\phi_2 \quad \text{where} \quad \phi_1 = \prod_{q \in U} (-1, x)_{m_w},\ \phi_2 = \prod_{q \in U} (-1, y)_{m_w} = \prod_{q \in U} (-1, zz^{\sigma})_{m_w}.$$

First, let us compute ϕ_2: notice that if $v \in V$ and w is any prime of E above v, then we have $(-1, z)_{m_w} = (-1, z)_{n_v}$, since $v(z) = w(z)$ (for v is unramified in E) and these two Hilbert symbols are tame and depend only on the valuation v (see [S1, Proposition 8, Part 3, Chap. XIV]). Therefore

$$\phi_2 = \prod_{q \in U} (-1, zz^{\sigma})_{m_w} = \prod_{v \in V} (-1, z)_{m_w} = \prod_{v \in V} (-1, z)_{n_v} = \prod_{v} (-1, z)_{n_v}^{n_v/n} = 1,$$

by reciprocity ($z \in F^*$).

Now, let us compute ϕ_1:

$$\phi_1 = \prod_{q \in U} (-1, x)_{m_w} = \prod_{q \in U} (-1, x)_{n_v}$$

by the same argument as previously for ϕ_2; and again in the first product, w denotes any prime of E above q and in the second one, v denotes any prime of F above q. But if $q \in U$, then $v \in V$ and $n_v = 2$; thus, if $(\cdot, \cdot)_{F_v}$ denotes the classical Hilbert symbol with value in μ_2, then $(-1, x)_{n_v} = (-1, x)_{F_v}$ and

$$\begin{aligned}\phi_1 &= \prod_{q \in U} (-1, x)_{F_v} = \prod_{q \in U} \left(-1, \frac{r + \sqrt{p}}{2s}\right)_{F_v} \\ &= \prod_{q \in U} (-1, (r + \sqrt{p})s)_{F_v} = \prod_{q \in U} (-1, r + \sqrt{p})_{F_v} \prod_{q \in U} (-1, s)_{F_v}.\end{aligned}$$

It may be remarked that if $q \in U$ and if v divides q then $F_v = \mathbb{Q}_q$, and moreover, if q divides s, then $q \in U$ (indeed, $p = r^2 - 8s^2$). As a result, we obtain

$$\prod_{q \in U} (-1, s)_{F_v} = (-1, s)_2 \prod_{q \mid s,\, q \,\mathrm{odd}} (-1, s)_q = 1,$$

by the reciprocity law. Hence

$$\phi_1 = \prod_{q \in U} (-1, r + \sqrt{p})_{F_v} = (-1, r + \sqrt{p})_2 \prod_{q \in U,\, q \,\mathrm{odd}} (-1, r + \sqrt{p})_q.$$

If $q \in U$ is odd, we have $(-1, r + \sqrt{p})_q = (-1, r - \sqrt{p})_q$ (computing the product) and it is easy to see that these symbols are trivial if $q \nmid s$. Now, if $q \mid s$, either $r + \sqrt{p}$ or $r - \sqrt{p}$ is a q-adic unit: otherwise, considering the sum, q would divide $2r$, which is impossible since q already divides s. Therefore, one of the previous symbols is trivial because it is tame, and so both are trivial. We get

$$\phi_1 = (-1, r + \sqrt{p})_2.$$

We write $p = u^2$ with $u \in \mathbb{Q}_2^*$. Since $p \equiv 9 \bmod 16$, in $\mathbb{Q}_2^*$ we have $u \equiv \pm 3 \bmod 8$. But the equality $(-1, r + u)_2 = (-1, r - u)_2$ enables us to make a choice of a square root of p: let us take $u \equiv 5 \bmod 8$. But $r \equiv 1 \bmod 8$, so that we can write $r + u = 2(3 + 4\lambda)$ with $\lambda \in \mathbb{Z}_2$. Finally, with [S2] we obtain

$$\phi_1 = (-1, r + u)_2 = -1.$$

Thus, $\phi = \phi_1 \phi_2 = -1$ and so $\varrho = 1$.

We have therefore shown

Proposition 3.3. *Let p be a prime with*

$$p \equiv 9 \bmod 16, \quad p \neq x^2 - 32y^2, \quad x > 0, \quad x \equiv 1 \bmod 4.$$

The only C_4-extension of $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{p})$ with trivial 2-primary Hilbert kernel is exactly $\mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big)$ where ε denotes the fundamental unit of $\mathbb{Q}(\sqrt{p})$.

Second step. Let E be the unique subfield of degree 8 in the cyclotomic field $\mathbb{Q}(\mu_p)$, where p is a prime with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$. The subfield of E of degree 4 is $F = \mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big)$ and we assume that 2 is totally decomposed in $E/\mathbb{Q}$. The question is: is $\mathrm{WK}_2(E)(2)$ trivial? Actually, $|T_{E/F}| = 1$ and the index appearing in the genus formula is also 1. So $\mathrm{WK}_2(E)(2)$ is trivial if and only if $\varrho = 1$; but, since F is totally real and E is totally complex, part (i) of the genus formula applies and $\varrho = 1$.

Moreover, 2 is totally decomposed in $E/\mathbb{Q}$ if and only if $2^{(p-1)/8} \equiv 1 \bmod p$.

Consequently, we have shown

THEOREM 3.4. *Let p be a prime with $p \equiv 1 \bmod 8$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$. The only cyclic 2-extensions of $\mathbb{Q}$ containing $\mathbb{Q}(\sqrt{p})$ with trivial 2-primary Hilbert kernel are*

- $\mathbb{Q}(\sqrt{p})$,
- *the unique subfield $\mathbb{Q}\big(\sqrt{\varepsilon\sqrt{p}}\big)$ of degree 4 of $\mathbb{Q}(\mu_p)$ where $p \equiv 9 \bmod 16$ and ε denotes the fundamental unit of $\mathbb{Q}(\sqrt{p})$,*
- *the unique subfield of degree 8 of $\mathbb{Q}(\mu_p)$ where $p \equiv 9 \bmod 16$ and $2^{(p-1)/8} \equiv 1 \bmod p$.*

Note that 73, 89, 233, 281, 601, 617, 937, 1049, 1097, 1193 are the first ten primes p such that $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$, and 73, 89, 233, 601, 937 are those which also satisfy $2^{(p-1)/8} \equiv 1 \bmod p$.

Proposition 2.4 and Theorem 3.4 lead to the complete list announced in the Introduction of all cyclic 2-extensions of $\mathbb{Q}$ which have trivial 2-primary Hilbert kernel.

REMARK. Comparing the two lists of 2-extensions with trivial 2-primary Hilbert kernel or positive tame kernel, we can even draw up the list of all cyclic 2-extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel, but with non-trivial 2-primary positive tame kernel:

(i) $\mathbb{Q}(\sqrt{-p})$ where p is a prime with $p \equiv 7 \bmod 8$,
(ii) $\mathbb{Q}(\sqrt{p})$ where p is a prime with $p \equiv 1 \bmod 8$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$,
(iii) $\mathbb{Q}(\sqrt{pq})$ where p, q are primes with $p \equiv q \equiv 3 \bmod 8$,
(iv) $\mathbb{Q}(\sqrt{-pq})$ where p, q are primes with $p \equiv -q \equiv 3 \bmod 8$,
(v) the unique subfield of degree 4 of $\mathbb{Q}(\mu_p)$ where p is a prime with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$,
(vi) the unique subfield of degree 8 of $\mathbb{Q}(\mu_p)$ where p is a prime with $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$ and $2^{(p-1)/8} \equiv 1 \bmod p$.

4. The general abelian case. The determination of all abelian 2-extensions with trivial 2-primary Hilbert kernel seems to be more complicated. Indeed, to get an insight into this issue, let us give an example: let E be the composite of the unique degree 8 subfield of $\mathbb{Q}(\mu_p)$ with $\mathbb{Q}(\sqrt{-q})$ where p and q are two primes satisfying the following conditions:

(i) $p \equiv 9 \bmod 16$,
(ii) $q \equiv 7 \bmod 8$,
(iii) $\left(\frac{p}{q}\right) = -1$ (the Legendre symbol),
(iv) $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$,
(v) $2^{(p-1)/8} \equiv 1 \bmod p$.

We are not able to decide whether the 2-primary Hilbert kernel of E is trivial or not. In fact, if F denotes the maximal real subfield of E, we know that $\mathrm{WK}_2(F)(2) \neq 0$, and we can even show that $\mathrm{WK}_2(E)(2) = 0$ if and only if $|\mathrm{WK}_2(F)(2)| = 2$.

However we are able to determine the list of totally real abelian 2-extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel. To begin with, the case of 2-extensions of degree less than 4 over $\mathbb{Q}$ is obtained using [BS], [KM] and our Sections 2 and 3. Now, before going into details about the degree ≥ 8, we recall that, according to [G], the set of all abelian 2-extensions with trivial 2-primary positive tame kernel is the set of those which are contained in some $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$ with $p \equiv \pm 3 \bmod 8$. Moreover, we note that such extensions have only one dyadic prime.

To complete the result, we want to prove the following:

Proposition 4.1. *Let E be a totally real abelian 2-extension of $\mathbb{Q}$ such that $[E : \mathbb{Q}] \geq 8$ and $\mathrm{WK}_2(E)(2) = 0$. Then there exists a prime $p \equiv \pm 3 \bmod 8$ such that $E \subset \mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$.*

Before giving a proof, we will quote a useful proposition stated in [KM] establishing a product formula for bi-quadratic extensions of totally real number fields:

Proposition 4.2. *Let E/F be a bi-quadratic extension of totally real number fields with quadratic subfields F_i, $i = 1, 2, 3$, such that $E/\mathbb{Q}$ is abelian. Let $\delta = \delta_{E/F}$ denote the number of undecomposed dyadic primes of F such that $|\mu(F_v)(2)| = 2$ and E_w is the first layer of the cyclotomic $\mathbb{Z}_2$-extension of $F_v(\sqrt{-1})$. Then*

$$2^\delta |\mathrm{WK}_2(F)(2)|^2 |\mathrm{WK}_2(E)(2)| = \prod_{i=1}^{3} |\mathrm{WK}_2(F_i)(2)|.$$

We are now ready for the

Proof of Proposition 4.1. Again we do induction on the degree $[E : \mathbb{Q}]$.

(a) The case $[E : \mathbb{Q}] = 8$.

(i) If $E/\mathbb{Q}$ is cyclic, this is obvious from Proposition 2.4 and Theorem 3.4.

(ii) If $E/\mathbb{Q}$ is tri-quadratic, it is known (cf. [Gr]) that $\mathrm{WK}_2(E)(2)$ cannot be trivial. Since the sources are not easily accessible, we will give the main ideas of the proof. We assume $\mathrm{WK}_2(E)(2) = 0$ and we will show a contradiction. We can write $E = \mathbb{Q}(\sqrt{d}, \sqrt{d_1}, \sqrt{d_2})$ where d, d_1, d_2 are integers ≥ 2; we also may assume that d is divisible by an odd prime number that divides neither d_1 nor d_2. Thus, $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ has an odd prime ramifying in E and so $\mathrm{WK}_2(K)(2) = 0$ (for $\mathrm{WK}_2(E)(2) = 0$). Now by [KM], K is in the following list:

$$\text{(L)} \qquad \begin{cases} \mathbb{Q}(\sqrt{2}, \sqrt{p}), & p \equiv \pm 3 \bmod 8, \\ \mathbb{Q}(\sqrt{2^a p}, \sqrt{2^a q}), & p \equiv q \equiv 3 \bmod 8, a \in \{0, 1\}, \\ \mathbb{Q}(\sqrt{pq}, \sqrt{qr}), & p \equiv q \equiv r \equiv 3 \bmod 8, \end{cases}$$

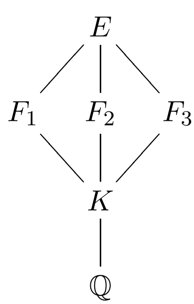
where p, q and r are distinct odd primes.

If K is of the first type in the above list then we may assume that neither 2 nor p divides d. Hence, by tame ramification, the field $\mathbb{Q}(\sqrt{2}, \sqrt{d})$ appears in (L), which implies that d is a prime $\equiv \pm 3 \bmod 8$. Thus in the product formula appearing in Proposition 4.2, we have $\delta_{E/\mathbb{Q}(\sqrt{2})} = 0$ and so all three intermediate fields have trivial 2-primary Hilbert kernel. Hence a contradiction since $\mathbb{Q}(\sqrt{2}, \sqrt{pd})$ does not appear in (L).

If K is of the second type in (L), we may assume that neither p nor q divides d. Thus, by tame ramification, the field $\mathbb{Q}(\sqrt{2^a p}, \sqrt{d})$ appears in (L), hence $d = 2^a t$ for a prime $t \equiv 3 \bmod 8$ distinct from p and q. In the product formula, $\delta_{E/\mathbb{Q}(\sqrt{2^a p})} = 0$ and so we have a contradiction since $\mathbb{Q}(\sqrt{2^a p}, \sqrt{qt})$ does not appear in (L).

If K is of the remaining type in (L), once again we find $\delta_{E/\mathbb{Q}(\sqrt{pq})} = 0$ and the product formula implies that $\mathbb{Q}(\sqrt{pq}, \sqrt{d})$ should appear in (L). Since $d \notin \{p, q, 2p, 2q\}$, we have, without loss of generality, $d = qt$ for an odd prime $t \equiv 3 \bmod 8$ distinct from p and q. But applying the product formula to $E/\mathbb{Q}(\sqrt{pr})$, we could show that the field $\mathbb{Q}(\sqrt{pr}, \sqrt{qt})$ should appear in (L). Hence a contradiction.

(iii) If $\mathrm{Gal}(E/\mathbb{Q}) \simeq C_4 \times C_2$, we need to introduce some notations. Let F_1 and F_2 denote the two C_4-extensions contained in E and F_3 the unique bi-quadratic extension contained in E. Let K denote the intersection of F_1 and F_2. We are in the following situation:



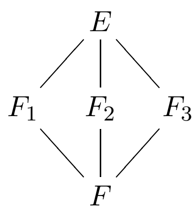
On the one hand, if $K \neq \mathbb{Q}(\sqrt{2})$, then E/F_3 is tamely ramified, and so $\mathrm{WK}_2(F_3)(2) = 0$. By [KM], we have $F_3 = \mathbb{Q}(\sqrt{2}, \sqrt{p})$, where $p \equiv 5 \bmod 8$. Now, $\delta_{E/K} = 0$ in the product formula and consequently we obtain $\mathrm{WK}_2(F_i)(2) = 0$, $i = 1, 2, 3$. Thus, $E \subset \mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$ by applying the results of Section 2 to F_1 and F_2.

On the other hand, if $K=\mathbb{Q}(\sqrt{2})$, the product formula gives

$$2^{\delta_{E/K}}=\prod_{i=1}^{3}|\mathrm{WK}_2(F_i)(2)|,$$

where $\delta_{E/K}$ is equal to 0 or 1. Assume that $\mathrm{WK}_2(F_3)(2)\neq 0$; then $\delta_{E/K}=1$ and $\mathrm{WK}_2(F_1)(2)=\mathrm{WK}_2(F_2)(2)=0$; thus, F_1 or F_2 is of the form $\mathbb{Q}\big(\sqrt{p(2+\sqrt{2})}\big)$ with $p\equiv\pm 3 \bmod 8$ and, for a dyadic prime w of E, $E_w=\mathbb{Q}_2(\mu_{16})$ should contain $\mathbb{Q}_2(\sqrt{\pm 3})$, which leads to a contradiction. Hence, $\mathrm{WK}_2(F_3)(2)=0$ and so $F_3=\mathbb{Q}(\sqrt{2},\sqrt{p})$, where p is a prime $\equiv\pm 3 \bmod 8$. Arguing as previously, we find $\delta_{E/K}=0$ in the product formula appearing in Proposition 4.2, which implies that $\mathrm{WK}_2(F_1)(2)=\mathrm{WK}_2(F_2)(2)=0$; it is now obvious to see by Proposition 2.4 that $E=F_1F_2\subset\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$.

(b) The case $[E:\mathbb{Q}]>8$. We can assume that $E/\mathbb{Q}$ is not cyclic. Furthermore, if E is unramified outside 2, E is contained in $\mathbb{Q}(\mu_{2^\infty})$ as desired. Otherwise, at least one odd prime ramifies in $E/\mathbb{Q}$; now, considering the inertia group at this prime, there exists a field F_1 such that $[E:F_1]=2$ and E/F_1 is tamely ramified. Since $E/\mathbb{Q}$ is not cyclic, there exists a subfield F of F_1 such that E/F is bi-quadratic. Moreover, let F_2 and F_3 be the other two intermediate fields. The situation is the following:



If $\mathrm{WK}_2(E)(2)=0$, then $\mathrm{WK}_2(F_1)(2)=0$ by tame ramification. Hence, by induction hypothesis, there exists a prime $p\equiv\pm 3 \bmod 8$ such that $F_1\subset\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$ and so $\mathrm{WK}_2(F)(2)=0$ (for $F\subset F_1$). The product formula gives

$$2^{\delta_{E/F}}=\prod_{i=2}^{3}|\mathrm{WK}_2(F_i)(2)|.$$

But, since F has only one dyadic prime, $\delta_{E/F}=0$ or 1, and so, without restriction, we may assume that $\mathrm{WK}_2(F_2)(2)=0$. Thus, there exists a prime $q\equiv\pm 3 \bmod 8$ such that $F_2\subset\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_q)$. It remains to be shown that $p=q$.

If $p\neq q$, then F is a totally real number field which is unramified outside 2 and consequently F is contained in the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$; in other words, if $[E:\mathbb{Q}]=2^{n+2}$ and if the sequence $(\alpha_k)_{k\in\mathbb{N}}$ is defined by $\alpha_0=0$ and $\alpha_{k+1}=\sqrt{2+\alpha_k}$, then $F=\mathbb{Q}(\alpha_n)$. Now, we may assume that F_1/F and F_2/F are respectively ramified at p and q (otherwise the result would become obvious). If $\delta_{E/F} = 1$, we would have $E_w = \mathbb{Q}_2(\mu_{2^{n+3}}) = \mathbb{Q}_2(\sqrt{-1}, \sqrt{2+\alpha_n})$ for a dyadic prime w of E; hence, the Galois group of $E/\mathbb{Q}$ would necessarily be $C_{2^{n+1}} \times C_2$ and so we might assume without loss of generality that F_1 is cyclic over $\mathbb{Q}$. Thus, we would obtain $F_1 = \mathbb{Q}(\sqrt{p(2+\alpha_n)})$ and so E_w would contain $\mathbb{Q}_2(\sqrt{p}) = \mathbb{Q}_2(\sqrt{\pm 3})$, hence a contradiction. We then deduce that $\delta_{E/F} = 0$ and consequently $\mathrm{WK}_2(F_3)(2) = 0$. Now, $F_3 \subset \mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_r)$ with $r \equiv \pm 3 \bmod 8$. Furthermore, $E = F_1F_2$ is unramified outside 2, p, q, which enables us to assume that $r = p$ (even if it requires replacing p by q). But now we have $E = F_1F_3 \subset \mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$, which contradicts the fact that q ramifies in F_2/F. Therefore, we really have $p = q$. ■

In other words, we have shown the following

THEOREM 4.3. *All totally real abelian* 2*-extensions of* $\mathbb{Q}$ *with trivial* 2*-primary Hilbert kernel are the following*:

(1) $\mathbb{Q}(\sqrt{p})$ *where* p *is a prime with* $p \equiv 1 \bmod 8$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$,

(2) $\mathbb{Q}(\sqrt{pq})$ *where* p, q *are primes with* $p \equiv q \equiv 3 \bmod 8$,

(3) *the unique subfield of degree* 4 *of* $\mathbb{Q}(\mu_p)$ *where* p *is a prime with* $p \equiv 9 \bmod 16$, $p \neq x^2 - 32y^2$, $x > 0$, $x \equiv 1 \bmod 4$,

(4) $\mathbb{Q}(\sqrt{2^a p}, \sqrt{2^a q})$ *where* p, q *are primes with* $p \equiv q \equiv 3 \bmod 8$, $a \in \{0, 1\}$,

(5) $\mathbb{Q}(\sqrt{pq}, \sqrt{qr})$ *where* p, q, r *are primes with* $p \equiv q \equiv r \equiv 3 \bmod 8$,

(6) *the totally real* 2*-extensions contained in the composite* $\mathbb{Q}(\mu_{2^\infty})\mathbb{Q}(\mu_p)$ *with* $p \equiv \pm 3 \bmod 8$.

REMARK. As a consequence of Proposition 4.1 and [G], we deduce that, for totally real abelian 2-extensions of degree ≥ 8, the triviality of the 2-primary Hilbert kernel is equivalent to that of the 2-primary positive tame kernel.

References

[BC] P. Barrucand and H. Cohn, *Note on primes of type* $x^2 + 32y^2$, *class number, and residuacity*, J. Reine Angew. Math. 238 (1969), 67–70.

[BS] J. Browkin and A. Schinzel, *On Sylow* 2*-subgroups of* K_2O_F *for quadratic number fields* F, ibid. 331 (1982), 104–113.

[C] A. Czogała, *The functor* K_2 *for multiquadratic number fields*, Ann. Math. Sil. 3 (1990), 7–17.

[FT] A. Fröhlich and M. J. Taylor, *Algebraic Number Theory*, Cambridge Univ. Press, 1991.

[Ga] H. Garland, *A finiteness theorem for* K_2 *of a number field*, Ann. of Math. 94 (1971), 534–548.

[G] G. Gras, *Remarks on K_2 of number fields*, J. Number Theory 23 (1986), 322–335.
[Gr] R. A. W. Griffiths, *Multi-quadratic extensions of $\mathbb{Q}$ with trivial 2-primary Hilbert kernel*, M.Sc. thesis, McMaster Univ., 2000.
[K] B. Kahn, *Deux théorèmes de comparaison en cohomologie étale; applications*, Duke Math. J. 69 (1993), 137–165.
[KM] M. Kolster and A. Movahhedi, *Bi-quadratic number fields with trivial 2-primary Hilbert kernels*, Proc. London Math. Soc. 87 (2003), 109–136.
[S1] J.-P. Serre, *Corps locaux*, Hermann, Paris, 1968.
[S2] —, *Cours d'arithmétique*, Presses Universitaires de France, 1970.
[S3] —, *Topics in Galois Theory*, Bartlett and Jones, 1992.
[St] M. Stinner, *Pell's equation and integers of the form $x^2 + 32y^2$*, M.Sc. thesis, McMaster Univ., 2001.
[T] J. Tate, *Relations between K_2 and Galois cohomology*, Invent. Math. 36 (1976), 257–274.

LACO (UMR 6090 CNRS)
Université de Limoges
123 avenue Albert Thomas
87060 Limoges Cedex, France
E-mail: mikael.lescop@unilim.fr

Current address:
Department of Mathematics
Science Lecture Building
UCD, Belfield
Dublin 4, Ireland

Received on 19.3.2003
and in revised form on 24.6.2003 (4492)