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On the diophantine equation $x(x-1)\ldots(x-(m-1)) = \lambda y(y-1)\ldots(y-(n-1)) + l$

by

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Dedicated to the memory of Professor Péter Kiss

1. Introduction. Many diophantine problems lead to equations of the form

$$f(x) = g(y), \tag{1}$$

where f and g are given polynomials with rational coefficients and x and y are unknown integers. In the special case $f(x) = y^n$ with integer $n \geq 2$, effective results were proved by several authors (see e.g. [3], [32], [37], [11], [17] and the references given there). Their proofs are based upon Baker's method. Extending some earlier works of Davenport, Lewis and Schinzel [13], Schinzel [30] and Fried [15], [16], Bilu and Tichy [7] gave an explicit finiteness criterion for (1). For certain applications of these results we refer to [5], [6], [10].

The title equation with $l = 0$ as well as equation (2) below were studied by several authors, including Saradha and Shorey [23]–[25] and Saradha, Shorey and Tijdeman [26]–[29]. For a survey of recent results we refer to [31]. Using an algebraic-geometrical approach, Beukers, Shorey and Tijdeman [4] proved the following

Theorem A. *Let m and n be integers with $1 < m \leq n$. Let d_1 and d_2 be positive rational numbers*, *with $d_1 \neq d_2$ if $m = n$. The equation*

$$x(x+d_1)\ldots(x+(m-1)d_1) = y(y+d_2)\ldots(y+(n-1)d_2) \tag{2}$$

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admits only finitely many integral solutions x, y except for the infinite class of solutions $x = y^2 + 3d_2y$, $-2d_2^2 - 3d_2y - y^2$ when $m = 2$, $n = 4$ and $d_1 = 2d_2^2$. Moreover, the equation admits infinitely many rational solutions x, y when $(m, n) = (2, 2), (2, 3), (2, 4), (3, 3)$ and $m = 2$, $n = 6$, $d_1 = 15d_2^3/4$. In all other cases there are only finitely many rational solutions.

In [4], the authors also established the following

Theorem B. *Let*

$$f(x) = x(x-1)\dots(x-(m-1)), \qquad g(y) = \lambda y(y-1)\dots(y-(n-1)),$$

where $m, n \in \mathbb{N}$ with $m \le n$ and $\lambda \in \mathbb{Q}$ with $\lambda \neq 0$. Then equation (1) *has only finitely many integer solutions apart from the following cases*:

- $m = n, \lambda = 1$ *or* $m = n$ *is odd*, $\lambda = -1$;
- $(m, n) = (2, 2)$; $(2, 4)$ *and* $\lambda = 1/4$.

Moreover, *equation* (1) *has only finitely many rational solutions except the above cases and the following ones*:

- $m = n = 4$, $\lambda = -\frac{9}{16}, -\frac{16}{9}$;
- $(m, n) = (2, 3)$; $(3, 3)$; $(2, 4)$ *and* $\lambda \neq \frac{1}{4}, -\frac{4}{9}$; $(2, 6)$ *and* $\lambda = \frac{16}{225}$.

In the proofs of Theorems A and B the main part was the characterization of the corresponding polynomials $f(x) - g(y)$ which are irreducible over $\mathbb{C}$ and for which the curves $f(x) - g(y) = 0$ have genus zero or one, respectively. Then the theorems of Siegel [33] and Faltings [14] implied the finiteness of the number of integer and rational solutions, respectively.

The purpose of this paper is to extend Theorems A and B to the inhomogeneous case, i.e. to the equation

$$x(x+d_1)\dots(x+(m-1)d_1) = \lambda y(y+d_2)\dots(y+(n-1)d_2) + l \tag{3}$$

in integers x, y and, more generally, in rationals x, y, where d_1, d_2 are positive rational numbers or $d_1 = d_2 = -1$, and $\lambda, l \in \mathbb{Q}$ with $\lambda \neq 0$ (cf. Theorems 2 and 3).

An important special case of (3) is the combinatorial diophantine equation

$$a\binom{x}{m} = b\binom{y}{n} + k \qquad \text{in integers } x \ge m,\ y \ge n, \tag{4}$$

where a, b are non-zero integers, and k, m, n are integers with $1 < m \le n$. For $k = 0$ and $l = 0$, $d_1 = d_2 = -1$, and for various special values of the parameters m, n, a, b and m, n, λ, respectively, all the integer solutions of (4) and (3) were given e.g. in [1], [8], [12], [18], [20], [21], [22], [35], [36], [38], [39]. In the trivial case $m = n$, $a = b$ and $k = 0$, equation (4) has obviously infinitely many solutions. In 1988, Kiss [19] showed that if p is a given odd prime, then the equation

$$\binom{x}{p} = \binom{y}{2}$$

has only finitely many integer solutions $x \geq p$, $y \geq 2$ which can be effectively determined. In 1991, this was generalized by Brindza [9] to the case when $p \geq 3$ is an arbitrary but fixed integer.

As an application of our Theorems 2, 3, we give a general finiteness result (cf. Theorem 1) for equation (4), which includes, in an ineffective form, the above-quoted results of [19] and [9] as a special case.

Our main results (cf. Theorems 1 to 3) on rational and integer solutions are formulated in Section 2. In Section 3, we characterize those polynomials $F(x,y) = x(x-1)\dots(x-(m-1)) - \lambda y(y-1)\dots(y-(n-1)) - l$ which are reducible in $\mathbb{C}[x,y]$ (cf. Theorem 4). Further, when F is reducible, we describe those cases when $F(x,y)=0$ has infinitely many rational or integer solutions (cf. Propositions 1 to 5). Section 4 is devoted to the study of curves $C : x(x-1)\dots(x-(m-1)) = \lambda y(y-1)\dots(y-(n-1)) + l$ having genus 0 or 1 (cf. Theorems 5, 6). The proofs can be found in Sections 5 to 7. To prove Theorems 4, 5 and 6, we generalize the method of proof of [4] from $l=0$ to arbitrary $l \in \mathbb{Q}$. Finally, as in [4], we also combine our Theorems 4 to 6 with results of Siegel and Faltings to establish our Theorems 2, 3 and 1.

2. The main results. Our Theorem 1 provides a characterization of those pairs (m,n) and parameters a,b,k for which equation (4) may have infinitely many solutions.

THEOREM 1. *Let* a, b, k, m, n *be integers with* $a \neq 0, b \neq 0$ *and* $1 < m \leq n$. *Apart from the cases*:

1) $m = n, a = b, k = 0$;
2) $(m,n) = (2,2)$;
3) $(m,n) = (2,4)$ *and* $(24k+3a)/b = 1$ *or* $-\frac{9}{16}$;
4) $(m,n) = (4,4)$ *and* $(24k+a)/b = 1$

(4) *has only finitely many integer solutions. Further*, *for each pair* (m,n) *listed in* 2) *to* 4), *the parameters* a,b *and* k *can be chosen so that* (4) *has infinitely many integer solutions.*

Our Theorem 1 includes as special cases the finiteness results on equation (4) mentioned in the Introduction.

Theorem 1 is a simple consequence of

THEOREM 2. *Let*

$$f(x) = x(x-1)\dots(x-(m-1)), \qquad g(y) = \lambda y(y-1)\dots(y-(n-1)) + l,$$

where $m, n \in \mathbb{N}$ with $m \leq n$ and $\lambda, l \in \mathbb{Q}$ with $\lambda \neq 0$. Then equation (1) *has only finitely many integer solutions apart from the following cases*:

1) $m = n$ *and* $\lambda = 1$, $l = 0$ *or* $m = n$ *is odd*, $\lambda = -1, l = 0$;
2) $(m, n) = (2, 2)$;
3) $(m, n) = (2, 4)$, $4\lambda - 4l = 1$ *or* $9\lambda + 16l = -4$;
4) $(m, n) = (4, 4)$, $\lambda - l = 1$.

Moreover, equation (1) *has only finitely many rational solutions except for cases* 1) *to* 4) *and the following ones*:

5) $(m, n) = (2, 3)$;
6) $(m, n) = (2, 4)$, $9\lambda + 16l \neq -4$;
7) $(m, n) = (2, 6)$, $-\frac{225}{64}\lambda + l = -\frac{1}{4}$;
8) $(m, n) = (3, 3)$;
9) $m = n = 4$, $-\lambda + l = \frac{9}{16}$ *and* $l \neq -\frac{7}{16}$, *or* $\frac{9}{16}\lambda + l = -1$ *and* $l \neq -\frac{7}{16}$.

Further, for each pair (m, n) *listed above, the parameters* λ, l *can be given in infinitely many ways such that equation* (1) *has infinitely many solutions* x, y.

It is easy to check that for $l = 0$, Theorem 2 yields Theorem B.

The following result is an inhomogeneous generalization of Theorem A.

Theorem 3. *Let d_1 and d_2 be positive rational numbers,* $\widetilde{\lambda} \in \mathbb{Q} \setminus \{0\}$, $\widetilde{l} \in \mathbb{Q}$. *Then the equation*

$$x(x + d_1) \dots (x + (m-1)d_1) = \widetilde{\lambda} y(y + d_2) \dots (y + (n-1)d_2) + \widetilde{l}, \tag{5}$$

where $m, n \in \mathbb{N}$ with $m \leq n$, has only finitely many integer and rational solutions, respectively, apart from the exceptional cases listed in Theorem 2 *with*

$$\lambda = (-1)^{m+n} \widetilde{\lambda} \, \frac{d_2^n}{d_1^m}, \qquad l = (-1)^m \, \frac{\widetilde{l}}{d_1^m}.$$

3. Study of irreducibility. The next results will be very important in our proofs. The first one describes those cases in which the polynomial $x(x-1)\dots(x-(m-1)) - \lambda y(y-1)\dots(y-(n-1)) - l$ is reducible in $\mathbb{C}[x, y]$. In Theorem 4 we deal with the more general situation when λ, l are not necessarily rational.

Theorem 4. *Let m and n be positive integers with $m \leq n$ and let $l \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$. If $F(x, y) = x(x-1)\dots(x-(m-1)) - \lambda y(y-1)\dots(y-(n-1)) - l$ is reducible in $\mathbb{C}[x, y]$, then one of the following conditions holds*:

1) $m = n$, $\lambda = 1$, $l = 0$, *in which case $x - y$ is a factor of $F(x, y)$*;
2) $m = n$ *is odd*, $\lambda = -1$, $l = 0$, *in which case $x + y - m + 1$ is a factor of $F(x, y)$*;

3) $(m,n)=(2,2)$, $\lambda-4l=1$, *in which case*

$$F(x,y)=\tfrac{1}{4}(2x-2Ay+A-1)(2x+2Ay-A-1)\quad \textit{where}\quad A=\sqrt{4l+1};$$

4) $(m,n)=(2,4)$, $4\lambda-4l=1$, *in which case*

$$F(x,y)=\tfrac{1}{4}(2x+2Ay^2-6Ay+2A-1)(2x-2Ay^2+6Ay-2A-1)$$

where $A=\sqrt{l+1/4}$;

5) $(m,n)=(4,4)$, $\lambda-l=1$, *in which case*

$$F(x,y)=(x^2-3x+Ay^2-3Ay+A+1)(x^2-3x-Ay^2+3Ay-A+1)$$

where $A=\sqrt{l+1}$;

6) $(m,n)=(4,4)$, $\lambda=-1$, $l=-\frac{7}{16}$ *in which case*

$$\begin{aligned}F(x,y)=&\big(x^2-\sqrt{2}xy-\big(3-\tfrac{3}{2}\sqrt{2}\big)x+y^2-\big(3-\tfrac{3}{2}\sqrt{2}\big)y+\tfrac{13}{4}-\tfrac{9}{4}\sqrt{2}\big)\\ &\times\big(x^2+\sqrt{2}xy-\big(3+\tfrac{3}{2}\sqrt{2}\big)x+y^2-\big(3+\tfrac{3}{2}\sqrt{2}\big)y+\tfrac{13}{4}+\tfrac{9}{4}\sqrt{2}\big);\end{aligned}$$

7) $(m,n)=(6,6)$, $\lambda=-1$, $l=-\frac{320}{27}$ *in which case*

$$\begin{aligned}F(x,y)=&\big(y^2-5y+x^2-5x+\tfrac{20}{3}\big)\big(y^4-10y^3-x^2y^2+5xy^2+\tfrac{85}{3}y^2\\ &+5x^2y-25xy-\tfrac{50}{3}y+x^4-10x^3+\tfrac{85}{3}x^2-\tfrac{50}{3}x+\tfrac{16}{9}\big).\end{aligned}$$

In [4, Theorem 2.1], Beukers, Shorey and Tijdeman characterized the polynomials

$$x(x+1)\dots(x+m-1)-\lambda y(y+1)\dots(y+n-1)$$

with $\lambda\in\mathbb{C}\backslash\{0\}$ which are reducible in $\mathbb{C}[x,y]$. For $\lambda\in\mathbb{C}\backslash\{0\}$, our Theorem 4 is an extension of Theorem 2.1 of [4] to the inhomogeneous case.

In the next propositions we consider those cases 3)–7) of Theorem 4 when $\lambda,l\in\mathbb{C}$ with $\lambda\neq 0$ and the polynomial $F(x,y)$ is reducible in $\mathbb{C}[x,y]$. In these cases we give necessary and sufficient conditions for $F(x,y)=0$ to have infinitely many rational or integer points, respectively.

PROPOSITION 1. *The equation*

$$\tfrac{1}{4}(2x-2Ay+A-1)(2x+2Ay-A-1)=0,$$

where $A=\sqrt{4l+1}$, $l\in\mathbb{C}$, *has infinitely many rational solutions* x,y *if and only if* $A\in\mathbb{Q}$. *Further, there are infinitely many integer solutions* x,y *if and only if* $A=\sqrt{4l+1}=c/d$ *with relatively prime, odd integers* c,d.

PROPOSITION 2. *The equation*

$$\tfrac{1}{4}(2x+2Ay^2-6Ay+2A-1)(2x-2Ay^2+6Ay-2A-1)=0,$$

where $A=\sqrt{l+1/4}, l\in\mathbb{C}$, *has infinitely many rational solutions* x,y *if and only if* $A\in\mathbb{Q}$. *Moreover, there are infinitely many integer solutions if and only if* $A=c/d$ *with coprime integers* c,d *such that* d *is even and the congruence* $2u^2-6u+2\equiv 0 \bmod d$ *is solvable.*

Proposition 3. *The equation*

$$(x^2-3x+Ay^2-3Ay+A+1)(x^2-3x-Ay^2+3Ay-A+1)=0,$$

where $A=\sqrt{l+1}, l\in\mathbb{C}$, *has infinitely many rational solutions* x,y *if and only if* $A\in\mathbb{Q}$ *and one of the Hilbert symbols* $\left(\frac{c}{5c+5d},\frac{d}{5c+5d}\right)$, $\left(\frac{-c}{5c+5d},\frac{d}{5c+5d}\right)$ *equals* 1, *where* $A=c/d$ *with coprime integers* c,d. *There are infinitely many integer solutions if and only if* c,d *are odd*, *and the equation*

$$d(2x-3)^2-c(2y-3)^2=5(d-c), \tag{6}$$

has infinitely many integer solutions.

We note that there are integers c, d, such that equation (6) has infinitely many integer solutions x,y. For example, put $c=1$ and $d=5$. Then $x=(b_n+3)/2$, $y=(a_n+3)/2$ are solutions of (6), where a_n and b_n are defined by $(a_0,b_0)=(5,3)$, $(a_{n+1},b_{n+1})=(9a_n+20b_n,4a_n+9b_n)$.

Proposition 4. *The equation*

$$\begin{aligned}&\left(x^2-\sqrt{2}xy-\left(3-\tfrac{3}{2}\sqrt{2}\right)x+y^2-\left(3-\tfrac{3}{2}\sqrt{2}\right)y+\tfrac{13}{4}-\tfrac{9}{4}\sqrt{2}\right)\\&\quad\times\left(x^2+\sqrt{2}xy-\left(3+\tfrac{3}{2}\sqrt{2}\right)x+y^2-\left(3+\tfrac{3}{2}\sqrt{2}\right)y+\tfrac{13}{4}+\tfrac{9}{4}\sqrt{2}\right)=0\end{aligned}$$

has no rational solution x,y.

Proposition 5. *The equation*

$$\begin{aligned}&\left(y^2-5y+\tfrac{20}{3}-5x+x^2\right)\left(y^4-10y^3-x^2y^2+5xy^2+\tfrac{85}{3}y^2+5x^2y\right.\\&\qquad\left.-25xy-\tfrac{50}{3}y+x^4-10x^3+\tfrac{85}{3}x^2-\tfrac{50}{3}x+\tfrac{16}{9}\right)=0\end{aligned}$$

has no rational solution x,y.

4. Study of the genus. Consider the curves

$$C:\quad x(x-1)\dots(x-(m-1))=\lambda y(y-1)\dots(y-(n-1))+l,$$

where $n\ge m>1$, and λ,l are not necessarily rational, but $\lambda\in\mathbb{C}\setminus\{0\}$ and $l\in\mathbb{C}$. In the following two theorems we list those curves C which have genus zero and one, respectively.

Theorem 5. *Suppose* C *is irreducible. Then its genus is zero in the following cases*:

1) $(m,n)=(2,2)$, $\lambda-4l\neq1$;
2) $(m,n)=(2,3)$, $t\lambda+l=-\frac{1}{4}$, $t^2=\frac{4}{27}$;
3) $(m,n)=(2,4)$, $\frac{9}{16}\lambda+l=-\frac{1}{4}$;
4) $(m,n)=(2,6)$, $t\lambda+l=-\frac{1}{4}$, $27t^2+320t-2304=0$;
5) $(m,n)=(3,3)$, $l\neq0$ *and* $t(\lambda\pm1)=-l$, $t^2=\frac{4}{27}$;
6) $(m,n)=(3,4)$, $\lambda=\frac{64}{225}t$, $l=\frac{14}{225}t$, $t^2=3$;
7) $(m,n)=(3,6)$, $\lambda=\frac{3}{392}t$ *and* $l=\frac{20}{441}t$, $t^2=21$.

THEOREM 6. *Assume that C is irreducible. Its genus is one in the following cases*:

1) $(m,n)=(2,3)$, $\frac{2}{9}t\lambda+l\neq-\frac{1}{4}$, $t^2=3$;
2) $(m,n)=(2,4)$, $\frac{9}{16}\lambda+l\neq-\frac{1}{4}$;
3) $(m,n)=(2,5)$, $t\lambda+l=-\frac{1}{4}$, $3125t^4-47500t^2+82944=0$;
4) $(m,n)=(2,6)$, $-\frac{225}{64}\lambda+l=-\frac{1}{4}$;
5) $(m,n)=(2,8)$, $t\lambda+l=-\frac{1}{4}$, $t^3+576t^2-54432t-4665600=0$;
6) $(m,n)=(3,3)$, $l=0$ *or* $l\neq0$ *and* $t(\lambda\pm1)\neq-l$, $t^2=\frac{4}{27}$;
7) $(m,n)=(3,4)$, $-\lambda+l=t$ *and* $(\lambda,l)\neq\left(-\frac{32}{25}t,-\frac{7}{25}t\right)$, $t^2=\frac{4}{27}$;
8) $(m,n)=(3,6)$,

$$\lambda=-\frac{256s}{2025+576t},\qquad l=\frac{2s}{9}-\frac{100s}{225+64t},$$

where $s^2=3$ *and* $27t^2+320t-2304=0$;

9) $m=n=4$, $-\lambda+l=\frac{9}{16}$ *and* $l\neq-\frac{7}{16}$, *or* $\frac{9}{16}\lambda+l=-1$ *and* $l\neq-\frac{7}{16}$.

For $l=0$, Theorems 5 and 6 give as a special case Theorem 2.2 of [4]. We recall that in [4] the curve $x(x+1)\dots(x+(m-1))=\lambda y(y+1)\dots(y+(n-1))$ is considered.

We remark that recently Avanzi and Zannier [2] classified the genus 1 curves of the form $f(x)=g(y)$, where the polynomials f and g have coprime degrees. From this result of [2] one can deduce the cases $(m,n)=(2,3),(2,5)$ and $(3,4)$ of our Theorem 6.

5. Proofs of Theorem 4 and Propositions 1 to 5. We introduce some further notation. Put

$$f(x)=x(x-1)\dots(x-(m-1)),\qquad g(y)=\lambda y(y-1)\dots(y-(n-1))+l,$$

where $1<m\leq n\in\mathbb{N}$ and $\lambda\in\mathbb{C}\backslash\{0\}$, $l\in\mathbb{C}$. To examine the irreducibility of the polynomial $f(x)-g(y)$, we apply a method and some results of Beukers, Shorey and Tijdeman [4].

Let $\widetilde{f},\widetilde{g}\in\mathbb{C}[x]$ and let $S_{\widetilde{f}}$ and $S_{\widetilde{g}}$ be the sets of stationary points of $\widetilde{f}$ and $\widetilde{g}$, respectively, which we assume to be simple. Let $m=\deg(\widetilde{f})$ and $n=\deg(\widetilde{g})$. For any $a\in\mathbb{C}$, put

$$m_a=\#\{\alpha\in S_{\widetilde{f}}\mid\widetilde{f}(\alpha)=a\},\qquad n_a=\#\{\beta\in S_{\widetilde{g}}\mid\widetilde{g}(\beta)=a\}.$$

Suppose that the polynomial $\widetilde{f}(x)-\widetilde{g}(y)$ is reducible over $\mathbb{C}$, i.e.

$$\widetilde{f}(x)-\widetilde{g}(y)=G_1(x,y)G_2(x,y)$$

with $G_1,G_2\in\mathbb{C}[x,y]$. Denote by δ the weighted degree defined by $\delta(x^ay^b)=na+mb$. We use the following lemmas from [4].

Lemma 1. *Let m_1, m_2 be the weighted degrees of G_1, G_2, respectively. Then*

$$m_1 m_2 \leq mn \sum_{a\in\mathbb{C}} m_a n_a.$$

Moreover, $m_1+m_2=mn$ and m_1 and m_2 are multiples of mn/d, where $d=\gcd(m,n)$.

Lemma 2. *Suppose that $n_a \leq 1$ for all $a\in\mathbb{C}$. Then $n=\gcd(m,n)$ and $\widetilde{f}(x)-\widetilde{g}(y)$ has a factor of degree 1 in y.*

Lemma 3. *Let $\widetilde{f}(x)=f_m(x)$. Then $m_a\leq 2$ for all $a\in\mathbb{C}$. Moreover, if m is odd, then $m_a\leq 1$ for all $a\in\mathbb{C}$.*

Lemma 4. *Let d be an even positive integer. Put*

$$\overline{f}(x)=(x-1^2)(x-3^2)\dots(x-(d-1)^2), \qquad \overline{m}_a=\#\{\alpha\in S_{\overline{f}} \mid \overline{f}(\alpha)=a\}.$$

Then $\overline{m}_a\leq 1$ for every $a\in\mathbb{C}$.

Proof of Theorem 4. Consider the polynomial $F(x,y)=f(x)-g(y)$, where $f(x)=x(x-1)\dots(x-(m-1))$ and $g(y)=\lambda y(y-1)\dots(y-(n-1))+l$. Suppose that $F(x,y)$ is reducible in $\mathbb{C}[x,y]$. Assuming that n is odd, from Lemma 2 we deduce that $n=d$ and $f(x)-g(y)$ has a linear factor in y. Thus from $m\leq n$ it follows that $m=n$, whence this factor is also linear in x. Hence we can write

$$f(x)-g(y)=(ax+y+c)T(x,y), \tag{7}$$

where $a,c\in\mathbb{C}$, $a\neq 0$, $T\in\mathbb{C}[x,y]$. Put $Q(x)=-ax-c$. From (7) it follows that

$$f(x)=g(Q(x)).$$

Since $f(x)=x(x-1)\dots(x-(m-1))$, therefore $Q(0),Q(1),\dots,Q(m-1)$ are roots of the polynomial g, but then $g\big(Q\big(\frac{m-1}{2}\big)\big)$ is zero. On the other hand, as

$$g(y)=\lambda y(y-1)\dots(y-(n-1))+l$$

and also

$$g(y)=\lambda(y+c)(y+a+c)\dots(y+(m-1)a+c),$$

we have

$$\lambda(c+a+c+2a+c+\ldots+(m-1)a+c)=-\lambda(1+2+\ldots+(m-1)).$$

Thus $c=(1-m)(1+a)/2$, and we obtain

$$Q(x)=-ax+\frac{m-1}{2}a+\frac{m-1}{2},$$

and so

$$Q\left(\frac{m-1}{2}\right)=\frac{m-1}{2}.$$

Now

$$g\left(Q\left(\frac{m-1}{2}\right)\right)=0$$

implies

$$\lambda\,\frac{m-1}{2}\left(\frac{m-1}{2}-1\right)\dots\left(\frac{m-1}{2}-(m-1)\right)+l=0,$$

whence $l=0$. This case is already described by Beukers, Shorey and Tijdeman [4].

Now suppose that n is even and $l\neq 0$. Since $g(y)=g(n-1-y)$, we get

$$g\left(\frac{n-1-y}{2}\right)=\lambda 2^{-n}(y^2-1^2)(y^2-3^2)\dots(y^2-(n-1)^2)+l.$$

Let

$$h(v)=\lambda 2^{-n}(v-1^2)(v-3^2)\dots(v-(n-1)^2)+l.$$

Then from Lemma 4, it follows that $\#\{\alpha\in S_h \mid h(\alpha)=a\}\leq 1$ for every $a\in\mathbb{C}$.

Suppose that $f(x)-g(y)$ has an irreducible factor $K(x,y)$ of degree $<n/2$ in y. Then either $K(x,y)$ or $K(x,n-1-y)K(x,y)$ is symmetric with respect to the transformation $y\mapsto n-1-y$ and is a non-trivial factor of $f(x)-g(y)$. Hence $f(x)-h(v)$ has a factor of degree one in v and $n/2=\gcd(m,n/2)$. So either $m=n$ or $n=2m$.

Suppose $m=n$. We know that $f(x)-h(v)$ has a factor which is linear in v and quadratic in x. Thus we have

$$f(x)-h(v)=(ax^2+bx+c+v)T(x,v),$$

where $a,b,c\in\mathbb{C}, a\neq 0, T\in\mathbb{C}[x,v]$. Let $Q(x)=-ax^2-bx-c$. Then $Q\in\mathbb{C}[x]$ is a quadratic polynomial such that $f(x)=h(Q(x))$. As $\deg(h)=m/2$, and Q is quadratic, among the complex numbers $Q(0),Q(1),\dots,Q(m-1)$ there are exactly $m/2$ different ones. Hence there exists an integer $j\in\{1,\dots,m-1\}$ such that $Q(0)=Q(j)$. So we have $-c=-aj^2-bj-c$, whence $j=-b/a$. Let $u\in\{1,\dots,m-1\}\setminus\{j\}$. Then there exists an integer $v\in\{1,\dots,m-1\}\setminus\{j,u\}$ such that $Q(u)=Q(v)$. We get

$$u+v=-b/a,\qquad -b/a=m-1.$$

These yield

$$\begin{gathered}Q(0)=Q(m-1),\\ Q(1)=Q(m-2),\\ \vdots\\ Q\left(\frac{m-2}{2}\right)=Q\left(\frac{m}{2}\right).\end{gathered}$$

So we can write the polynomial $h(v)$ in the form

$$h(v) = \lambda 2^{-m}(v - Q(0))(v - Q(1)) \dots \left(v - Q\left(\frac{m-2}{2}\right)\right).$$

On the other hand, by the definition of $h(v)$ and $m = n$ we also have

$$h(v) = \lambda 2^{-m}(v - 1^2)(v - 3^2) \dots (v - (m-1)^2) + l.$$

Assume that $m \geq 8$. Comparing the coefficients of $v^{m/2-1}$, $v^{m/2-2}$ and $v^{m/2-3}$ in the two formulas for $h(v)$, we get the following system of equations:

$$\sum_{i=0}^{k-1} (2i+1)^2 = \sum_{i=0}^{k-1} Q(i),$$

$$\sum_{i=0}^{k-1} \sum_{j=0}^{i-1} (2i+1)^2 (2j+1)^2 = \sum_{i=0}^{k-1} \sum_{j=0}^{i-1} Q(i)Q(j),$$

$$\sum_{i=0}^{k-1} \sum_{j=0}^{i-1} \sum_{t=0}^{j-1} (2i+1)^2 (2j+1)^2 (2t+1)^2 = \sum_{i=0}^{k-1} \sum_{j=0}^{i-1} \sum_{t=0}^{j-1} Q(i)Q(j)Q(t),$$

where $k = m/2$. However, a simple calculation shows that this system has no solutions, that is, $f(x) - g(y)$ has no irreducible factor of degree $< n/2$ in y. In the remaining cases $(m, n) = (2, 2), (4, 4), (6, 6)$, a straightforward computation gives that $f(x) - g(y)$ has such a factor if and only if $(m, n) = (6, 6)$ when

$$f(x) - g(y) = \left(y^2 - 5y + x^2 - 5x + \tfrac{20}{3}\right) \times \left(y^4 - 10y^3 - x^2y^2 + 5xy^2 + \tfrac{85}{3}y^2 + 5x^2y - 25xy - \tfrac{50}{3}y + x^4 - 10x^3 + \tfrac{85}{3}x^2 - \tfrac{50}{3}x + \tfrac{16}{9}\right).$$

Suppose that $n = 2m$. We know that $f(x) - h(v)$ has a factor which is linear both in x and in v, that is, we may write

$$f(x) - h(v) = (ax + b + v)T(x, v) \tag{8}$$

with $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$, $T \in \mathbb{C}[x, v]$. Put $L(x) = -ax - b$. Then $L \in \mathbb{C}[x]$ is a linear polynomial such that $f(x) = h(L(x))$. This implies that $L(0), L(1), \dots, L(m-1)$ are different roots of the polynomial h. Thus

$$h(v) = \lambda 2^{-n}(v - L(0))(v - L(1)) \dots (v - L(m-1)).$$

Using the definition of $h(v)$, as $n = 2m$ we get

$$h(v) = \lambda 2^{-2m}(v - 1^2)(v - 3^2) \dots (v - (2m-1)^2) + l.$$

Suppose that $m \geq 4$. If we compare again the coefficients of v^{m-1}, v^{m-2} and v^{m-3} in the two formulas for $h(v)$, we are led to the following system of equations:

$$\sum_{i=0}^{m-1}(2i+1)^2 = \sum_{i=0}^{m-1} L(i),$$

$$\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}(2i+1)^2(2j+1)^2 = \sum_{i=0}^{m-1}\sum_{j=0}^{i-1} L(i)L(j),$$

$$\sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\sum_{t=0}^{j-1}(2i+1)^2(2j+1)^2(2t+1)^2 = \sum_{i=0}^{m-1}\sum_{j=0}^{i-1}\sum_{t=0}^{j-1} L(i)L(j)L(t).$$

A simple computation shows that this system is unsolvable.

In the remaining cases $(m,n) = (2,4),(3,6)$, $f(x) - g(y)$ has no irreducible factor of degree $< n/2$ in y.

We are left with the case when $f(x) - g(y)$ has only factors of degree $n/2$ in y. Then we can apply Lemma 1 with $m_1 = m_2 = mn/2$. Hence by $n_a \le 2$,

$$\left(\frac{mn}{2}\right)^2 \le mn \sum_a m_a n_a \le 2mn(m-1).$$

This yields

$$n \le 8\left(1 - \frac{1}{m}\right) < 8,$$

whence $n = 2,4,6$. Since $mn/2$ is a multiple of mn/d, we can see that d is even, thus $m = 2,4,6$. We consider the remaining cases separately.

First let $(m,n) = (2,2)$. If $f(x) - g(y)$ is reducible, then since

$$\left(\frac{mn}{2}\right)^2 \le mn \sum_a m_a n_a,$$

we get

$$4 \le 4 \sum_a m_a n_a.$$

As

$$m_a = \begin{cases} 1 & \text{if } a = -\frac{1}{4}, \\ 0 & \text{otherwise,} \end{cases}$$

if $n_{-1/4} = 0$ then $f(x) - g(y)$ is irreducible. However, $n_{-1/4} = 0$ if and only if $\lambda\left(-\frac{1}{4}\right) + l \ne -\frac{1}{4}$, that is, $\lambda - 4l \ne 1$.

On the other hand, if $\lambda - 4l = 1$ then $f(x) - g(y)$ is reducible, and

$$f(x) - g(y) = \tfrac{1}{4}(2x - 2Ay + A - 1)(2x + 2Ay - A - 1),$$

where $A = \sqrt{l+1}$.

Now let $(m,n) = (2,4)$. As above, if $f(x) - g(y)$ is reducible, then $16 \le 8n_{-1/4}$. Hence if $-\lambda + l \ne -\frac{1}{4}$, then $f(x) - g(y)$ is irreducible.

If $-\lambda+l=-\frac{1}{4}$ then

$$f(x)-g(y)=\tfrac{1}{4}(2x+2Ay^2-6Ay+2A-1)(2x-2Ay^2+6Ay-2A-1),$$

where $A=\sqrt{l+1/4}$.

Put $(m,n)=(4,4)$. We have

$$64\le 16m_{-1}n_{-1}+16m_{9/16}n_{9/16}=32n_{-1}+16n_{9/16},$$

whence $m_{-1}=2$, $m_{9/16}=1$. Thus $f(x)-g(y)$ can be reducible only if $n_{-1}=1$ and $n_{9/16}=2$, or $n_{-1}=2$.

In the former case, we get $\lambda=-1$, $l=-\frac{7}{16}$, and

$$f(x)-g(y)=(x^2-3x+Ay^2-3Ay+A+1)(x^2-3x-Ay^2+3Ay-A+1),$$

where $A=\sqrt{l+1}$.

In the latter case, we obtain $-\lambda+l=1$, and

$$\begin{aligned}f(x)-g(y)=&\left(\tfrac{13}{4}-\tfrac{9}{4}\sqrt{2}-\left(3-\tfrac{3}{2}\sqrt{2}\right)y-\left(3-\tfrac{3}{2}\sqrt{2}\right)x+x^2+y^2-\sqrt{2}xy\right)\\&\times\left(\tfrac{13}{4}+\tfrac{9}{4}\sqrt{2}-\left(3+\tfrac{3}{2}\sqrt{2}\right)y-\left(3+\tfrac{3}{2}\sqrt{2}\right)x+x^2+y^2+\sqrt{2}xy\right).\end{aligned}$$

For $(m,n)=(2,6),(4,6)$ one can repeat the above method to conclude that $f(x)-g(y)$ does not have a factor of degree $n/2$ in y. Hence the theorem follows. ∎

In what follows we study the exceptional cases in Theorem 4. In fact, these cases include the reducible curves.

Proof of Proposition 1. Our equation is

$$(2x-2Ay+A-1)(2x+2Ay-A-1)=0,$$

where $A=\sqrt{4l+1}$, $l\in\mathbb{Q}$. Obviously, there exist infinitely many rational solutions x,y if and only if $\sqrt{4l+1}$ is rational.

Suppose that $A=\sqrt{4l+1}=c/d$, where c,d are integers with $\gcd(c,d)=1$. There are infinitely many integer solutions x,y only if infinitely many integers x,y satisfy

$$2x-2Ay+A-1=0\quad\text{or}\quad 2x+2Ay-A-1=0,$$

that is,

$$2x-2\,\frac{c}{d}\,y+\frac{c-d}{d}=0\quad\text{or}\quad 2x+2\,\frac{c}{d}\,y-\frac{c+d}{d}=0.$$

It is easy to check that each of these equations has infinitely many integer solutions if and only if c and d are odd. ∎

Proof of Proposition 2. Let

$$(2x+2Ay^2-6Ay+2A-1)(2x-2Ay^2+6Ay-2A-1)=0,$$

where $A=\sqrt{l+1/4}$, $l\in\mathbb{Q}$. It is clear that there exist infinitely many rational solutions x,y if and only if A is rational.

Suppose that $A = c/d$, where $c, d \in \mathbb{Z}$ with $\gcd(c, d) = 1$. If x, y is an integer solution, then

$$x + \frac{c}{d}\, y^2 - 3\, \frac{c}{d}\, y + \frac{c}{d} - \frac{1}{2} = 0 \tag{9}$$

or

$$x - \frac{c}{d}\, y^2 + 3\, \frac{c}{d}\, y - \frac{c}{d} - \frac{1}{2} = 0. \tag{10}$$

Equation (9) gives $2dx + 2cy^2 - 6cy + 2c - d = 0$. This implies that d is even, and $2y^2 - 6y + 2 \equiv 0 \bmod d$. Further, $y^2 - 3y + 1$ is always odd, hence $2 \parallel d$.

It is easy to see that for each solution y of the congruence $2u^2 - 6u + 2 \equiv 0 \bmod d$, where d is even,

$$x = \frac{d - 2c(y^2 - 3y + 1)}{2d}, y$$

is an integer solution of (9).

Concerning (10), we can use a similar argument. So we find that for $m = 2$, $n = 4$ and $4\lambda - 4l = 1$ there are infinitely many integer solutions if and only if $A = \sqrt{l + 1/4}$ is rational. Further, if we write $A = c/d$, with $\gcd(c, d) = 1$, then d must be even and the congruence $2u^2 - 6u + 2 \equiv 0 \bmod d$ must be solvable. ∎

Proof of Proposition 3. Consider the equation

$$(x^2 - 3x + Ay^2 - 3Ay + A + 1)(x^2 - 3x - Ay^2 + 3Ay - A + 1) = 0,$$

where $A = \sqrt{l + 1}$. This equation has infinitely many rational solutions x, y if and only if A is rational and one of the Hilbert symbols

$$\left(\frac{c}{5c + 5d}, \frac{d}{5c + 5d}\right), \ \left(\frac{-c}{5c + 5d}, \frac{d}{5c + 5d}\right)$$

equals 1, where $A = c/d$ with $\gcd(c, d) = 1$.

Moreover, if x, y is an integer solution then we have

$$x^2 - 3x + \frac{c}{d}\, y^2 - 3\, \frac{c}{d}\, y + \frac{c}{d} + 1 = 0 \tag{11}$$

or

$$x^2 - 3x - \frac{c}{d}\, y^2 + 3\, \frac{c}{d}\, y - \frac{c}{d} + 1 = 0. \tag{12}$$

In the case (11) we get $d \mid y^2 - 3y + 1$, whence d is odd. Further, $c \mid x^2 - 3x + 1$, thus c is also odd. So we obtain

$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} + \frac{c}{d}\left(\left(y^2 - \frac{3}{2}\right)^2 - \frac{5}{4}\right) = 0,$$

which yields $d(2x - 3)^2 + c(2y - 3)^2 = 5(c + d)$. It follows that in this case there are only finitely many integer solutions.

In the case (12) we have

$$\left(x-\frac{3}{2}\right)^2-\frac{5}{4}-\frac{c}{d}\left(\left(y^2-\frac{3}{2}\right)^2-\frac{5}{4}\right)=0,$$

and so $d(2x-3)^2-c(2y-3)^2=5(d-c)$.

Summarising, we have infinitely many integer solutions if and only if there are infinitely many integer solutions of the equation

$$d(2u-3)^2-c(2v-3)^2=5(d-c),$$

where d and c are odd. ■

Proof of Proposition 4. Suppose that

$$\begin{aligned}&\left(\tfrac{13}{4}-\tfrac{9}{4}\sqrt{2}-\left(3-\tfrac{3}{2}\sqrt{2}\right)y-\left(3-\tfrac{3}{2}\sqrt{2}\right)x+x^2+y^2-\sqrt{2}xy\right)\\ &\qquad\times\left(\tfrac{13}{4}+\tfrac{9}{4}\sqrt{2}-\left(3+\tfrac{3}{2}\sqrt{2}\right)y-\left(3+\tfrac{3}{2}\sqrt{2}\right)x+x^2+y^2+\sqrt{2}xy\right)=0.\end{aligned}$$

If $(x,y)\in\mathbb{Q}^2$ is a solution, then either

$$\tfrac{13}{4}-\tfrac{9}{4}\sqrt{2}-\left(3-\tfrac{3}{2}\sqrt{2}\right)y-\left(3-\tfrac{3}{2}\sqrt{2}\right)x+x^2+y^2-\sqrt{2}xy=0 \tag{13}$$

or

$$\tfrac{13}{4}+\tfrac{9}{4}\sqrt{2}-\left(3+\tfrac{3}{2}\sqrt{2}\right)y-\left(3+\tfrac{3}{2}\sqrt{2}\right)x+x^2+y^2+\sqrt{2}xy=0. \tag{14}$$

We deduce from (13) that

$$\tfrac{13}{4}-3y-3x+x^2+y^2+\sqrt{2}\left(\tfrac{3}{2}y+\tfrac{3}{2}x-xy-\tfrac{9}{4}\right)=0,$$

thus

$$\tfrac{13}{4}-3y-3x+x^2+y^2=0\quad\text{and}\quad\tfrac{3}{2}y+\tfrac{3}{2}x-xy-\tfrac{9}{4}=0.$$

These yield $(x-y)^2=\frac{5}{4}$, which is a contradiction.

We get a similar contradiction in the case (14). ■

Proof of Proposition 5. Suppose that $(x,y)\in\mathbb{Q}^2$ is a solution of the equation

$$y^2-5y+\tfrac{20}{3}-5x+x^2=0.$$

This yields

$$u^2+v^2=\tfrac{35}{6},$$

where $u=a/b$, $v=c/d$ with $a,b\in\mathbb{Z}$, $\gcd(a,b)=\gcd(c,d)=1$. Hence we obtain

$$(6ad)^2+(6bc)^2=2\cdot3\cdot5\cdot7\cdot b^2d^2,$$

which is impossible.

Suppose that the rational numbers $x=a/b$ and $y=c/d$ with $a,b\in\mathbb{Z}$, $(a,b)=(c,d)=1$ satisfy

$$\begin{aligned}y^4-10y^3-x^2y^2+5xy^2+\tfrac{85}{3}y^2+5x^2y-25xy-\tfrac{50}{3}y+x^4-10x^3\\ +\tfrac{85}{3}x^2-\tfrac{50}{3}x+\tfrac{16}{9}=0.\end{aligned} \tag{15}$$

From (15) we get

$$(16)\quad 3b^4c^4-30b^4c^3d-3a^2b^2c^2d^2+15ab^3c^2d^2+85b^4c^2d^2+15a^2b^2cd^3 \\ -75ab^3cd^3-50b^4cd^3+\tfrac{16}{3}b^4d^4+85a^2b^2d^4-50ab^3d^4+3a^4d^4-30a^3bd^4=0.$$

We obtain $3 \mid b^4d^4$. Since (15) is symmetric in x, y, we may assume that $3 \mid b$ and so $3 \nmid a$. Now let $b = 3^k e$, $d = 3^t f$ where $\gcd(3, e) = \gcd(3, f) = 1$ and $k \geq 1$, $t \geq 0$. Then in (16) each term is divisible by $3^{\min\{4k+1,4t+1\}}$.

Suppose that $t < k$. Then every term in (16) is divisible by 3^{4t+2}, except $3a^4d^4$, which is a contradiction.

So let $t \geq k$. Since $k \geq 1$, we have $t \geq 1$, and hence $3 \mid d$. Moreover, $k = t$. It follows that

$$3^{5k+1} \mid 3b^4c^4 - 3a^2b^2c^2d^2 + 3a^4d^4 = 3^{4k+1}(e^4c^4 - a^2e^2c^2f^2 + a^4f^4).$$

Since $3 \mid acef$, the last relation gives a contradiction. ■

6. Proofs of Theorems 5 and 6. We now prove Theorems 5 and 6. For this purpose we need the following result from Beukers, Shorey and Tijdeman [4].

LEMMA 5. *Let $f, g \in \mathbb{C}[X]$ be polynomials of degrees m and n respectively, and suppose that $f(X) - g(Y)$ is irreducible. Assume that the stationary points of f and g are simple. For $\alpha \in S_f$ put $r_\alpha = \#\{y \in S_g \mid f(\alpha) = g(y)\}$. Let g_C be the genus of the curve $C : f(X) = g(Y)$. Then*

$$2g_C = \sum_{\alpha \in S_f} (n - 2r_\alpha) - m + 2 - \gcd(m, n).$$

Proof of Theorems 5 and 6. Let

$$f(x) = x(x-1)\ldots(x-(m-1)) \quad\text{and}\quad g(y) = \lambda y(y-1)\ldots(y-(n-1)) + l,$$

where we assume that $1 < m \leq n$, $\lambda, l \in \mathbb{C}$ with $\lambda \neq 0$.

From Lemma 3 we get $r_\alpha \leq 2$ in all cases and $r_\alpha \leq 1$ if n is odd. Let $\delta(n) = 2$ if n is odd, and 4 if n is even. Since $|S_f| = m - 1$, we obtain

$$\begin{aligned} 2g_C &= \sum_{\alpha \in S_f} (n - 2r_\alpha) - m + 2 - \gcd(m, n) \\ &\geq (n - \delta(n))(m-1) - 2(m-1) + m - \gcd(m, n) \\ &= (n - \delta(n) - 2)(m-1) + m - \gcd(m, n). \end{aligned}$$

Suppose $n \geq 9$ or $n = 7$. Then $n - \delta(n) - 2 \geq 3$ and we get $2g_C \geq 3(m-1)$, hence $g_C > 1$. If $n = 8$, then $n - \delta(n) - 2 = 2$ and $2g_C \geq 2(m-1) + m - \gcd(m, 8) \geq 2(m-1)$. Hence $g_C > 1$ for $m \geq 3$. This leaves us with the case $m = 2$, $n = 8$. If $n = 5$, then $2g_C \geq m - 1 + m - \gcd(m, 5)$. So for $m = 3, 4, 5$

we have $g_C > 1$. There remains $m = 2$, $n = 5$. Thus we must consider the following cases only: $m = 2$, $n = 2, 3, 4, 5, 6, 8$; $m = 3$, $n = 3, 4, 6$; $m = 4$, $n = 4, 6$; $m = 5$, $n = 6$; $m = 6$, $n = 6$.

We work out only $(m, n) = (2, 2)$, $(2, 3)$ and $(2, 4)$ in detail. One can apply the same argument to check the other cases.

First let $(m, n) = (2, 2)$. Then $S_f = S_g = \left\{\frac{1}{2}\right\}$ and

$$2g_C = 2 - 2r_{1/2} - 2 + 2 - 2 = -2r_{1/2}.$$

If $-\frac{1}{4}\lambda + l \neq -\frac{1}{4}$, then $g_C = 0$. Otherwise, the curve is reducible.

For $(m, n) = (2, 3)$ we have $S_f = \left\{\frac{1}{2}\right\}$, $S_g = \left\{1 + \frac{\sqrt{3}}{3}, 1 - \frac{\sqrt{3}}{3}\right\}$. In this case $2g_C = 2 - 2r_{1/2}$. So, if $r_{1/2} = 0$ then $g_C = 1$, and if $r_{1/2} = 1$ then $g_C = 0$. However, $r_{1/2} = 1$ if and only if $\pm\frac{2\sqrt{3}}{9}\lambda + l = -\frac{1}{4}$.

In case of $(m, n) = (2, 4)$, $S_f = \left\{\frac{1}{2}\right\}$, $S_g = \left\{\frac{3}{2}, \frac{3}{2} + \frac{\sqrt{5}}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}\right\}$. Now $2g_C = 2 - 2r_{1/2}$. Thus, if $r_{1/2} = 0$ then $g_C = 1$, and if $r_{1/2} = 1$ then $g_C = 0$. If $r_{1/2} = 2$, then $-\lambda + l = -\frac{1}{4}$, and the curve is reducible. On the other hand, $r_{1/2} = 1$ if and only if $\frac{9}{16}\lambda + l = -\frac{1}{4}$. ∎

7. Proofs of Theorems 2, 3 and 1. We are ready to prove the main result of this paper. The proof of Theorem 2 is based upon the following two well-known ineffective theorems.

Theorem C (Siegel [33]). *The number of integral points on an irreducible algebraic curve of genus > 0 is finite.*

Theorem D (Faltings [14]). *The number of rational points on an irreducible algebraic curve of genus > 1 is finite.*

Proof of Theorem 2. Consider the algebraic curve

$$x(x-1)\dots(x-(m-1)) = \lambda y(y-1)\dots(y-(n-1)) + l, \tag{17}$$

where $m, n \in \mathbb{N}$ with $m \leq n$ and $\lambda, l \in \mathbb{Q}$ with $\lambda \neq 0$. Combining Theorems 4 to 6, C and D shows easily that apart from the enumerated exceptional cases, equation (17) has only finitely many rational and integer solutions respectively.

In what follows in each excluded case we show that the parameters λ, l can be given on infinitely many values such that equation (17) has infinitely many solutions x, y.

In case of 1) our curve is reducible with a simple factor (see Theorem 4), and we are done. When $(m, n) = (2, 2)$, we can reduce our equation to a Pellian equation. For example, one can easily see that if $\lambda = 2$ and $l = k(k-1)$, where k is odd, then equation (17) has infinitely many integer solutions. For $(m,n)=(2,4)$ and $4\lambda-4l=1$ our curve is also reducible and we considered this case in Proposition 2. We deal with the case $(m,n)=(4,4), \lambda-l=1$ in Proposition 3. In the remaining cases 5), 7), 8) and 9) our curve has genus one and for an appropriate choice of λ, l, we can transform it to an elliptic curve by a birational transformation. We can make this transformation in such a way that we can easily find a rational point on the original curve, which will be a non-torsion rational point on the transformed elliptic curve. We give the details only in the case 5).

Study the curve

$$x(x-1)-\lambda y(y-1)(y-2)-l=0, \tag{18}$$

where $\lambda, l \in \mathbb{Z}$ and $\lambda \neq 0$. If we solve the equation $x(x-1)-l=0$ for x we get

$$x_{1,2}=\frac{1\pm\sqrt{1+4l}}{2}.$$

Let $l=(k^2-1)/4$ with $k\in\mathbb{Z}$, k is odd and $\gcd(\lambda,k)=1$. Then (18) leads to

$$x(x-1)-\lambda y(y-1)(y-2)-\tfrac{1}{4}k^2+\tfrac{1}{4}=0. \tag{19}$$

It is plain that the point $P=((1-k)/2,2)$ belongs to this curve. Transform the curve (19) to the elliptic curve

$$V^2=U^3-256\lambda^2U+1024\lambda^2k^2 \tag{20}$$

by the birational transformation

$$x=\frac{32\lambda-V}{64\lambda},\qquad y=\frac{U+16\lambda}{16\lambda}.$$

The discriminant of $U^3-256\lambda^2U+1024\lambda^2k^2$ is

$$\varDelta=-67108864\lambda^6+2811552\lambda^4k^4.$$

The image of P is $W=(16\lambda,32\lambda k)$, which is a rational point. It is obvious that $32\lambda k$ divides the discriminant $\varDelta$ if and only if k divides $2^{21}\lambda^5$. Since $\gcd(\lambda,k)=1$ and k is odd, the well-known theorem of Nagell–Lutz [34] implies that W is not a torsion point on curve (20). As the transformation is birational and W is not a torsion point, we infer that there are infinitely many rational points on the curve (18).

In each of the remaining cases, we give infinitely many curves having infinitely many rational points. By a similar computation to case 5) one can check that the equations below have infinitely many rational solutions indeed.

6) $(m,n)=(2,4)$, $9\lambda+16l\neq -4$, and

$$x(x-1)-3ty(y-1)(y-2)(y-3)-\tfrac{1}{4}k^2+\tfrac{1}{4}=0,$$

where $k,t\in\mathbb{N}$ and $\gcd(t,k)=1$, $\gcd(k,208)=1$.

7) $(m,n)=(2,6)$, $-\frac{1}{4}=-\frac{225}{64}\lambda+l$, and

$$x(x-1)-\tfrac{16}{225}k^2y(y-1)(y-2)(y-3)(y-4)(y-5)-\tfrac{1}{4}k^2+\tfrac{1}{4}=0,$$

where $k\in\mathbb{N}$.

8) $(m,n)=(3,3)$, and

$$x(x-1)(x-2)-y(y-1)(y-2)-3k^2+9k-6=0,$$

where $k\in\mathbb{N}$ with $(2k-3)\nmid 2^9 3^5 13$.

9) $(m,n)=(4,4)$, $-\lambda+l=\frac{9}{16}$, $l\neq-\frac{7}{16}$, and

$$\begin{aligned}x(x-1)(x-2)(x-3)&-\big(k(k-1)(k-2)(k-3)-\tfrac{9}{16}\big)\\&\times y(y-1)(y-2)(y-3)-k(k-1)(k-2)(k-3)=0,\end{aligned}$$

where $k>4$.

Further, $(m,n)=(4,4)$, $\frac{9}{16}\lambda+l=-1$, $l\neq-\frac{7}{16}$, and

$$\begin{aligned}x(x-1)(x-2)(x-3)&-\big(-\tfrac{16}{9}k(k-1)(k-2)(k-3)-\tfrac{16}{9}\big)\\&\times y(y-1)(y-2)(y-3)-k(k-1)(k-2)(k-3)=0,\end{aligned}$$

where $k\nmid 3^{13}5^{16}$. ∎

Proof of Theorem 3. Our equation is

$$x(x+d_1)\dots(x+(m-1)d_1)=\widetilde{\lambda}y(y+d_2)\dots(y+(n-1)d_2)+\widetilde{l},$$

where $m\leq n$ and $m,n\in\mathbb{N}$. From this equation we obtain

$$X(X-1)\dots(X-(m-1))=\lambda Y(Y-1)\dots(Y-(n-1))+l,$$

where

$$X=-\frac{x}{d_1},\quad Y=-\frac{y}{d_2},\quad \lambda=(-1)^{m+n}\widetilde{\lambda}\,\frac{d_2^n}{d_1^m},\quad l=(-1)^m\,\frac{\widetilde{l}}{d_1^m}.$$

Application of Theorem 2 yields the statement. ∎

Proof of Theorem 1. Suppose that a, b, k are integers with $a\neq0$, $b\neq0$, and m, n are positive integers with $m\leq n$. Equation (4) can be written as

$$x(x-1)\dots(x-(m-1))=\lambda y(y-1)\dots(y-(n-1))+l, \tag{21}$$

where

$$\lambda=\frac{bm!}{an!}\quad\text{and}\quad l=\frac{km!}{a}.$$

We can now apply Theorem 2 to (21). It is easy to see that apart from the case $m=n$, $a=b$ and $k=0$, equation (21) may have infinitely many solutions only if $(m,n)\in\{(2,2),(2,4),(4,4)\}$.

When $(m,n)=(2,2)$, (21) leads to the equation

$$a(2x-1)^2-b(2y-1)^2=8k+a-b, \tag{22}$$

which, for an appropriate choice of a,b and k, has infinitely many solutions in x,y. For example, let $a=1$, $b=6$, $k=1$. It is easy to check (e.g. mod 4) that any solutions u,v of $u^2-6v^2=3$ are odd and $(u,v)=(27,11)$ is an integer solution. Hence (22) has infinitely many solutions indeed.

In the case $(m,n)=(2,4)$, the equation (21) takes the form

$$x(x-1)=\lambda y(y-1)(y-2)(y-3)+l, \tag{23}$$

where $\lambda=b/(12a)$, $l=2k/a$. From Theorems 4, 5 we know that if

$$4\lambda-4l\neq 1 \quad \text{and} \quad 9\lambda+16l\neq -4,$$

that is, when

$$\frac{24k+3a}{b}\neq 1 \quad \text{and} \quad \frac{24k+3a}{b}\neq -\frac{9}{16},$$

then (23) has only finitely many solutions in integers x,y. Moreover, for $(24k+3a)/b=1$ our curve is reducible. Proposition 2 shows that our equation has infinitely many integer solutions x,y if and only if we can write

$$\frac{b}{3a}=\left(\frac{c}{d}\right)^2,$$

where $c,d\in\mathbb{Z}$, $(c,d)=1$ and the congruence $u^2-3u+1\equiv 0 \bmod d$ is solvable.

If $a=25$, $b=147$, $k=3$ then $(24k+3a)/b=1$, and

$$x=(1+7(5t^2+5t+1))/2, \qquad y=4+5t$$

for any $t\geq 0$ gives infinitely many solutions to (4) with $x\geq 2$, $y\geq 4$.

In the case when $(24k+3a)/b=-9/16$, our equation leads to

$$(2x-1)^2=\frac{b}{48a}(4y^2-12y-1)(2y-3)^2. \tag{24}$$

There are infinitely many integers a,b,k such that (24), and so our original equation, has infinitely many integer solutions. For example, if $a=1$, $b=720$, $k=-17$, then $(24k+3a)/b=-9/16$. Now, (4) has infinitely many solutions $x\geq 2$, $y\geq 4$, namely

$$x=(15a_{n+1}b_{n+1}+1)/2, \qquad y=(a_{n+1}+3)/2,$$

where a_n and b_n are defined by

$$(a_0,b_0)=(5,1), \qquad (a_{n+1},b_{n+1})=(4a_n+15b_n,a_n+4b_n).$$

For $(m,n)=(4,4)$, (23) can have infinitely many solutions only if $\lambda-l=1$, i.e. if $(24k+a)/b=1$. Then equation (23) is of the form

$$z^2 = \frac{16b}{a}\,(y^2 - 3y + 1)^2, \tag{25}$$

where $z = (2x-3)^2 - 5$. In this case one can also give integers a, b, k for which (25), and hence our original equation, has infinitely many integer solutions. Let, for instance $a = 25$, $b = 1$, $k = -1$, whence $(24k + a)/b = 1$. Then (4) has again infinitely many integer solutions $x \geq 2$, $y \geq 4$, given by

$$x = (a_{n+1} + 3)/2, \qquad y = (5b_{n+1} + 3)/2,$$

where a_n and b_n are defined by

$$(a_0, b_0) = (3, 1), \qquad (a_{n+1}, b_{n+1}) = (9a_n + 20b_n, 4a_n + 9b_n). \ \blacksquare$$

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