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On higher-power moments of $\Delta(x)$

by

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1. Introduction and main results. In this paper we shall study the higher-power moments of some error terms in analytic number theory, including $\Delta(x)$, $E(t)$, $P(x)$, $A(x)$ and $\Delta_a(x)$.

1.1. *Higher-power moments of* $\Delta(x)$. We begin with the Dirichlet divisor problem. Let $d(n)$ denote the divisor function. Dirichlet first proved that the error term

$$\Delta(x) := \sum_{n\le x}{}' d(n) - x\log x - (2\gamma-1)x, \qquad x \ge 2,$$

satisfies $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result is due to Huxley [10], who showed that

$$\Delta(x) \ll x^{23/73}(\log x)^{315/146}. \tag{1.1}$$

For a survey of the history of this problem, see Krätzel [19].

For the lower bounds, the best results read

$$\Delta(x) = \Omega_+(x^{1/4}(\log x)^{1/4}(\log\log x)^{(3+\log 4)/4} \times \exp(-c\sqrt{\log\log\log x})) \qquad (c>0) \tag{1.2}$$

due to Hafner [6], and

$$\Delta(x) = \Omega_-(x^{1/4}\exp(c'(\log\log x)^{1/4}(\log\log\log x)^{-3/4})) \qquad (c'>0) \tag{1.3}$$

due to Corrádi and Kátai [3]. It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}),$$

which is supported by the classical mean-square result

$$\int_2^T \Delta^2(x)\,dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}T^{3/2} + O(T\log^5 T) \tag{1.4}$$

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proved by Tong [24]. On the other hand, Voronoï [26] proved that

$$\int_2^T \Delta(x)\,dx = T/4 + O(T^{3/4}), \tag{1.5}$$

which in conjunction with (1.4) shows that $\Delta(x)$ has a lot of sign changes and cancellations between the positive and negative portions.

Tsang [25] studied the third- and fourth-power moments of $\Delta(x)$. He proved that

$$\int_2^T \Delta^3(x)\,dx = \frac{3c_1}{28\pi^3}T^{7/4} + O(T^{7/4-\delta_1+\varepsilon}), \tag{1.6}$$

$$\int_2^T \Delta^4(x)\,dx = \frac{3c_2}{64\pi^4}T^{2} + O(T^{2-\delta_2+\varepsilon}), \tag{1.7}$$

where $\delta_1 = 1/14$, $\delta_2 = 1/23$,

$$c_1 := \sum_{\alpha,\beta,h\in\mathbb{N}} (\alpha\beta(\alpha+\beta))^{-3/2}h^{-9/4}|\mu(h)|d(\alpha^2h)d(\beta^2h)d((\alpha+\beta)^2h),$$

$$c_2 := \sum_{\substack{n,m,k,l\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} (nmkl)^{-3/4}d(n)d(m)d(k)d(l),$$

and $\mu(h)$ is the Möbius function. (1.6) shows, just as Tsang [25] stated, that "$\Delta^3(x)$ is biased strongly towards the positive side and does not even out as much as $\Delta(x)\ldots$ this suggests that $\Delta(x)$ frequently attains exceptionally large values. This is also consistent with the fact that the Ω_+ result in (1.2) is stronger than the Ω_- result in (1.3) (here)."

In this paper we shall improve (1.6) and (1.7) further. We shall also study the fifth-power moment of $\Delta(x)$.

For the third-power moment of $\Delta(x)$, we prove the following

THEOREM 1. *We have*

$$\int_2^T \Delta^3(x)\,dx = \frac{3c_1}{28\pi^3}T^{7/4} + O(T^{3/2+\varepsilon}). \tag{1.8}$$

For the fourth-power moment of $\Delta(x)$, we prove the following

THEOREM 2. *Suppose (κ,λ) is any exponent pair. Then the asymptotic formula*

$$\int_2^T \Delta^4(x)\,dx = \frac{3c_2}{64\pi^4}T^{2} + O(T^{2-\delta_2(\kappa,\lambda)+\varepsilon}) \tag{1.9}$$

holds, *where*

$$\delta_2(\kappa,\lambda) := \frac{1-\eta(\kappa,\lambda)}{7}, \qquad \eta(\kappa,\lambda) := \frac{2\lambda+2\kappa}{2+2\kappa}.$$

Throughout this paper we shall use the definition $\eta(\kappa,\lambda) = (2\lambda+2\kappa)/(2+2\kappa)$, which is well known in the theory of exponent pairs. If (κ,λ) is an exponent pair, then

$$A(\kappa,\lambda) := \left(\frac{\kappa}{2+2\kappa}, \frac{\lambda}{2+2\kappa}+\frac{1}{2}\right), \qquad B(\kappa,\lambda) := (\lambda-1/2, \kappa+1/2)$$

are both exponent pairs. Now take

$$\begin{aligned}(\kappa,\lambda) &= BA^2(ABA)(AB)^2(ABA)(AB)^2(ABA)(ABA^3)\left(\frac{1}{2},\frac{1}{2}\right)\\ &= \left(\frac{141841}{368018}, \frac{193668}{368018}\right).\end{aligned}$$

Then

$$(\kappa_0,\lambda_0) = A(\kappa,\lambda) = \left(\frac{141841}{1019718}, \frac{703527}{1019718}\right)$$

is Rankin's exponent pair [23] such that

$$\eta(\kappa,\lambda) = 2\kappa_0 + 2\lambda_0 - 1 = 0.65804\ldots$$

See also p. 58 of Krätzel [19].

The above exponent pair yields

Corollary 1. *We have*

$$\int_2^T \Delta^4(x)\,dx = \frac{3c_2}{64\pi^4}\,T^2 + O(T^{2-2/41}). \tag{1.10}$$

If the exponent pair hypothesis is true, namely, if $(\varepsilon, 1/2+\varepsilon)$ is an exponent pair, then

$$\int_2^T \Delta^4(x)\,dx = \frac{3c_2}{64\pi^4}\,T^2 + O(T^{2-1/14+\varepsilon}). \tag{1.11}$$

For the fifth-power moment of $\Delta(x)$, Heath-Brown [9] proved that

$$\int_2^T \Delta^5(x)\,dx = CT^{9/4}(1+o(1)) \tag{1.12}$$

for some constant C. But Heath-Brown did not give C explicitly. In this paper we shall prove

THEOREM 3. *Suppose* (κ,λ) *is any exponent pair with* $4\lambda+\kappa<3$. *Then*

$$\int_2^T \Delta^5(x)\,dx = \frac{5(2c_3-c_4)}{288\pi^5}\,T^{9/4} + O(T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon}), \tag{1.13}$$

where

$$\delta_3(\kappa,\lambda) := \frac{1}{15}\left(\frac{3}{4}-\eta(\kappa,\lambda)\right),$$

$$c_3 := \sum_{\substack{n,m,k,l,r\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}}} (nmklr)^{-3/4}d(n)d(m)d(k)d(l)d(r),$$

$$c_4 := \sum_{\substack{n,m,k,l,r\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}=\sqrt{r}}} (nmklr)^{-3/4}d(n)d(m)d(k)d(l)d(r).$$

As shown in Section 5, both series above are convergent.

The above exponent pair again yields

COROLLARY 2. *We have*

$$\int_2^T \Delta^5(x)\,dx = \frac{5(2c_3-c_4)}{288\pi^5}\,T^{9/4} + O(T^{9/4-5/816}). \tag{1.14}$$

If the exponent pair hypothesis is true, *then*

$$\int_2^T \Delta^5(x)\,dx = \frac{5(2c_3-c_4)}{288\pi^5}\,T^{9/4} + O(T^{9/4-1/60+\varepsilon}). \tag{1.15}$$

REMARK 1. Numerical computation shows that $c_3-c_4>0$ and hence $2c_3-c_4>c_3$. Thus Theorem 3 means that $\Delta^5(x)$ also has the properties similar to $\Delta^3(x)$.

REMARK 2. For the third-power moment of $\Delta(x)$, it is the most important thing to study the distribution of the values of $\sqrt{n}+\sqrt{m}-\sqrt{k}$ for $(n,m,k)\in\mathbb{N}^3$. The points (n,m,k) with $\sqrt{n}+\sqrt{m}-\sqrt{k}=0$ provide the main term. For other points, we need two things. First, we need a good lower bound of $|\sqrt{n}+\sqrt{m}-\sqrt{k}|$ if $\sqrt{n}+\sqrt{m}-\sqrt{k}\neq 0$, which was established by Lemma 2 of Tsang [25]. Secondly, we need a good upper bound of the number of solutions of the inequality $|\sqrt{n}+\sqrt{m}-\sqrt{k}|<\Delta$ for small $\Delta>0$, which will be given in Lemma 2.5 below. Note that Lemma 2.5 is best possible when Δ is very small. Maybe the exponent $3/2$ in Theorem 1 is also best possible.

Lemma 2.5 also plays an important role in the proof of the fifth-power moment of $\Delta(x)$.

1.2. *Higher-power moments of $E(t)$.* Let

$$E(t) := \int_0^t |\zeta(1/2+iu)|^2\,du - t\log(t/2\pi) - (2\gamma-1)t, \qquad t \ge 2. \tag{1.16}$$

Many results for $E(t)$ parallel to those for $\Delta(x)$ have been obtained (see, for example, Heath-Brown [8, 9], Jutila [14, 15], Hafner and Ivić [7], Meurman [21]). In particular, Meurman [21] proved that

$$\int_2^T E^2(t)\,dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}}\,T^{3/2} + O(T\log^5 T), \tag{1.17}$$

which is an analogue of (1.4). See Ivić [11] for a survey.

Tsang [25] studied the third- and fourth-power moment of $E(t)$ by using Atkinson's formula (see [1]) and proved that

$$\int_2^T E^3(t)\,dt = \frac{6}{7}(2\pi)^{-3/4}c_1T^{7/4} + O(T^{7/4-\delta_4+\varepsilon}), \tag{1.18}$$

$$\int_2^T E^4(t)\,dt = \frac{3}{8\pi}\,c_2T^2 + O(T^{2-\delta_5+\varepsilon}) \tag{1.19}$$

with $\delta_4 > 0$ and $\delta_5 > 0$. On p. 83 of [25], Tsang mentioned that (1.18) holds for $\delta_4 = 1/36$, but did not specify the permissible value of δ_5 in (1.19). Ivić [13] proved in a different way that (1.18) holds with $\delta_4 = 1/14$ and (1.19) holds with $\delta_5 = 1/23$.

We prove the following

THEOREM 4. *We have*

$$\int_2^T E^3(t)\,dt = \frac{6}{7}(2\pi)^{-3/4}c_1T^{7/4} + O(T^{7/4-1/12}\log^{3/2} T), \tag{1.20}$$

$$\int_2^T E^4(t)\,dt = \frac{3}{8\pi}\,c_2T^2 + O(T^{2-2/41}), \tag{1.21}$$

$$\int_2^T E^5(t)\,dt = \frac{5(2c_3-c_4)}{9(2\pi)^{5/4}}\,T^{9/4} + O(T^{9/4-5/816}). \tag{1.22}$$

REMARK 4. The exponent $1/12$ in (1.20) comes from Theorem 2 of Ivić [13]. We believe that it could be replaced by $1/4$ in view of the analogy between $E(t)$ and the Dirichlet divisor problem (Jutila [14, 15]). But we have not been able to prove this.

1.3. *Higher-power moments of $P(x)$.* The Gauss circle problem is to estimate the error term defined by

$$P(x) := \sum_{n\le x}{}' r(n) - \pi x,$$

where $r(n)$ denotes the number of ways n can be written as $n = x^2 + y^2$ for $x, y \in \mathbb{Z}$. It has been shown that $P(x)$ resembles $\Delta(x)$ in many respects. See Krätzel [19] for a survey of the circle problem.

Kátai [17] proved that

$$\int_2^T P^2(x)\,dx = \left(\frac{1}{3\pi^2}\sum_{n=1}^{\infty} r^2(n)n^{-3/2}\right)T^{3/2} + O(T\log^2 T). \tag{1.23}$$

Tsang [25] also studied the third- and the fourth-power moments of $P(x)$. He proved that

$$\int_2^T P^3(x)\,dx = -\frac{3c_5}{7\sqrt{2}\,\pi^3}\,T^{7/4} + O(T^{7/4-\delta_6}), \tag{1.24}$$

$$\int_2^T P^4(x)\,dx = \frac{3c_6}{16\pi^4}\,T^2 + O(T^{2-\delta_7}), \tag{1.25}$$

where $\delta_6 > 0$ and $\delta_7 > 0$ are unspecified constants, while c_5 and c_6 are constants defined respectively by the formulas for c_1 and c_2 with $d(\cdot)$ replaced by $r(\cdot)$.

Lemma 3 of Müller [22] yields a truncated Voronoï formula similar to that of $\Delta(x)$. So by Tsang's arguments for $\Delta(x)$, we know that (1.24) is true with $\delta_6 = 1/14 - \varepsilon$ and (1.25) is true with $\delta_7 = 1/23 - \varepsilon$.

We prove the following

THEOREM 5. *We have*

$$\int_2^T P^3(x)\,dx = -\frac{3c_5}{7\sqrt{2}\,\pi^3}\,T^{7/4} + O(T^{3/2+\varepsilon}), \tag{1.26}$$

$$\int_2^T P^4(x)\,dx = \frac{3c_6}{16\pi^4}\,T^2 + O(T^{2-2/41}), \tag{1.27}$$

$$\int_2^T P^5(x)\,dx = -\frac{5(2c_7 - c_8)}{36\sqrt{2}\,\pi^5}\,T^{9/4} + O(T^{9/4-5/816}), \tag{1.28}$$

where c_7 and c_8 are constants defined respectively by the formulas for c_3 and c_4 with $d(\cdot)$ replaced by $r(\cdot)$.

1.4. *Higher-power moments of $A(x)$.* Let $a(n)$ be the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \ge 12$ for the full modular group and define

$$A(x) := \sum_{n\le x}{}' a(n), \qquad x \ge 2.$$

It is well known that $A(x)$ has no main term and $A(x) \ll x^{\kappa/2-1/6+\varepsilon}$. Ivić [12] proved that

$$\int_1^T A^2(x)\,dx = B_2 T^{\kappa+1/2} + O(T^\kappa \log^5 T), \tag{1.29}$$

where

$$B_2 = \frac{1}{4\kappa+2}\sum_{n=1}^{\infty} a^2(n) n^{-\kappa-1/2}.$$

Cai [2] studied the third- and fourth-power moments of $A(x)$. He proved that

$$\int_1^T A^3(x)\,dx = B_3 T^{(6\kappa+1)/4} + O(T^{(6\kappa+1)/4-1/14+\varepsilon}), \tag{1.30}$$

$$\int_1^T A^4(x)\,dx = B_4 T^{2\kappa} + O(T^{2\kappa-1/23+\varepsilon}), \tag{1.31}$$

where

$$B_3 := \frac{3}{4(6\kappa+1)\pi^3} \sum_{\substack{n,m,k\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}=\sqrt{k}}} (nmk)^{-\kappa/2-1/4} a(n)a(m)a(k),$$

$$B_4 := \frac{3}{64\kappa\pi^4} \sum_{\substack{n,m,k,l\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} (nmkl)^{-\kappa/2-1/4} a(n)a(m)a(k)a(l).$$

We prove the following

THEOREM 6. *We have*

$$\int_1^T A^3(x)\,dx = B_3 T^{(6\kappa+1)/4} + O(T^{3\kappa/2+\varepsilon}), \tag{1.32}$$

$$\int_1^T A^4(x)\,dx = B_4 T^{2\kappa} + O(T^{2\kappa-2/41}), \tag{1.33}$$

$$\int_1^T A^5(x)\,dx = B_5 T^{(10\kappa-1)/4} + O(T^{(10\kappa-1)/4-5/816}), \tag{1.34}$$

where

$$B_5 = \frac{5(2c_9 - c_{10})}{32(10\kappa-1)\pi^5},$$

$$c_9 = \sum_{\substack{n,m,k,l,r\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}}} (nmklr)^{-\kappa/2-1/4}a(n)a(m)a(k)a(l)a(r),$$
$$c_{10} = \sum_{\substack{n,m,k,l,r\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}=\sqrt{r}}} (nmklr)^{-\kappa/2-1/4}a(n)a(m)a(k)a(l)a(r).$$

1.5. *Higher-power moments of $\Delta_a(x)$.* Let $-1/2 < a < 0$ be a fixed real number and set

$$\Delta_a(x) := {\sum_{n\le x}}' \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a}\, x^{1+a} + \frac{1}{2}\,\zeta(-a),$$

where $\sigma_a(n) := \sum_{d\mid n} d^a$. Kiuchi [18] proved that

$$\int_2^T \Delta_a^2(x)\,dx = C_2(a)T^{3/2+a} + O(T^{5/4+a/2+\varepsilon}) \quad (-1/2 < a < 0) \tag{1.35}$$

with

$$C_2(a) := \frac{\zeta^2(3/2)}{2\pi^2(6+4a)\zeta(3)}\,\zeta(3/2-a)\zeta(3/2+a).$$

Meurman [20] refined (1.35) to

$$\int_2^T \Delta_a^2(x)\,dx = C_2(a)T^{3/2+a} + O(T) \quad (-1/2 < a < 0). \tag{1.36}$$

For higher-power moments of $\Delta_a(x)$, we have the following theorems:

THEOREM 7. *Suppose* $0 > a > (2-\sqrt{13})/6 = -0.267\ldots$ *Then*

$$\int_2^T \Delta_a^3(x)\,dx = C_3(a)T^{(7+6a)/4} + O(T^{(7+6a)/4-\delta_1(a)+\varepsilon}), \tag{1.37}$$

where

$$C_3(a) := \frac{3c_1(a)}{(28+24a)\pi^3},$$
$$c_1(a) := \sum_{\substack{n,m,k\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}=\sqrt{k}}} (nmk)^{-(3+2a)/4}\sigma_a(n)\sigma_a(m)\sigma_a(k),$$
$$\delta_1(a) := (3+8a-12a^2)/(12-24a) > 0.$$

THEOREM 8. *Suppose* $0 > a > (3-\sqrt{17})/8 = -0.140\ldots$ *Then*

$$\int_2^T \Delta_a^4(x)\,dx = C_4(a)T^{2+2a} + O(T^{2+2a-\delta_2(a)+\varepsilon}), \tag{1.38}$$

where

$$C_4(a) := \frac{3c_2(a)}{64(1+a)\pi^4},$$

$$c_2(a) := \sum_{\substack{n,m,k,l\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}}} (nmkl)^{-(3+2a)/4}\sigma_a(n)\sigma_a(m)\sigma_a(k)\sigma_a(l),$$

$$\delta_2(a) := \min\left(\frac{1+6a-8a^2}{8-16a}, \frac{1}{7}\left(\frac{1}{3}+2a\right)\right) > 0.$$

THEOREM 9. *Suppose* $0 > a > -1/30$. *Then*

$$\int_2^T \Delta_a^5(x)\,dx = C_5(a)T^{(9+10a)/4} + O(T^{(9+10a)/4-\delta_3(a)+\varepsilon}), \tag{1.39}$$

where

$$C_5(a) := \frac{5(2c_3(a)-c_4(a))}{(288+320a)\pi^5},$$

$$c_3(a) := \sum_{\substack{n,m,k,l,r\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}}} (nmklr)^{-(3+2a)/4}\sigma_a(n)\sigma_a(m)\sigma_a(k)\sigma_a(l)\sigma_a(r),$$

$$c_4(a) := \sum_{\substack{n,m,k,l,r\in\mathbb{N}\\ \sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}=\sqrt{r}}} (nmklr)^{-(3+2a)/4}\sigma_a(n)\sigma_a(m)\sigma_a(k)\sigma_a(l)\sigma_a(r),$$

$$\delta_3(a) := (1+30a)/180 > 0.$$

Both (1.35) and (1.36) are true for all $-1/2 < a < 0$. However for higher-power moments, we can only get asymptotics in shorter intervals. We propose the following conjecture, which is partly confirmed by the above three theorems.

CONJECTURE. *Suppose* $-1/2 < a < 0, k = 3, 4, 5$. *Then*

$$\int_2^T \Delta_a^k(x)\,dx = C_k(a)T^{(4+k+2ka)/4}(1+o(1)). \tag{1.40}$$

Notations. $\mathbb{N}$ denotes the set of all natural numbers; $n \sim N$ means $N < n \le 2N$; $n \asymp N$ means there exist two absolute positive constants C_1, C_2 such that $C_1N \le n \le C_2N$; $\#G$ denotes the number of elements of a finite set G; $\|t\|$ denotes the distance between t and its nearest integer; ε always denotes a sufficiently small positive constant which may be different at different places. We will use the inequality $d(n) \ll n^\varepsilon$ freely. $\mathrm{SC}(\sum)$ denotes the summation condition of the sum $\sum$, and $\sum'_{n\le x}$ means that the final term should be weighted with $1/2$ if x is an integer.

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2. Some preliminary lemmas. The following lemmas will be needed in our proof.

LEMMA 2.1 ([25, Lemma 2]). *If n, m, k are natural numbers such that $\sqrt{n} + \sqrt{m} - \sqrt{k} \neq 0$, then*

$$|\sqrt{n} + \sqrt{m} - \sqrt{k}| \geq \frac{1}{27} \max(n, m, k)^{-3/2}. \blacksquare$$

LEMMA 2.2 ([25, Lemma 3]). *If $n, m, k, l \in \mathbb{N}$ such that $\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l} \neq 0$ or $\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l} \neq 0$, then respectively,*

$$|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| \gg \max(n, m, k, l)^{-7/2}$$

or

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l}| \gg \max(n, m, k, l)^{-7/2}. \blacksquare$$

LEMMA 2.3. *If $n, m, k, l, r \in \mathbb{N}$ such that $\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l} - \sqrt{r} \neq 0$ or $\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} - \sqrt{r} \neq 0$, then respectively,*

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l} - \sqrt{r}| \gg \max(n, m, k, l, r)^{-15/2}$$

or

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} - \sqrt{r}| \gg \max(n, m, k, l, r)^{-15/2}.$$

Proof. The proof is the same as that of Lemma 2 of [25]. ■

LEMMA 2.4. *Suppose $K \geq 10$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ and $0 < \delta < 1/2$. Then for any exponent pair (κ, λ), we have*

$$\#\{k \sim K : \|\beta + \alpha\sqrt{k}\| < \delta\} \ll K\delta + |\alpha|^{\kappa/(1+\kappa)} K^{(2\lambda+\kappa)/(2+2\kappa)} + |\alpha|^{-1} K^{1/2}.$$

The implied constant is absolute.

Proof. Suppose $K^{-1/2} \leq |\alpha| \leq K^{(2+\kappa-2\lambda)/2\kappa}$; otherwise the estimate is trivial. We begin with the formula (3.9) of [25], namely,

$$\#\{k \sim K : \|\beta + \alpha\sqrt{k}\| < \delta\} \leq 2K\delta + KH^{-1} + \sum_{1 \leq h \leq H} h^{-1} |S(h)|,$$

where

$$S(h) = \sum_{k \sim K} e(h\alpha\sqrt{k}).$$

This formula follows from the Erdős–Turán inequality [4].

If $1 \leq h \leq \sqrt{K}/2|\alpha|$, by the Kuz'min–Landau inequality [5, Theorem 2.1] we get

$$S(h) \ll \frac{K^{1/2}}{h|\alpha|}.$$

For $h > \sqrt{K}/2|\alpha|$, by the exponent pair (κ, λ) we get

$$S(h) \ll (h|\alpha|)^{\kappa} K^{\lambda - \kappa/2}.$$

Hence Lemma 2.4 follows by taking $H = [|\alpha|^{-\kappa/(1+\kappa)} K^{(2+\kappa-2\lambda)/(2+2\kappa)}]$. ∎

LEMMA 2.5. *Suppose that* $1 \le N \le M \asymp K$ *and* $0 < \Delta < K^{1/2}$. *Let* $\mathcal{A}_1(N, M, K; \Delta)$ *denote the number of solutions of the inequality*

$$|\sqrt{n} + \sqrt{m} - \sqrt{k}| < \Delta$$

with $n \sim N$, $m \sim M$, $k \sim K$. *Then*

$$\mathcal{A}_1(N, M, K; \Delta) \ll \Delta K^{1/2} (MN)^{1+\varepsilon} + (MN)^{1/2+\varepsilon}.$$

In particular, *if* $\Delta K^{1/2} \gg 1$, *then*

$$\mathcal{A}_1(N, M, K; \Delta) \ll \Delta K^{1/2} NM.$$

Proof. We suppose $M \le K$; the case $M > K$ is the same. Suppose (n, m, k) satisfies $|\sqrt{n} + \sqrt{m} - \sqrt{k}| < \Delta$. Then $\sqrt{n} + \sqrt{m} = \sqrt{k} + \theta\Delta$ for some $|\theta| < 1$. Thus $n + m + 2\sqrt{nm} = k + u$ with

$$|u| = |2k^{1/2}\theta\Delta + \theta^2\Delta^2| < 2k^{1/2}\Delta + \Delta^2 < 10K^{1/2}\Delta.$$

So $\mathcal{A}_1(N, M, K; \Delta)$ does not exceed the number of solutions of the inequality

$$|n + m + 2\sqrt{nm} - k| < 10K^{1/2}\Delta \tag{2.1}$$

with $n \sim N$, $m \sim M$, $k \sim K$.

If $K^{1/2}\Delta \gg 1$, then for fixed (n, m), the number of k for which (2.1) holds is $\ll 1 + K^{1/2}\Delta \ll K^{1/2}\Delta$. Hence

$$\mathcal{A}_1(N, M, K; \Delta) \ll \Delta K^{1/2} NM.$$

Now suppose $K^{1/2}\Delta \le 1/40$. Then for fixed n, m, there is at most one k such that (2.1) holds. If such a k exists, then $\|2\sqrt{nm}\| < 10K^{1/2}\Delta$. Let

$$\mathcal{G} = \{(n, m) \in \mathbb{N}^2 : \|2\sqrt{nm}\| < 10K^{1/2}\Delta,\ n \sim N,\ m \sim M\},$$
$$\mathcal{G}' = \{n \in \mathbb{N} : \|2\sqrt{n}\| < 10K^{1/2}\Delta,\ MN < n \le 4MN\}.$$

Then

$$\mathcal{A}_1(N, M, K; \Delta) \le \#\mathcal{G} \ll \#\mathcal{G}'(MN)^{\varepsilon}.$$

By Lemma 2.4 with $\alpha = 2, \beta = 0$ and $(\kappa, \lambda) = (1/2, 1/2)$ we get

$$\#\mathcal{G}' \ll \Delta K^{1/2} MN + (MN)^{1/2}.$$

Thus

$$\mathcal{A}_1(N, M, K; \Delta) \le \Delta K^{1/2} (MN)^{1+\varepsilon} + (MN)^{1/2+\varepsilon}. \ ∎$$

LEMMA 2.6. *Suppose* $1 \le N \le M$, $1 \le K \le L$, $N \le K$, $M \asymp L$, $0 < \Delta \ll L^{1/2}$. *Let* $\mathcal{A}_2(N, M, K, L; \Delta)$ *denote the number of solutions of the inequality*

$$|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| < \Delta \tag{2.2}$$

with $n \sim N,\ m \sim M,\ k \sim K,\ l \sim L$. *Then for any exponent pair* (κ,λ) *we have*

$$\mathcal{A}_2(N,M,K,L;\Delta) \ll \Delta L^{1/2}NMK + NL + NKM^{\eta(\kappa,\lambda)}.$$

In particular, *if* $\Delta L^{1/2} \gg 1$, *then*

$$\mathcal{A}_2(N,M,K,L;\Delta) \ll \Delta L^{1/2}NMK.$$

REMARK. The term NL appears only when the equation $n = k$ has solutions with $n \sim N,\ k \sim K$.

Proof. If (n,m,k,l) satisfies $|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}| < \Delta$, then

$$l = m + 2m^{1/2}(\sqrt{n}-\sqrt{k}) + (\sqrt{n}-\sqrt{k})^2 + u$$

with $|u| \le C\Delta L^{1/2}$ for some absolute constant $C > 0$. Hence the quantity $\mathcal{A}_2(N,M,K,L;\Delta)$ does not exceed the number of solutions of

$$|2m^{1/2}(\sqrt{n}-\sqrt{k}) + (\sqrt{n}-\sqrt{k})^2 + m - l| < C\Delta L^{1/2} \tag{2.3}$$

with $n \sim N,\ m \sim M,\ k \sim K,\ l \sim L$.

If $\Delta L^{1/2} \gg 1$, then for fixed (n,m,k), the number of l for which (2.3) holds is $\ll 1 + \Delta L^{1/2} \ll \Delta L^{1/2}$. Hence

$$\mathcal{A}_2(N,M,K,L;\Delta) \ll \Delta L^{1/2}NMK.$$

Now suppose $\Delta L^{1/2} \le 1/4C$. Let $\mathcal{S}_1$ denote the set of solutions of (2.3) such that $n = k$, and $\mathcal{S}_2$ the set of solutions such that $n \ne k$, respectively. Then

$$\mathcal{A}_2(N,M,K,L;\Delta) \le \#\mathcal{S}_1 + \#\mathcal{S}_2.$$

Obviously, $\#\mathcal{S}_1 \ll NL$. It remains to estimate $\#\mathcal{S}_2$. For fixed (n,m,k), there is at most one l such that (2.3) holds. If such an l exists, then

$$\|2m^{1/2}(\sqrt{n}-\sqrt{k}) + (\sqrt{n}-\sqrt{k})^2\| < C\Delta L^{1/2}.$$

By Lemma 2.4 with $\alpha = 2(\sqrt{n}-\sqrt{k}), \beta = (\sqrt{n}-\sqrt{k})^2$ we get

$$\#\mathcal{S}_2 \ll \Delta L^{1/2}NMK + C_1M^{(2\lambda+\kappa)/(2+2\kappa)} + C_2M^{1/2},$$
$$C_1 := \sum_{n\ne k} |\sqrt{n}-\sqrt{k}|^{\kappa/(1+\kappa)}, \qquad C_2 := \sum_{n\ne k} |\sqrt{n}-\sqrt{k}|^{-1}.$$

Trivially, we have

$$C_1 \ll NK^{1+\kappa/(2+2\kappa)} \ll NKM^{\kappa/(2+2\kappa)}.$$

Moreover,

$$C_2 \ll K^{1/2} \sum_{N<n<k\le 2K} 1/(k-n) \ll K^{1/2}N\log K \ll NK.$$

Now Lemma 2.6 follows from the above estimates. ∎

LEMMA 2.7. *Suppose* $1 \le N \le M \le K \asymp L$ *and* $0 < \Delta \ll L^{1/2}$. *Let* $\mathcal{A}_3(N, M, K, L; \Delta)$ *denote the number of solutions of the inequality*

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l}| < \Delta \tag{2.4}$$

with $n \sim N,\ m \sim M,\ k \sim K,\ l \sim L$. *Then for any exponent pair* (κ, λ) *we have*

$$\mathcal{A}_3(N, M, K, L; \Delta) \ll \Delta L^{1/2} NMK + NMK^{\eta(\kappa,\lambda)}.$$

In particular, *if* $\Delta L^{1/2} \gg 1$, *then*

$$\mathcal{A}_3(N, M, K, L; \Delta) \ll \Delta L^{1/2} NMK.$$

Proof. We omit the proof since it is similar to that of Lemma 2.6. ∎

LEMMA 2.8. *Suppose* $1 \le N \le M \le K$, $1 \le L \le R$, $K \asymp R$ *and* $0 < \Delta \ll R^{1/2}$. *Let* $\mathcal{A}_4(N, M, K, L, R; \Delta)$ *denote the number of solutions of the inequality*

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l} - \sqrt{r}| < \Delta \tag{2.5}$$

with $n \sim N,\ m \sim M,\ k \sim K,\ l \sim L,\ r \sim R$. *Then for any exponent pair* (κ, λ) *we have*

$$\mathcal{A}_4(N, M, K, L, R; \Delta) \ll \Delta R^{1/2} NMKL + R(MN)^{1/2+\varepsilon} + NMLK^{\eta(\kappa,\lambda)}.$$

In particular, *if* $\Delta R^{1/2} \gg 1$, *then*

$$\mathcal{A}_4(N, M, K, L, R; \Delta) \ll \Delta R^{1/2} NMKL.$$

REMARK. The term $R(MN)^{1/2+\varepsilon}$ appears only when the equation $\sqrt{n} + \sqrt{m} = \sqrt{l}$ has solutions with $n \sim N, m \sim M, l \sim L$.

Proof. If (n, m, k, l, r) satisfies (2.5), then

$$r = k + 2k^{1/2}(\sqrt{n} + \sqrt{m} - \sqrt{l}) + (\sqrt{n} + \sqrt{m} - \sqrt{l})^2 + u$$

with $|u| \le C^* \Delta R^{1/2}$ for some absolute constant $C^* > 0$. Hence the quantity $\mathcal{A}_4(N, M, K, L, R; \Delta)$ does not exceed the number of solutions of

$$|2k^{1/2}(\sqrt{n} + \sqrt{m} - \sqrt{l}) + (\sqrt{n} + \sqrt{m} - \sqrt{l})^2 + k - r| < C^* \Delta R^{1/2} \tag{2.6}$$

with $n \sim N,\ m \sim M,\ k \sim K,\ l \sim L,\ r \sim R$.

If $\Delta R^{1/2} \gg 1$, then for fixed (n, m, k, l), the number of r for which (2.6) holds is $\ll 1 + \Delta R^{1/2} \ll \Delta R^{1/2}$. Hence

$$\mathcal{A}_4(N, M, K, L, R; \Delta) \ll \Delta R^{1/2} NMKL.$$

Now suppose $\Delta R^{1/2} \le 1/4C^*$. Let $\mathcal{T}_1$ denote the set of solutions of (2.6) such that $\sqrt{n} + \sqrt{m} = \sqrt{l}$, and $\mathcal{T}_2$ the set of solutions such that $\sqrt{n} + \sqrt{m} \ne \sqrt{l}$. Then

$$\mathcal{A}_4(N, M, K, L, R; \Delta) \le \#\mathcal{T}_1 + \#\mathcal{T}_2.$$

We estimate $\#\mathcal{T}_1$ first. Suppose that the equation $\sqrt{n}+\sqrt{m}=\sqrt{l}$ has solutions; otherwise $\#\mathcal{T}_1=0$. Since $\sqrt{n}+\sqrt{m}=\sqrt{l}$, the inequality (2.6) becomes $k=r$ and hence

$$\#\{(k,r):|\sqrt{k}-\sqrt{r}|<\Delta\}\ll R.$$

The equation $\sqrt{n}+\sqrt{m}=\sqrt{l}$ implies $\sqrt{mn}\in\mathbb{N}$, that is, mn is a square. Thus

$$\begin{aligned}\#\{(n,m,l):\sqrt{n}+\sqrt{m}=\sqrt{l}\}&\ll\#\{(n,m):nm \text{ is a square}\}\\&\ll(MN)^{\varepsilon}\#\{n:n \text{ is a square}\}\ll(MN)^{1/2+\varepsilon}.\end{aligned}$$

The above two estimates imply

$$\#\mathcal{T}_1\ll R(MN)^{1/2+\varepsilon}.$$

Now we estimate $\#\mathcal{T}_2$. For fixed (n,m,k,l), there is at most one r such that (2.6) holds. If such an r exists, then

$$\|2k^{1/2}(\sqrt{n}+\sqrt{m}-\sqrt{l})+(\sqrt{n}+\sqrt{m}-\sqrt{l})^2\|<C\Delta R^{1/2}.$$

By Lemma 2.4 with $\alpha=2(\sqrt{n}+\sqrt{m}-\sqrt{l}),\beta=(\sqrt{n}+\sqrt{m}-\sqrt{l})^2$ we get

$$\begin{aligned}\#\mathcal{T}_2&\ll\Delta R^{1/2}NMKL+C_3K^{(2\lambda+\kappa)/(2+2\kappa)}+C_4K^{1/2},\\C_3&:=\sum_{\substack{n,m,l\\ \sqrt{n}+\sqrt{m}\neq\sqrt{l}}}|\sqrt{n}+\sqrt{m}-\sqrt{l}|^{\kappa/(1+\kappa)},\\C_4&:=\sum_{\substack{n,m,l\\ \sqrt{n}+\sqrt{m}\neq\sqrt{l}}}|\sqrt{n}+\sqrt{m}-\sqrt{l}|^{-1}.\end{aligned}$$

Trivially we have

$$C_3\ll NML^{1+\kappa/(2+2\kappa)}\ll NMLK^{\kappa/(2+2\kappa)}.$$

Write $C_4=C_{41}+C_{42}$, where

$$\begin{aligned}C_{41}&=\sum_{|\sqrt{n}+\sqrt{m}-\sqrt{l}|\geq L^{1/2}/50}|\sqrt{n}+\sqrt{m}-\sqrt{l}|^{-1},\\C_{42}&=\sum_{0<|\sqrt{n}+\sqrt{m}-\sqrt{l}|\leq L^{1/2}/50}|\sqrt{n}+\sqrt{m}-\sqrt{l}|^{-1}.\end{aligned}$$

Trivially we have

$$C_{41}\ll NML^{1/2}.$$

If the inequality

$$(*)\qquad|\sqrt{n}+\sqrt{m}-\sqrt{l}|\leq L^{1/2}/50$$

has no solutions, then $C_{42}=0$. So we suppose $(*)$ has solutions, which implies that $M\asymp L$. By Lemma 2.1, $|\sqrt{n}+\sqrt{m}-\sqrt{l}|\gg L^{-3/2}$ for any such

(n,m,l). By a splitting argument and Lemma 2.5 we find that for some $L^{-3/2} \ll \delta \ll L^{1/2}$,

$$\begin{aligned} C_{42} &\ll \frac{\log 2L}{\delta} \sum_{\delta<|\sqrt{n}+\sqrt{m}-\sqrt{l}|\le 2\delta} 1 \\ &\ll \frac{\log 2L}{\delta}\,(\delta L^{1/2}(MN)^{1+\varepsilon} + (MN)^{1/2+\varepsilon}) \\ &\ll L^{1/2}(MN)^{1+\varepsilon} + L^{3/2}(MN)^{1/2+\varepsilon} \ll N^{1/2+\varepsilon}M^{1+\varepsilon}L. \end{aligned}$$

From the above estimates we get

$$\#\mathcal{T}_2 \ll \Delta R^{1/2}NMKL + NMLK^{\eta(\kappa,\lambda)}.$$

Now Lemma 2.8 follows from the estimates of $\#\mathcal{T}_1$ and $\#\mathcal{T}_2$. ∎

LEMMA 2.9. *Suppose* $1 \le N \le M \le K \le L \asymp R$, $0 < \Delta \ll R^{1/2}$. *Let* $\mathcal{A}_5(N,M,K,L,R;\Delta)$ *denote the number of solutions of the inequality*

$$|\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}-\sqrt{r}| < \Delta \tag{2.7}$$

with $n \sim N,\ m \sim M,\ k \sim K,\ l \sim L,\ r \sim R$. *Then for any exponent pair* (κ,λ) *we have*

$$\mathcal{A}_5(N,M,K,L,R;\Delta) \ll \Delta R^{1/2}NMKL + NMKL^{\eta(\kappa,\lambda)}.$$

In particular, *if* $\Delta R^{1/2} \gg 1$, *then*

$$\mathcal{A}_5(N,M,K,L,R;\Delta) \ll \Delta R^{1/2}NMKL.$$

Proof. We omit the proof since it is similar to that of Lemma 2.8 and much easier. ∎

3. The third-power moment of $\Delta(x)$. In this section we prove Theorem 1. We begin with the following truncated form of Voronoï's formula [11, (2.25)]

$$\Delta(x) = (\pi\sqrt{2})^{-1}\sum(x) + O(x^{1/2+\varepsilon}y^{-1/2}), \tag{3.1}$$

where

$$\sum(x) = \sum_{n\le y} d(n)n^{-3/4}x^{1/4}\cos(4\pi\sqrt{nx}-\pi/4)$$

and $1 \le y \ll x$.

Suppose $T \ge 10$ and take $y = T$ in (3.1). From the elementary formula $(a+b)^3 - a^3 \ll |b|a^2 + |b|^3$ and (1.4) we get

$$\int_T^{2T} \Delta^3(x)\,dx = \int_T^{2T} \Big(\sum(x)\Big)^3\,dx + O(T^{3/2+\varepsilon}). \tag{3.2}$$

We shall prove

$$\int_T^{2T} \Big(\sum(x)\Big)^3 dx = \frac{3c_1}{4\sqrt{2}} \int_T^{2T} x^{3/4}\, dx + O(T^{3/2+\varepsilon}). \tag{3.3}$$

Theorem 1 follows easily from (3.2), (3.3).

Let

$$g = g(n,m,k) := (nmk)^{-3/4} d(n)d(m)d(k) \quad \text{for } n,m,k \le T$$

and $g = 0$ otherwise. We can write (equation (2.7) of Tsang [25])

$$\Big(\sum(x)\Big)^3 = S_0(x) + S_1(x) + S_2(x), \tag{3.4}$$

where

$$S_0(x) := \frac{3}{4\sqrt{2}} \sum_{\sqrt{n}+\sqrt{m}=\sqrt{k}} g x^{3/4},$$

$$S_1(x) := \frac{3}{4} \sum_{\sqrt{n}+\sqrt{m}\ne\sqrt{k}} g x^{3/4} \cos(4\pi(\sqrt{n}+\sqrt{m}-\sqrt{k})\sqrt{x} - \pi/4),$$

$$S_2(x) := \frac{1}{4} \sum g x^{3/4} \cos(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k})\sqrt{x} - 3\pi/4).$$

From (2.12) of [25] we get

$$\int_T^{2T} S_0(x)\, dx = \frac{3c_1}{4\sqrt{2}} \int_T^{2T} x^{3/4}\, dx + O(T^{3/4+\varepsilon}). \tag{3.5}$$

From (2.14) of [25] we get

$$\int_T^{2T} S_2(x)\, dx \ll T^{5/4+\varepsilon} y^{1/4} \ll T^{3/2+\varepsilon}. \tag{3.6}$$

Now we estimate $\int_T^{2T} S_1(x)\, dx$. By the second mean-value theorem we get

$$\begin{aligned} \int_T^{2T} S_1(x)\, dx &\ll \sum_{\substack{n,m,k\le T \\ \sqrt{n}+\sqrt{m}\ne\sqrt{k}}} g \min\Big(T^{7/4}, \frac{T^{5/4}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}|}\Big) \\ &\ll T^{\varepsilon} H(N,M,K), \end{aligned} \tag{3.7}$$

where

$$H(N,M,K) = \sum_{\substack{n\sim N, m\sim M, k\sim K \\ \sqrt{n}+\sqrt{m}\ne\sqrt{k}}} g \min\Big(T^{7/4}, \frac{T^{5/4}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}|}\Big)$$

with $1 \ll N \le M$.

If $K < M/10$, then $|\sqrt{n}+\sqrt{m}-\sqrt{k}| \gg M^{1/2}$ and trivially we have

$$H(N,M,K) \ll \frac{T^{5/4+\varepsilon}NMK}{(NMK)^{3/4}M^{1/2}} \ll T^{5/4+\varepsilon}y^{1/4} \ll T^{3/2+\varepsilon}.$$

Similarly if $K > 10M$, we also have

$$H(N,M,K) \ll T^{3/2+\varepsilon}.$$

Later we always suppose $M \asymp K$. Write

$$H(N,M,K) = H_1(N,M,K) + H_2(N,M,K) + H_3(N,M,K), \tag{3.8}$$

where

$$H_1(N,M,K) = T^{7/4} \sum_{0<|\sqrt{n}+\sqrt{m}-\sqrt{k}|\le T^{-1/2}} g,$$
$$H_2(N,M,K) = T^{5/4} \sum_{T^{-1/2}<|\sqrt{n}+\sqrt{m}-\sqrt{k}|\le (40E^{1/2})^{-1}} \frac{g}{|\sqrt{n}+\sqrt{m}-\sqrt{k}|},$$
$$H_3(N,M,K) = T^{5/4} \sum_{|\sqrt{n}+\sqrt{m}-\sqrt{k}|\ge (40E^{1/2})^{-1}} \frac{g}{|\sqrt{n}+\sqrt{m}-\sqrt{k}|},$$
$$E = \max(M,K) \asymp M \asymp K.$$

By Lemma 2.5 we get

$$\begin{aligned} H_1(N,M,K) &\ll \frac{T^{7/4+\varepsilon}}{(NMK)^{3/4}}\,\mathcal{A}_1(N,M,K;T^{-1/2}) \\ &\ll \frac{T^{7/4+\varepsilon}}{(NMK)^{3/4}}\,(T^{-1/2}K^{1/2}MN + (MN)^{1/2}) \\ &\ll T^{5/4+\varepsilon}y^{1/4} + T^{7/4+\varepsilon}(MN)^{-1/4}K^{-3/4} \ll T^{3/2+\varepsilon}, \end{aligned} \tag{3.9}$$

where we used the estimate $E \gg T^{1/3}$ which follows from Lemma 2.1.

By a splitting argument and Lemma 2.5 we get (notice $\delta \gg K^{-1/2}$)

$$\begin{aligned} H_3(N,M,K) &\ll \frac{T^{5/4+\varepsilon}}{(NMK)^{3/4}\delta} \sum_{\delta<|\sqrt{n}+\sqrt{m}-\sqrt{k}|\le 2\delta} 1 \\ &\ll \frac{T^{5/4+\varepsilon}}{(NMK)^{3/4}}\,K^{1/2}MN \ll T^{5/4+\varepsilon}y^{1/4} \ll T^{3/2+\varepsilon}. \end{aligned} \tag{3.10}$$

Finally we estimate $H_2(N,M,K)$. We consider two cases: $NMK^3 \ll T$ and $NMK^3 \gg T$. If $NMK^3 \ll T$, then by Lemma 2.1 and the estimate

$$\sum_{|\sqrt{n}+\sqrt{m}-\sqrt{k}|\le (40E^{1/2})^{-1}} 1 \ll NM$$

we get

$$H_2(N,M,K) \ll \frac{T^{5/4+\varepsilon}K^{3/2}MN}{(NMK)^{3/4}} \ll T^{5/4+\varepsilon}(MN)^{1/4}K^{3/4} \ll T^{3/2+\varepsilon}. \tag{3.11}$$

Now suppose $NMK^3 \gg T$. By the splitting argument and Lemma 2.5 again we get

$$\begin{aligned} H_2(N,M,K) &\ll \frac{T^{5/4+\varepsilon}}{(NMK)^{3/4}\delta} \sum_{\delta<|\sqrt{n}+\sqrt{m}-\sqrt{k}|\le 2\delta} 1 \\ &\ll \frac{T^{5/4+\varepsilon}}{(NMK)^{3/4}} (K^{1/2}MN + (MN)^{1/2}\delta^{-1}) \\ &\ll T^{5/4+\varepsilon}y^{1/4} + T^{7/4+\varepsilon}(MN)^{-1/4}K^{-3/4} \ll T^{3/2+\varepsilon}. \end{aligned} \tag{3.12}$$

Thus from (3.7)–(3.12) we get

$$\int_T^{2T} S_1(x)\,dx \ll T^{3/2+\varepsilon}. \tag{3.13}$$

Now (3.3) follows from (3.4)–(3.6) and (3.13).

4. The fourth-power moment of $\Delta(x)$. In this section we prove Theorem 2. Suppose $T \ge 10$. From (3.1) and the inequality $(a+b)^4 - a^4 \ll |b|\,|a|^3 + |b|^4$, we get

$$\begin{aligned} &\int_T^{2T} \Delta^4(x)\,dx \\ &= \frac{1}{(\pi\sqrt{2})^4}\int_T^{2T}\Big(\sum(x)\Big)^4 dx + O\bigg(\frac{T^{1/2+\varepsilon}}{y^{1/2}}\int_T^{2T}\Big|\sum(x)\Big|^3 dx + \frac{T^{3+\varepsilon}}{y^2}\bigg) \\ &= \frac{1}{(\pi\sqrt{2})^4}\int_T^{2T}\Big(\sum(x)\Big)^4 dx + O\bigg(\frac{T^{9/4+\varepsilon}}{y^{1/2}}\bigg) \end{aligned} \tag{4.1}$$

for $T^{1/2} \ll y \ll T$. Take $y = T^{3/4}$. We shall prove that

$$\int_T^{2T}\Big(\sum(x)\Big)^4 dx = \frac{3c_2}{8}\int_T^{2T} x\,dx + O(T^{2-\delta_2(\kappa,\lambda)+\varepsilon}) \tag{4.2}$$

for any exponent pair (κ, λ). Theorem 2 follows easily from (4.1), (4.2).

Let

$$g_1 = g_1(n,m,k,l) := (nmkl)^{-3/4}d(n)d(m)d(k)d(l) \quad \text{for } n,m,k,l \le y,$$

and $g_1 = 0$ otherwise.

Equation (3.4) of Tsang [25] reads

$$\Big(\sum(x)\Big)^4 = S_3(x) + S_4(x) + S_5(x) + S_6(x), \tag{4.3}$$

where

$$S_3(x) := \frac{3}{8} \sum_{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}} g_1 x,$$
$$S_4(x) := \frac{3}{8} \sum_{\sqrt{n}+\sqrt{m}\neq\sqrt{k}+\sqrt{l}} g_1 x \cos(4\pi(\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l})\sqrt{x}),$$
$$S_5(x) := \frac{1}{2} \sum g_1 x \sin(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l})\sqrt{x}),$$
$$S_6(x) := -\frac{1}{8} \sum g_1 x \cos(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l})\sqrt{x}).$$

From (3.7) of [25] we get

$$\int_T^{2T} S_3(x)\,dx = \frac{3c_2}{8} \int_T^{2T} x\,dx + O(T^{2-3/16+\varepsilon}). \tag{4.4}$$

By (3.8) of [25] we get

$$\int_T^{2T} S_6(x)\,dx \ll T^{3/2+\varepsilon} y^{1/2} \ll T^{2-1/8+\varepsilon}. \tag{4.5}$$

Now let us consider the contribution of $S_4(x)$. By the second mean-value theorem we get

$$\begin{aligned}\int_T^{2T} S_4(x)\,dx &\ll \sum_{\substack{n,m,k,l\le y\\ \sqrt{n}+\sqrt{m}\neq\sqrt{k}+\sqrt{l}}} g_1 \min\bigg(T^2, \frac{T^{3/2}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}|}\bigg)\\ &\ll T^{\varepsilon} G(N,M,K,L),\end{aligned} \tag{4.6}$$

where

$$G(N,M,K,L) = \textstyle\sum_1 g_1 \min\bigg(T^2, \dfrac{T^{3/2}}{|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}|}\bigg),$$
$$\begin{aligned}\mathrm{SC}(\textstyle\sum_1) : {}&\sqrt{n}+\sqrt{m}\neq\sqrt{k}+\sqrt{l}, 1 \le N \le M \le y, 1 \le K \le L \le y,\\ &n \sim N, m \sim M, k \sim K, l \sim L.\end{aligned}$$

If $M \ge 100L$, then $|\sqrt{n}+\sqrt{m}-\sqrt{k}-\sqrt{l}| \gg M^{1/2}$, so the trivial estimate yields

$$G(N,M,K,L) \ll \frac{T^{3/2+\varepsilon} NMKL}{(NMKL)^{3/4} M^{1/2}} \ll T^{3/2+\varepsilon} y^{1/2} \ll T^{2-1/8+\varepsilon}.$$

If $L > 100M$, we get the same estimate. So later we always suppose that $M \asymp L$. Let $\eta_1 = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}$. Write

$$G(N, M, K, L, R) = G_1 + G_2 + G_3, \tag{4.7}$$

where

$$G_1 := T^2 \sum_{|\eta_1| \le T^{-1/2}} g_1,$$
$$G_2 := T^{3/2} \sum_{T^{-1/2} < |\eta_1| \le 1} g_1 |\eta_1|^{-1},$$
$$G_3 := T^{3/2} \sum_{|\eta_1| \gg 1} g_1 |\eta_1|^{-1}.$$

We estimate G_1 first. From $|\eta_1| \le T^{-1/2}$ we get $M \asymp L \gg T^{1/7}$ via Lemma 2.2. By Lemma 2.6 (suppose $N \le K$; the case $N > K$ is the same) we get

$$\begin{aligned} G_1 &\ll \frac{T^{2+\varepsilon}}{(NMKL)^{3/4}} \mathcal{A}_2(N, M, K, L; T^{-1/2}) \\ &\ll \frac{T^{2+\varepsilon}}{(NMKL)^{3/4}} (T^{-1/2} L^{1/2} NMK + NL + NKM^{\eta(\kappa,\lambda)}) \\ &\ll T^{3/2+\varepsilon} (NMK)^{1/4} L^{-1/4} + T^{2+\varepsilon} N^{1/4} K^{-3/4} L^{-1/2} \\ &\quad + T^{2+\varepsilon} (NK)^{1/4} M^{-(3/2-\eta(\kappa,\lambda))} \\ &\ll T^{3/2+\varepsilon} y^{1/2} + T^{2+\varepsilon} L^{-1/2} + T^{2+\varepsilon} M^{-(1-\eta(\kappa,\lambda))} \\ &\ll T^{2-1/14+\varepsilon} + T^{2-\delta_2(\kappa,\lambda)+\varepsilon} \ll T^{2-\delta_2(\kappa,\lambda)+\varepsilon}. \end{aligned} \tag{4.8}$$

Now we estimate G_2. Suppose also $N \le K$. By a splitting argument and Lemma 2.6 again we see for some $T^{-1/2} \ll \delta < 1$ that

$$\begin{aligned} G_2 &\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} \mathcal{A}_2(N, M, K, L; 2\delta) \\ &\ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} (\delta L^{1/2} NMK + NL + NKM^{\eta(\kappa,\lambda)}) \\ &\ll T^{3/2+\varepsilon} y^{1/2} + T^{3/2+\varepsilon} L^{-1/2} \delta^{-1} + T^{3/2+\varepsilon} M^{-(1-\eta(\kappa,\lambda))} \delta^{-1}. \end{aligned} \tag{4.9}$$

We consider two cases: $M \asymp L \ll T^{1/7}$ and $M \asymp L \gg T^{1/7}$. If $M \ll T^{1/7}$, from Lemma 2.2 we get $\delta^{-1} \ll M^{7/2}$. Thus (4.9) gives

$$\begin{aligned} G_2 &\ll T^{3/2+\varepsilon} y^{1/2} + T^{3/2+\varepsilon} M^3 + T^{3/2+\varepsilon} M^{5/2+\eta(\kappa,\lambda)} \\ &\ll T^{2-\delta_2(\kappa,\lambda)+\varepsilon}. \end{aligned} \tag{4.10}$$

If $M \asymp L \gg T^{1/7}$, using $\delta^{-1} \ll T^{1/2}$ (4.9) yields

$$G_2 \ll T^{3/2+\varepsilon} y^{1/2} T^{2+\varepsilon} L^{-1/2} + T^{2+\varepsilon} M^{-(1-\eta(\kappa,\lambda))} \ll T^{2-\delta_2(\kappa,\lambda)+\varepsilon}. \tag{4.11}$$

For G_3, by a splitting argument and Lemma 2.6 again (notice $|\eta_1| \gg 1$) we get

$$G_3 \ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}\delta} \sum_{\delta<|\eta_1|\le 2\delta,\, \delta\gg 1} 1 \ll \frac{T^{3/2+\varepsilon}}{(NMKL)^{3/4}} L^{1/2}NMK \ll T^{3/2+\varepsilon}y^{1/2} \ll T^{2-1/8+\varepsilon}. \tag{4.12}$$

Combining (4.7)–(4.12), we get

$$\int_T^{2T} S_4(x)\,dx \ll T^{2-\delta_2(\kappa,\lambda)+\varepsilon}. \tag{4.13}$$

In the same way we can show that

$$\int_T^{2T} S_5(x)\,dx \ll T^{2-\delta_2(\kappa,\lambda)+\varepsilon} \tag{4.14}$$

if we use Lemma 2.7 instead of Lemma 2.6. Now (4.2) follows from (4.4), (4.5), (4.13) and (4.14).

5. The fifth-power moment of $\Delta(x)$. In this section we prove Theorem 3. Suppose $T \ge 10$. From (3.1) and the inequality $(a+b)^5 - a^5 \ll |b|a^4 + |b|^5$, we get

$$\begin{aligned}\int_T^{2T} \Delta^5(x)\,dx &= \frac{1}{(\pi\sqrt{2})^5}\int_T^{2T}\Big(\sum(x)\Big)^5 dx + O\Big(\frac{T^{1/2+\varepsilon}}{y^{1/2}}\int_T^{2T}\Big(\sum(x)\Big)^4 dx + \frac{T^{7/2+\varepsilon}}{y^{5/2}}\Big)\\ &= \frac{1}{(\pi\sqrt{2})^5}\int_T^{2T}\Big(\sum(x)\Big)^5 dx + O\Big(\frac{T^{5/2+\varepsilon}}{y^{1/2}}\Big)\end{aligned} \tag{5.1}$$

for $T^{1/2} \ll y \ll T$. Take $y = T^{3/5}$. We shall prove

$$\frac{1}{(\pi\sqrt{2})^5}\int_T^{2T}\Big(\sum(x)\Big)^5 dx = \frac{5(2c_3-c_4)}{288\pi^5}T^{9/4} + O(T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon}), \tag{5.2}$$

where (κ,λ) is any exponent pair with $4\lambda+\kappa<3$. Theorem 2 follows easily from (5.1), (5.2).

Let

$$\begin{aligned} g_2 &= g_2(n,m,k,l,r)\\ &:= (nmklr)^{-3/4}d(n)d(m)d(k)d(l)d(r) \quad \text{for } n,m,k,l,r \le y,\end{aligned}$$

and $g_2 = 0$ otherwise.

Similar to equation (2.7) of Tsang [25], we can write

$$\Big(\sum(x)\Big)^5 = S_7(x) + S_8(x) + S_9(x) + S_{10}(x) + S_{11}(x), \tag{5.3}$$

where

$$S_7(x) := \frac{5\cos(\pi/4)}{8} \sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}} g_2 x^{5/4},$$

$$S_8(x) := \frac{5}{8} \sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}\neq\sqrt{l}+\sqrt{r}} g_2 x^{5/4}$$
$$\times \cos(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l}-\sqrt{r})\sqrt{x}-\pi/4),$$

$$S_9(x) := \frac{5\cos(-3\pi/4)}{16} \sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}=\sqrt{r}} g_2 x^{5/4},$$

$$S_{10}(x) := \frac{5}{16} \sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}\neq\sqrt{r}} g_2 x^{5/4}$$
$$\times \cos(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}-\sqrt{r})\sqrt{x}-3\pi/4),$$

$$S_{11}(x) := \frac{1}{16} \sum g_2 x^{5/4} \cos(4\pi(\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}+\sqrt{r})\sqrt{x}-5\pi/4).$$

Let us consider the sum $S_7(x)$ first. The classical result of Besicovitch says that the square roots of squarefree numbers are linearly independent over the integers. From this result we know that $\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}$ if and only if (n,m,k,l,r) satisfies one of the following cases:

Case 1.1: $l=n$, $m=m_*^2h$, $k=k_*^2h$, $r=r_*^2h$, $m_*+k_*=r_*$, $\mu(h)\neq 0$;
Case 1.2: $l=m$, $n=n_*^2h$, $k=k_*^2h$, $r=r_*^2h$, $n_*+k_*=r_*$, $\mu(h)\neq 0$;
Case 1.3: $l=k$, $m=m_*^2h$, $n=n_*^2h$, $r=r_*^2h$, $m_*+n_*=r_*$, $\mu(h)\neq 0$;
Case 1.4: $r=n$, $m=m_*^2h$, $k=k_*^2h$, $l=l_*^2h$, $m_*+k_*=l_*$, $\mu(h)\neq 0$;
Case 1.5: $r=m$, $n=n_*^2h$, $k=k_*^2h$, $l=l_*^2h$, $n_*+k_*=l_*$, $\mu(h)\neq 0$;
Case 1.6: $r=k$, $m=m_*^2h$, $n=n_*^2h$, $l=l_*^2h$, $m_*+n_*=l_*$, $\mu(h)\neq 0$;
Case 2: $n=n_*^2h$, $m=m_*^2h$, $k=k_*^2h$, $l=l_*^2h$, $r=r_*^2h$, $n_*+m_*+k_*=l_*+r_*$, $\mu(h)\neq 0$, $l_*\neq n_*$, $l_*\neq m_*$, $l_*\neq k_*$, $r_*\neq n_*$, $r_*\neq m_*$, $r_*\neq k_*$.

So in the sum

$$\sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}} g_2$$
$$= \sum_{\substack{n,m,k,l,r\le y\\ \sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}}} (nmklr)^{-3/4} d(n)d(m)d(k)d(l)d(r),$$

if we let the variables n, m, k, l, r run over all natural numbers, the error is

$$\ll \sum_{n>y} n^{-3/2} d^2(n) + \Big| \sum_{\substack{n,m,k\le y \\ \sqrt{n}+\sqrt{m}=\sqrt{k}}} (nmk)^{-3/4} d(n)d(m)d(k) - c_1 \Big|$$

$$+ \sum_{n^2h>y,\, l^2h\gg y} h^{-15/4}(nmklr)^{-3/2} d(n^2)d(m^2)d(k^2)d(l^2)d(r^2)d^5(h)$$

$$\ll y^{-1/2+\varepsilon} + \sum_{n^2h>y,\, l^2h\gg y} h^{-15/4}(nl)^{-3/2} d(n^2)d(l^2)d^5(h)$$

$$\ll y^{-1/2+\varepsilon}.$$

Thus we get

$$\int_T^{2T} S_7(x)\,dx = \frac{5\sqrt{2}}{16}\, c_3 \int_T^{2T} x^{5/4}\,dx + O(T^{9/4-3/10+\varepsilon}). \tag{5.4}$$

Similarly, we get

$$\int_T^{2T} S_9(x)\,dx = -\frac{5\sqrt{2}}{32}\, c_4 \int_T^{2T} x^{5/4}\,dx + O(T^{9/4-3/10+\varepsilon}). \tag{5.5}$$

The contribution of $S_{11}(x)$ is

$$\int_T^{2T} S_{11}(x)\,dx \ll \sum_{n,m,k,l,r\le y} \frac{g_2 T^{7/4}}{\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}+\sqrt{r}} \tag{5.6}$$

$$\ll T^{7/4+\varepsilon} \sum_{n\le m\le k\le l\le r\le y} (nmklr)^{-3/4} r^{-1/2}$$

$$\ll T^{7/4+\varepsilon} y^{3/4} \ll T^{11/5+\varepsilon}.$$

Now let us consider the contribution of $S_8(x)$. By the second mean-value theorem we get

$$\int_T^{2T} S_8(x)\,dx \tag{5.7}$$

$$\ll \sum_{\substack{n,m,k,l,r\le y \\ \sqrt{n}+\sqrt{m}+\sqrt{k}\ne\sqrt{l}+\sqrt{r}}} g_2 \min\left(T^{9/4}, \frac{T^{7/4}}{|\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l}-\sqrt{r}|}\right)$$

$$\ll T^{\varepsilon} F(N, M, K, L, R),$$

where

$$F(N, M, K, L, R) = \sum\nolimits_2 g_2 \min\left(T^{9/4}, \frac{T^{7/4}}{|\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l}-\sqrt{r}|}\right),$$

$$\mathrm{SC}(\textstyle\sum_2) : \sqrt{n}+\sqrt{m}+\sqrt{k}\neq\sqrt{l}+\sqrt{r},\ 1\le N\le M\le K\le y,\ 1\le L\le R\le y,$$
$$n\sim N,\ m\sim M,\ k\sim K,\ l\sim L,\ r\sim R.$$

If $R < K/100$, then $|\sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l}-\sqrt{r}| \gg K^{1/2}$, so the trivial estimate yields

$$F(N,M,K,L,R) \ll \frac{T^{7/4+\varepsilon}NMKLR}{(NMKLR)^{3/4}K^{1/2}} \ll T^{7/4+\varepsilon}y^{3/4} \ll T^{11/5+\varepsilon}.$$

If $R > 100K$, we get the same estimate. So later we always suppose that $R \asymp K$. Let $\eta_2 = \sqrt{n}+\sqrt{m}+\sqrt{k}-\sqrt{l}-\sqrt{r}$. Write

$$F(N,M,K,L,R) = F_1 + F_2 + F_3, \tag{5.8}$$

where

$$F_1 := T^{9/4} \sum_{|\eta_2|\le T^{-1/2}} g_2,$$
$$F_2 := T^{7/4} \sum_{T^{-1/2}<|\eta_2|\le 1} g_2|\eta_2|^{-1},$$
$$F_3 := T^{7/4} \sum_{|\eta_2|\gg 1} g_2|\eta_2|^{-1}.$$

We estimate F_1 first. From $|\eta_2| \le T^{-1/2}$ we get $R \gg T^{1/15}$ via Lemma 2.3. By Lemma 2.8 (suppose $M \le L$; the case $L < M$ is the same) we get

$$\begin{aligned} F_1 &\ll \frac{T^{9/4+\varepsilon}}{(NMKLR)^{3/4}}\,\mathcal{A}_4(N,M,K,L,R;T^{-1/2}) \\ &\ll \frac{T^{9/4+\varepsilon}}{(NMKLR)^{3/4}}\,(T^{-1/2}R^{1/2}NMKL + R(MN)^{1/2} \\ &\qquad\qquad + NMLK^{\eta(\kappa,\lambda)}) \\ &\ll T^{7/4+\varepsilon}y^{3/4} + \frac{T^{9/4+\varepsilon}}{(MN)^{1/4}L^{3/4}R^{1/2}} + \frac{T^{9/4+\varepsilon}}{R^{3/4-\eta(\kappa,\lambda)}} \\ &\ll T^{9/4-1/30+\varepsilon} + T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon} \ll T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon}. \end{aligned} \tag{5.9}$$

Now we estimate F_2. Suppose also $M \le L$. By a splitting argument and Lemma 2.8 again we infer for some $T^{-1/2} \ll \delta < 1$ that

$$\begin{aligned} F_2 &\ll \frac{T^{7/4+\varepsilon}}{(NMKLR)^{3/4}\delta}\,\mathcal{A}_2(N,M,K,L,R;2\delta) \\ &\ll T^{7/4+\varepsilon}y^{3/4} + \frac{T^{7/4+\varepsilon}}{(MN)^{1/4}L^{3/4}R^{1/2}\delta} + \frac{T^{7/4+\varepsilon}}{R^{3/4-\eta(\kappa,\lambda)}\delta}. \end{aligned} \tag{5.10}$$

We consider two cases: $K \asymp R \ll T^{1/15}$ and $K \asymp R \gg T^{1/15}$. If $R \ll T^{1/15}$, from Lemma 2.3 we get $\delta^{-1} \ll M^{15/2}$. Thus (5.10) gives

$$F_2 \ll T^{7/4+\varepsilon}y^{3/4} + T^{7/4+\varepsilon}R^7 + T^{7/4+\varepsilon}R^{27/4+\eta(\kappa,\lambda)} \ll T^{9/4-1/30+\varepsilon} + T^{2-\delta_3(\kappa,\lambda)+\varepsilon} \ll T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon}. \tag{5.11}$$

If $R \gg T^{1/15}$, using $\delta^{-1} \ll T^{1/2}$ and (5.10) yields

$$F_2 \ll T^{7/4+\varepsilon}y^{3/4} + \frac{T^{9/4+\varepsilon}}{R^{1/2}} + \frac{T^{9/4+\varepsilon}}{R^{3/4-\eta(\kappa,\lambda)}} \ll T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon}. \tag{5.12}$$

For F_3, by a splitting argument and Lemma 2.8 again (notice $|\eta_2| \gg 1$) we get

$$\begin{aligned} F_3 &\ll \frac{T^{7/4+\varepsilon}}{(NMKLR)^{3/4}\delta} \sum_{\delta<|\eta|\le 2\delta,\, \delta\gg 1} 1 \\ &\ll \frac{T^{7/4+\varepsilon}}{(NMKLR)^{3/4}} R^{1/2}NMKL \\ &\ll T^{7/4+\varepsilon}y^{3/4} \ll T^{11/5+\varepsilon}. \end{aligned} \tag{5.13}$$

Combining (5.7)–(5.13), we get

$$\int_T^{2T} S_8(x)\,dx \ll T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon}. \tag{5.14}$$

In the same way we can show that

$$\int_T^{2T} S_{10}(x)\,dx \ll T^{9/4-\delta_3(\kappa,\lambda)+\varepsilon} \tag{5.15}$$

if we use Lemma 2.9 instead of Lemma 2.8.

Now (5.2) follows from (5.4)–(5.6), (5.14) and (5.15).

6. Proofs of Theorems 4–9. $P(x)$ has the following truncated Voronoï formula:

$$P(x) = -\frac{1}{\pi}\sum_{n\le y} r(n)n^{-3/4}x^{1/4}\cos(4\pi\sqrt{nx} + \pi/4) + O(x^{1/2+\varepsilon}y^{-1/2}) \tag{6.1}$$

for $1 \le y \ll x$, which follows from Lemma 3 of Müller [22]. $A(x)$ has the following truncated Voronoï formula:

$$A(x) = \frac{1}{\pi\sqrt{2}}\, x^{\kappa/2-1/4}\sum_{n\le y} a(n)n^{-\kappa/2-1/4}\cos(4\pi\sqrt{nx} - \pi/4) + O(x^{\kappa/2+\varepsilon}y^{-1/2}) \tag{6.2}$$

for $1 \le y \ll x$, which is a special case of Theorem 1.1 of Jutila [16]. $\Delta_a(x)$

has the following truncated Voronoï formula [18]:

$$(6.3)\quad \Delta_a(x) = \frac{1}{\pi\sqrt{2}} \sum_{n\le y} \sigma_a(n) n^{-3/4-a/2} x^{1/4+a/2} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} y^{-1/2})$$

for $1 \le y \ll x$. So by the same arguments of $\Delta(x)$, we get Theorems 5–9 immediately. Note that in the proofs of Theorems 8 and 9, only the exponent pair $(1/2, 1/2)$ was used.

Now we prove Theorem 4. We shall follow Ivić [13]. Let

$$(6.4)\quad \Delta^*(x) := \frac{1}{2} \sum_{n\le 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1), \qquad x \ge 1.$$

Then for $1 \ll N \ll x$, we have [13, (7)]

$$(6.5)\quad \Delta^*(x) = \frac{1}{\pi\sqrt{2}} \sum_{n\le N} (-1)^n d(n) n^{-3/4} x^{1/4} \cos(4\pi\sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}).$$

Jutila [15] proved that

$$(6.6)\quad \int_0^T \left(E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right)\right)^2 dt \ll T^{4/3} \log^3 T,$$

which means that $E(t)$ is well approximated by $2\pi\Delta^*(t/2\pi)$ at least in the mean square sense.

Ivić [13] proved that

$$(6.7)\quad \int_0^T E^3(t)\, dt = (2\pi)^4 \int_0^{T/2\pi} (\Delta^*(t))^3\, dt + O(T^{5/3} \log^{3/2} T),$$

$$(6.8)\quad \int_0^T E^4(t)\, dt = (2\pi)^5 \int_0^{T/2\pi} (\Delta^*(t))^4\, dt + O(T^{23/12} \log^{3/2} T).$$

Using Ivić's argument we can get

$$(6.9)\quad \int_0^T E^5(t)\, dt = (2\pi)^6 \int_0^{T/2\pi} (\Delta^*(t))^5\, dt + O(T^{13/6} \log^{3/2} T).$$

We need the estimates

$$(6.10)\quad \int_0^T |E(t)|^A\, dt \ll T^{1+A/4}, \qquad \int_0^T |\Delta^*(t)|^A\, dt \ll T^{1+A/4} \qquad (0 \le A \le 9),$$

which follow from Heath-Brown [9].

By (6.6), (6.10) and Cauchy's inequality we get

$$\int_0^T E^5(t)\,dt - (2\pi)^6 \int_0^{T/2\pi} (\Delta^*(t))^5\,dt$$
$$= \int_0^T \left(E^5(t) - \left(2\pi\Delta^*\left(\frac{t}{2\pi}\right)\right)^5\right) dt$$
$$\ll \int_0^T \left|E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right)\right| \left(E^4(t) + \Delta^*\left(\frac{t}{2\pi}\right)^4\right) dt$$
$$\ll \left\{\int_0^T \left|E(t) - 2\pi\Delta^*\left(\frac{t}{2\pi}\right)\right|^2 dt\right\}^{1/2} \left\{\int_0^T \left(E^8(t) + \Delta^*\left(\frac{t}{2\pi}\right)^8\right) dt\right\}^{1/2}$$
$$\ll (T^{4/3}\log^3 T)^{1/2} T^{3/2} \ll T^{13/6}\log^{3/2} T,$$

that is, (6.9) holds.

Now the problem is reduced to evaluating the integral $\int_0^T (\Delta^*(t))^k\,dt$ ($k = 3, 4, 5$). By the same arguments as those for $\Delta(x)$, we get

$$\int_0^T (\Delta^*(t))^3\,dt = \frac{3c_1^*}{28\pi^3} T^{7/4} + O(T^{3/2+\varepsilon}), \tag{6.11}$$

$$\int_0^T (\Delta^*(t))^4\,dt = \frac{3c_2^*}{64\pi^4} T^2 + O(T^{2-2/41}), \tag{6.12}$$

$$\int_0^T (\Delta^*(t))^5\,dt = \frac{5(2c_3^* - c_4^*)}{288\pi^5} T^{9/4} + O(T^{9/4-5/816}), \tag{6.13}$$

where

$$c_1^* := \sum_{\sqrt{n}+\sqrt{m}=\sqrt{k}} (-1)^{n+m+k}(nmk)^{-3/4} d(n)d(m)d(k),$$
$$c_2^* := \sum_{\sqrt{n}+\sqrt{m}=\sqrt{k}+\sqrt{l}} (-1)^{n+m+k+l}(nmkl)^{-3/4} d(n)d(m)d(k)d(l),$$
$$c_3^* := \sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}=\sqrt{l}+\sqrt{r}} (-1)^{n+m+k+l+r}(nmklr)^{-3/4} d(n)d(m)d(k)d(l)d(r),$$
$$c_4^* := \sum_{\sqrt{n}+\sqrt{m}+\sqrt{k}+\sqrt{l}=\sqrt{r}} (-1)^{n+m+k+l+r}(nmklr)^{-3/4} d(n)d(m)d(k)d(l)d(r).$$

Ivić [13] proved that $c_1^* = c_1$, $c_2^* = c_2$. Now we prove that $c_3^* = c_3$. Suppose $(n,m,k,l,r) \in \mathbb{N}^5$ is such that $\sqrt{n}+\sqrt{m}+\sqrt{k} = \sqrt{l}+\sqrt{r}$. We shall prove that $n+m+k+l+r \in 2\mathbb{N}$. In Section 4 we concluded that

(n,m,k,l,r) must satisfy one of Cases 1.1 to 1.6 or Case 2. We only consider Case 1.1 and Case 2. Suppose $n=l$. Then $\sqrt{m}+\sqrt{k}=\sqrt{r}$. By the result of Besicovitch again we get

$$m=\alpha^2 h,\qquad k=\beta^2 h,\qquad r=\gamma^2 h,\qquad \alpha+\beta=\gamma.$$

Hence $n+m+k+l+r=2n+h(2\alpha^2+2\beta^2+2\alpha\beta)\in 2\mathbb{N}$. Now suppose that (n,m,k,l,r) satisfies Case 2. Then

$$n=n_*^2 h,\ \ m=m_*^2 h,\ \ k=k_*^2 h,\ \ l=l_*^2 h,\ \ r=r_*^2 h,\ \ n_*+m_*+k_*=l_*+r_*.$$

Using the simple congruence $n^2\equiv n \pmod 2$, we get

$$\begin{aligned} n+m+k+l+r &= (n_*^2+m_*^2+k_*^2+l_*^2+r_*^2)h \\ &\equiv (n_*+m_*+k_*+l_*+r_*)h \\ &= (2l_*+2r_*)h \equiv 0 \pmod 2, \end{aligned}$$

that is, $n+m+k+l+r\in 2\mathbb{N}$. Thus $c_3^*=c_3$. Similarly we get $c_4^*=c_4$.

Now Theorem 4 follows from (6.7)–(6.9), (6.11)–(6.13).

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