ACTA ARITHMETICA
XCVIII.4 (2001)

The universality of zeta-functions attached to certain cusp forms

by

Antanas Laurinčikas (Vilnius) and Kohji Matsumoto (Nagoya)

1. Introduction. The universality property was first discovered by Voronin [21] in the case of the Riemann zeta-function. Denote by $\mathbb{C}$ the complex plane, $s = \sigma + it$ a complex variable, $\zeta(s)$ the Riemann zeta-function, and $\mathrm{meas}\{A\}$ the Lebesgue measure of the set A. We use the notation

$$\nu_T(\ldots) = T^{-1}\,\mathrm{meas}\{\tau \in [0,T] : \ldots\}$$

for $T > 0$, where in place of dots we write a condition satisfied by τ. The modern statement of Voronin's universality theorem is as follows (see Chapter 6 of [12]):

Let K be a compact subset of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complement. Let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\big(\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\big) > 0.$$

This remarkable result raised much attention among specialists, and Reich [18], [19], Gonek [4], Good [5], Bagchi [1], [2], and the first author [8]–[12], [14] improved and generalized the Voronin theorem to various other Dirichlet series including Dirichlet L, and Dedekind, Hurwitz, and Lerch zeta-functions.

It is the purpose of the present paper to prove the universality theorem for zeta-functions attached to certain cusp forms. Let $F(z)$ be a holomorphic cusp form of weight κ for the full modular group $\mathrm{SL}(2,\mathbb{Z})$, and assume that

2000 *Mathematics Subject Classification*: 11F66, 11M41, 11K99.

The first version of this paper was written during the stay of the first author at Nagoya University, November 1997, financially supported by Scientific Research Grant 09440009 of Japanese Ministry of Education, Science, and Culture.

$F(z)$ is a normalized eigenform. Then $F(z)$ has the Fourier series expansion

$$F(z)=\sum_{n=1}^{\infty} c(n)e^{2\pi i n z}, \qquad c(1)=1.$$

A classical result of Hecke [6] says that the Dirichlet series

$$\varphi(s,F)=\sum_{n=1}^{\infty} c(n)n^{-s}$$

is absolutely convergent in $\sigma > (\kappa+1)/2$, and can be continued analytically to an entire function. Moreover it satisfies the functional equation

$$(2\pi)^{-s}\Gamma(s)\varphi(s,F)=(-1)^{\kappa/2}(2\pi)^{s-\kappa}\Gamma(\kappa-s)\varphi(\kappa-s,F),$$

which implies that the critical strip for $\varphi(s,F)$ is $(\kappa-1)/2\le\sigma\le(\kappa+1)/2$. Let $D=\{s\in\mathbb{C}:\kappa/2<\sigma<(\kappa+1)/2\}$. Then we shall prove

THEOREM. *Let $F(z)$ be a normalized eigenform of weight κ for* $\mathrm{SL}(2,\mathbb{Z})$. *Let K be a compact subset of D with connected complement, and let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then for any $\varepsilon>0$ we have*

$$\liminf_{T\to\infty}\nu_T(\sup_{s\in K}|\varphi(s+i\tau,F)-f(s)|<\varepsilon)>0.$$

Before the present work, the universality of $\varphi(s,F)$ was obtained by Kačėnas–Laurinčikas [7], and also as a special case of the theorem given in [15], but both papers require rather strong assumptions. For instance, the universality theorem of Kačėnas–Laurinčikas [7] is proved under the assumption of the existence of $\eta>0$ such that

$$\sum_{\substack{p \text{ prime}\\ |c_p|<\eta}} p^{-\delta}<\infty \tag{1.1}$$

for $\delta>1/2$, where $c_p=c(p)p^{(1-\kappa)/2}$. However it seems to be hopeless to verify (1.1). Now, our theorem assures the universality property of $\varphi(s,F)$ unconditionally.

Bagchi [1] gave a new proof of the universality theorem for $\zeta(s)$, which is presented in Chapter 6 of [12]. In this paper we apply Bagchi's method to $\varphi(s,F)$, but some new ideas are necessary to complete the proof. A key lemma of Bagchi's method is Theorem 6.4.14 of [12], whose proof is based on the well known fact (see, e.g., [16])

$$\sum_{p\le x}\frac{1}{p}=\log\log x+c_1+O(\exp(-c_2\sqrt{\log x})) \tag{1.2}$$

for $x>1$, with some constants c_1 and $c_2>0$, where p runs over all primes $\le x$. If we try to apply Bagchi's method directly to our case, we need the corresponding asymptotic result for the sum $\sum_{p\le x}|c_p|/p$, but it is quite difficult to obtain such a formula. Instead, we use the asymptotic formula

$$\sum_{p\le x} c_p^2 = \pi(x)(1+o(1)), \qquad x\to\infty, \tag{1.3}$$

which is equivalent to Theorem 2 of Rankin [17]. Here, $\pi(x)$ denotes the number of primes up to x. From (1.3), we can deduce a vanishing lemma. This is Lemma 6 stated in Section 3, and plays an essential role in our argument. The proof of Lemma 6 will be given in Section 4, and this is the most novel part of the present paper. From Lemma 6 we can obtain Lemma 2, which corresponds to Lemma 6.5.4 of [12]. The deduction of our theorem from Lemma 2 is essentially the same as Bagchi's argument.

2. A limit theorem for the function $\varphi(s,F)$. Since $F(z)$ is a normalized eigenform, the function $\varphi(s,F)$ for $\sigma>(\kappa+1)/2$ has the Euler product expansion

$$\varphi(s,F)=\prod_p\left(1-\frac{\alpha(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)}{p^s}\right)^{-1},$$

with

$$c(p)=\alpha(p)+\beta(p) \tag{2.1}$$

and

$$|\alpha(p)|\le p^{(\kappa-1)/2}, \qquad |\beta(p)|\le p^{(\kappa-1)/2} \tag{2.2}$$

(Deligne [3]). From (2.1) and (2.2) we have

$$|c_p|\le 2. \tag{2.3}$$

Let $N>0$, $D_N=\{s\in\mathbb{C}:\kappa/2<\sigma<(\kappa+1)/2,\ |t|<N\}$, and denote by $H(D_N)$ the space of analytic functions on D_N equipped with the topology of uniform convergence on compacta. Let $\mathcal{B}(S)$ stand for the class of Borel subsets of the space S. Define on $(H(D_N),\mathcal{B}(H(D_N)))$ the probability measure

$$P_T(A)=\nu_T(\varphi(s+i\tau,F)\in A), \qquad A\in\mathcal{B}(H(D_N)).$$

For our purpose we need a limit theorem in the sense of the weak convergence of probability measures for P_T as $T\to\infty$, with an explicit form of the limit measure. Let $\gamma=\{s\in\mathbb{C}:|s|=1\}$, and let

$$\Omega=\prod_p\gamma_p,$$

where $\gamma_p=\gamma$ for all primes p. The infinite-dimensional torus Ω is a compact topological Abelian group. Denote by m_H the probability Haar measure on $(\Omega,\mathcal{B}(\Omega))$; thus we obtain the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Let $\omega(p)$

be the projection of $\omega \in \Omega$ to the coordinate space γ_p, and define the $H(D_N)$-valued random element $\varphi(s,\omega,F)$ on $(\Omega, \mathcal{B}(\Omega), m_H)$ by the formula

$$\varphi(s,\omega,F) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}$$

for $s \in D_N$. Denote by P_φ the distribution of the random element $\varphi(s,\omega,F)$, i.e.

$$P_\varphi(A) = m_H(\omega \in \Omega : \varphi(s,\omega,F) \in A), \qquad A \in \mathcal{B}(H(D_N)).$$

Then we have

LEMMA 1. *The probability measure P_T converges weakly to P_φ as $T \to \infty$.*

Kačėnas–Laurinčikas [7] proved this limit theorem on the space $H(\widetilde{D})$, where $\widetilde{D} = \{s \in \mathbb{C} : \sigma > \kappa/2\}$, from which Lemma 1 follows immediately. Lemma 1 can also be regarded as a special case of the result proved in [13].

3. A denseness lemma. Let, for $|z| < 1$,

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots,$$

and define

$$f_p(s) = f_p(s;a_p) = -\log\left(1 - \frac{\alpha(p)a_p}{p^s}\right) - \log\left(1 - \frac{\beta(p)a_p}{p^s}\right)$$

for $s \in D_N$ and $a_p \in \gamma$. We shall prove

LEMMA 2. *The set of all convergent series $\sum_p f_p(s;a_p)$ is dense in the space $H(D_N)$.*

In the proof of this lemma we will use the following three lemmas.

LEMMA 3. *Let $\{z_m\}$ be a sequence of complex numbers such that*

$$\sum_{m=1}^{\infty} |z_m|^2 < \infty.$$

Let $\{\varepsilon_m\}$ be a sequence of independent random variables on a certain probability space $(S, \mathcal{B}(S), \mathbb{P})$ such that $\mathbb{P}(\varepsilon_m = 1) = \mathbb{P}(\varepsilon_m = -1) = 1/2$ for any m. Then the series $\sum_{m=1}^{\infty} \varepsilon_m z_m$ converges almost surely.

The assertion of this lemma is included in the proof of Lemma 6.5.3 of [12].

LEMMA 4. *Let $\{f_m\}$ be a sequence in $H(D_N)$ which satisfies*:

(a) *If μ is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_N such that $\sum_{m=1}^{\infty} |\int_{\mathbb{C}} f_m\, d\mu| < \infty$, then $\int_{\mathbb{C}} s^r\, d\mu(s) = 0$ for any non-negative integer r.*

(b) *The series* $\sum_{m=1}^{\infty} f_m$ *converges in* $H(D_N)$.

(c) *For any compact* $K \subset D_N$, $\sum_{m=1}^{\infty} \sup_{s\in K} |f_m(s)|^2 < \infty$.

Then the set of all convergent series $\sum_{m=1}^{\infty} a_m f_m$, $a_m \in \gamma$, *is dense in* $H(D_N)$.

LEMMA 5. *Let* μ *be a complex measure on* $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *with compact support contained in the half-plane* $\sigma > \sigma_0$, *and let* $f(z) = \int_{\mathbb{C}} e^{sz}\, d\mu(s)$. *If* $f(z) \not\equiv 0$, *then*

$$\limsup_{r\to\infty} \frac{\log|f(r)|}{r} > \sigma_0.$$

Both of Lemmas 4 and 5 are due to Bagchi [1]. For the proofs, see Theorem 6.3.10 and Lemma 6.4.10, respectively, of [12].

Now we start the proof of Lemma 2, which we divide into three steps.

The first step. Let

$$\widetilde{f}_p = \widetilde{f}_p(s) = -\log\left(1 - \frac{\alpha(p)}{p^s}\right) - \log\left(1 - \frac{\beta(p)}{p^s}\right),$$

and let $p_0 > 0$. Define

$$\widehat{f}_p = \widehat{f}_p(s) = \begin{cases} \widetilde{f}_p(s) & \text{if } p > p_0, \\ 0 & \text{if } p \le p_0. \end{cases}$$

We claim that there exists a sequence $\{\widehat{a}_p : \widehat{a}_p \in \gamma\}$ such that the series

$$\sum_p \widehat{a}_p \widehat{f}_p \tag{3.1}$$

converges in $H(D_N)$.

To prove this claim, we observe that in view of (2.1) and (2.2),

$$\widetilde{f}_p(s) = \frac{\alpha(p) + \beta(p)}{p^s} + r_p(s) = \frac{c(p)}{p^s} + r_p(s)$$

with

$$r_p(s) = O(p^{\kappa - 2\sigma - 1}). \tag{3.2}$$

The series

$$\sum_p r_p(s) \tag{3.3}$$

converges uniformly on any compact subset of D_N. Next, let $\{\sigma(j)\}$ be a sequence of real numbers, $\sigma(1) > \sigma(2) > \ldots$ and $\sigma(j) \to \kappa/2$ as $j \to \infty$. For each j, the series $\sum_p \varepsilon_p c(p) p^{-\sigma(j)}$ converges almost surely by Lemma 3. Hence we can find a sequence $\{\widehat{a}_p : \widehat{a}_p = \pm 1\}$ such that $\sum_p \widehat{a}_p c(p) p^{-\sigma(j)}$ converges for any j. By a well known property of Dirichlet series, $\sum_p \widehat{a}_p c(p) p^{-s}$ converges uniformly on any compact subset of D_N. This and the convergence of (3.3) imply our claim on the series (3.1).

The second step. We now claim that the set of all convergent series

$$\sum_p a_p \widehat{f}_p, \qquad a_p \in \gamma, \tag{3.4}$$

is dense in $H(D_N)$. For this purpose we apply Lemma 4. Obviously it suffices to show that the set of all convergent series

$$\sum_p a_p g_p, \qquad a_p \in \gamma, \tag{3.5}$$

is dense in $H(D_N)$, where $g_p = \widehat{a}_p \widehat{f}_p$.

We have already shown that the series $\sum_p g_p$ converges in $H(D_N)$. Also it is easy to see that $\sum_p \sup_{s\in K} |g_p(s)|^2 < \infty$ for any compact subset $K \subset D_N$. Thus it remains to verify the condition (a) of Lemma 4.

Let μ be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in D_N such that

$$\sum_p \Big| \int_{\mathbb{C}} g_p(s)\, d\mu(s) \Big| < \infty. \tag{3.6}$$

We put $h_p(s) = \widehat{a}_p c(p) p^{-s}$. Then in virtue of (3.2) we have

$$\sum_p \sup_{s\in K} |g_p(s) - h_p(s)| < \infty.$$

From this and (3.6) we have $\sum_p |\int_{\mathbb{C}} h_p(s)\, d\mu(s)| < \infty$, so

$$\sum_p |c(p)| \Big| \int_{\mathbb{C}} p^{-s}\, d\mu(s) \Big| < \infty. \tag{3.7}$$

Let $D_{1,N} = \{s \in \mathbb{C} : 1/2 < \sigma < 1,\ |t| < N\}$ and let $h(s) = s - (\kappa - 1)/2$. Then

$$\mu h^{-1}(A) = \mu(h^{-1}(A)), \qquad A \in \mathcal{B}(\mathbb{C}),$$

is a complex measure with compact support contained in $D_{1,N}$. From (3.7) it follows that

$$\sum_p |c_p| \Big| \int_{\mathbb{C}} p^{-s}\, d\mu h^{-1}(s) \Big| < \infty. \tag{3.8}$$

(Recall $c_p = c(p)p^{(1-\kappa)/2}$.) Define

$$\varrho(z) = \int_{\mathbb{C}} e^{-sz}\, d\mu h^{-1}(s), \qquad z \in \mathbb{C}.$$

Then (3.8) can be written as

$$\sum_p |c_p| \cdot |\varrho(\log p)| < \infty. \tag{3.9}$$

From (3.9) we can deduce

LEMMA 6. $\varrho(z) \equiv 0$.

The proof of this fact is the most novel part of the present paper, and will be given in the next section.

Let r be a non-negative integer. Differentiating r-times the equality $\varrho(z) \equiv 0$ with respect to z, and then putting $z = 0$, we see that $\int_{\mathbb{C}} s^r \, d\mu h^{-1}(s) = 0$, hence $\int_{\mathbb{C}} s^r \, d\mu(s) = 0$. Consequently, all hypotheses of Lemma 4 are satisfied, and we obtain the denseness of the set of all convergent series (3.5), hence (3.4).

The third step. Let $x_0 \in H(D_N)$, K be a compact subset of D_N, and $\varepsilon > 0$. We choose a p_0 for which

$$\sup_{s\in K}\left(\sum_{p>p_0}\sum_{l=2}^{\infty}\frac{|\alpha(p)|^l+|\beta(p)|^l}{lp^{l\sigma}}\right)<\frac{\varepsilon}{4}. \tag{3.10}$$

By the claim proved in the second step we find a sequence $\{\widetilde{a}_p : \widetilde{a}_p \in \gamma\}$ such that

$$\sup_{s\in K}\left|x_0(s)-\sum_{p\le p_0}\widetilde{f}_p(s)-\sum_{p>p_0}\widetilde{a}_p\widetilde{f}_p(s)\right|<\frac{\varepsilon}{2}. \tag{3.11}$$

We put

$$a_p=\begin{cases}1 & \text{if } p\le p_0,\\ \widetilde{a}_p & \text{if } p>p_0.\end{cases}$$

Then (3.10) and (3.11) yield

$$\begin{aligned}\sup_{s\in K}\left|x_0(s)-\sum_{p}f_p(s;a_p)\right| &\le \sup_{s\in K}\left|x_0(s)-\sum_{p\le p_0}\widetilde{f}_p(s)-\sum_{p>p_0}\widetilde{a}_p\widetilde{f}_p(s)\right|\\ &\quad+\sup_{s\in K}\left|\sum_{p>p_0}\widetilde{a}_p\widetilde{f}_p(s)-\sum_{p>p_0}f_p(s;a_p)\right|\\ &<\frac{\varepsilon}{2}+2\sup_{s\in K}\left(\sum_{p>p_0}\sum_{l=2}^{\infty}\frac{|\alpha(p)|^l+|\beta(p)|^l}{lp^{l\sigma}}\right)<\varepsilon.\end{aligned}$$

Therefore the proof of Lemma 2 is now reduced to the validity of Lemma 6.

4. Proof of Lemma 6. An essential ingredient of the proof is the following

LEMMA 7. *Let $f(s)$ be an entire function of exponential type, and let $\{\lambda_m\}$ be a sequence of complex numbers. Let α, β and δ be positive numbers such that*

$$\limsup_{y\to\infty}\frac{\log|f(\pm iy)|}{y}\le\alpha, \tag{a}$$

(b) $$|\lambda_m - \lambda_n| \geq \delta|m-n|,$$

(c) $$\lim_{m\to\infty} \frac{\lambda_m}{m} = \beta,$$

(d) $$\alpha\beta < \pi.$$

Then

$$\limsup_{m\to\infty} \frac{\log|f(\lambda_m)|}{|\lambda_m|} = \limsup_{r\to\infty} \frac{\log|f(r)|}{r}.$$

This is a variant of the Bernstein theorem, and is given as Theorem 6.4.12 of [12] with a proof.

To prove Lemma 6, we apply Lemma 7 with $f = \varrho$. Since the support of the measure μh^{-1} is included in $D_{1,N}$, we see that

$$|\varrho(\pm iy)| \leq e^{Ny} \int_{\mathbb{C}} |d\mu h^{-1}(s)|$$

for $y > 0$, hence we can take $\alpha = N$ in the condition (a) of Lemma 7. Let us take a fixed positive number β satisfying

$$\beta < \pi/N. \tag{4.1}$$

Consider the set A of all positive integers m such that there exists $r \in ((m-1/4)\beta, (m+1/4)\beta]$ with $|\varrho(r)| \leq e^{-r}$.

We fix a number μ satisfying $0 < \mu < 1$, and put

$$\mathcal{P}_\mu = \{p : \text{primes},\ |c_p| > \mu\}.$$

Then from (3.9) it follows that

$$\sum_{p\in\mathcal{P}_\mu} |\varrho(\log p)| < \infty. \tag{4.2}$$

On the other hand, we have

$$\sum_{p\in\mathcal{P}_\mu} |\varrho(\log p)| \geq \sum_{m\notin A} {\sum_m}' |\varrho(\log p)| \geq \sum_{m\notin A} {\sum_m}' p^{-1}, \tag{4.3}$$

where ${\sum_m}'$ denotes the sum running over all primes $p \in \mathcal{P}_\mu$ satisfying

$$(m-1/4)\beta < \log p \leq (m+1/4)\beta.$$

Therefore, putting

$$a = \exp((m-1/4)\beta), \qquad b = \exp((m+1/4)\beta),$$

from (4.2) and (4.3) we obtain

$$\sum_{m\notin A} \sum_{\substack{p\in\mathcal{P}_\mu \\ a<p\leq b}} p^{-1} < \infty. \tag{4.4}$$

Let $\pi_\mu(x)$ be the number of primes $p \in \mathcal{P}_\mu$ up to x. Then, using (2.3), we have, for $a \le u \le b$,

$$(4.5)\qquad \sum_{a<p\le u} c_p^2 \le 4 \sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\le u}} 1 + \mu^2 \sum_{\substack{p\notin\mathcal{P}_\mu\\ a<p\le u}} 1$$
$$= 4(\pi_\mu(u) - \pi_\mu(a)) + \mu^2((\pi(u) - \pi_\mu(u)) - (\pi(a) - \pi_\mu(a)))$$
$$= (4-\mu^2)(\pi_\mu(u) - \pi_\mu(a)) + \mu^2(\pi(u) - \pi(a)).$$

On the other hand, by Rankin's formula (1.3), we have

$$(4.6)\qquad \sum_{a<p\le u} c_p^2 = \pi(u)(1+o(1)) - \pi(a)(1+o(1))$$

as $m \to \infty$.

We fix a positive parameter δ satisfying $1+\delta < e^{\beta/2}$, and let $0 < \varepsilon < \delta/100$. If $m \ge m_0(\varepsilon)$, then, for any $u \ge a(1+\delta)$, we obtain

$$\pi(u)(1+o(1)) \ge \pi(u)(1-\varepsilon), \qquad \pi(a)(1+o(1)) \le \pi(a)(1+\varepsilon).$$

Hence

$$(4.7)\qquad \pi(u)(1+o(1)) - \pi(a)(1+o(1)) \ge (\pi(u) - \pi(a)) - \varepsilon(\pi(u) + \pi(a)).$$

Since $u \ge a(1+\delta)$, we have, for $m \ge m_0(\varepsilon)$,

$$(4.8)\qquad \pi(u) - \pi(a) \ge \frac{u}{\log u}(1-\varepsilon) - \frac{a}{\log a}(1+\varepsilon)$$
$$\ge \frac{a(1+\delta)}{\log a + \log(1+\delta)}(1-\varepsilon) - \frac{a}{\log a}(1+\varepsilon)$$
$$\ge \frac{a}{\log a}(1+\delta)(1-2\varepsilon) - \frac{a}{\log a}(1+\varepsilon)$$
$$\ge \frac{a}{\log a}(\delta - 4\varepsilon) \ge \frac{a}{\log a}\cdot\frac{\delta}{2}.$$

On the other hand, if $u \le b = Ba$ where $B = e^{\beta/2}$, then, for $m \ge m_0(\varepsilon)$,

$$\pi(u) + \pi(a) \le \pi(b) + \pi(a) \le \frac{b}{\log b}(1+\varepsilon) + \frac{a}{\log a}(1+\varepsilon)$$
$$\le \frac{Ba}{\log a}(1+\varepsilon)^2 + \frac{a}{\log a}(1+\varepsilon) \le \frac{a}{\log a}(2B+2).$$

Therefore this and (4.8) yield

$$\pi(u) + \pi(a) \le \frac{4B+4}{\delta}(\pi(u) - \pi(a)).$$

From this and (4.7) we find that for the same u as above and $m \to \infty$,

$$\pi(u)(1+o(1)) - \pi(a)(1+o(1)) \ge \pi(u) - \pi(a) - \varepsilon\frac{4B+4}{\delta}(\pi(u) - \pi(a))$$
$$= (\pi(u) - \pi(a))(1+o(1)).$$

Hence, by (4.5) and (4.6), we find

$$(\pi(u)-\pi(a))(1+o(1)) \le (4-\mu^2)(\pi_\mu(u)-\pi_\mu(a)) + \mu^2(\pi_\mu(u)-\pi_\mu(a)),$$

so

$$\pi_\mu(u)-\pi_\mu(a) \ge \frac{1-\mu^2}{4-\mu^2}(\pi(u)-\pi(a))(1+o(1))$$

for $u \ge a(1+\delta)$, $m \to \infty$. Therefore, by partial summation,

$$\begin{aligned}
(4.9)\quad \sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\le b}} \frac{1}{p} &= \Big(\sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\le b}} 1\Big) + \int_a^b \Big(\sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\le b}}\Big)\frac{du}{u^2}\\
&= (\pi_\mu(b)-\pi_\mu(a))\frac{1}{b} + \int_a^b (\pi_\mu(u)-\pi_\mu(a))\frac{du}{u^2}\\
&\ge (\pi_\mu(b)-\pi_\mu(a))\frac{1}{b} + \int_{a(1+\delta)}^b (\pi_\mu(u)-\pi_\mu(a))\frac{du}{u^2}\\
&\ge \frac{1-\mu^2}{4-\mu^2}\Big((\pi(b)-\pi(a))\frac{1}{b} + \int_{a(1+\delta)}^b (\pi(u)-\pi(a))\frac{du}{u^2}\Big)(1+o(1))\\
&\ge \frac{1-\mu^2}{4-\mu^2}\Big((\pi(b)-\pi(a(1+\delta)))\frac{1}{b}\\
&\quad + \int_{a(1+\delta)}^b (\pi(u)-\pi(a(1+\delta)))\frac{du}{u^2}\Big)(1+o(1))\\
&= \frac{1-\mu^2}{4-\mu^2}\Big(\sum_{a(1+\delta)<p\le b}\frac{1}{p}\Big)(1+o(1))
\end{aligned}$$

as $m \to \infty$.

From (1.2) it follows that, as $m \to \infty$,

$$\sum_{a(1+\delta)<p\le b}\frac{1}{p} = \Big(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\Big)\frac{1}{m} + O\Big(\frac{1}{m^2}\Big);$$

hence and from (4.9),

$$(4.10)\qquad \sum_{\substack{p\in\mathcal{P}_\mu\\ a<p\le b}} \frac{1}{p} \ge \frac{1-\mu^2}{4-\mu^2}\Big(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\Big)\frac{1}{m}(1+o(1)) + O\Big(\frac{1}{m^2}\Big).$$

Since $0<\mu<1$ and $1+\delta < e^{\beta/2}$, we see that

$$\frac{1-\mu^2}{4-\mu^2}\Big(\frac{1}{2} - \frac{\log(1+\delta)}{\beta}\Big) > 0.$$

Therefore, from (4.4) and (4.10), we obtain

$$\sum_{m\notin A}\frac{1}{m}<\infty. \tag{4.11}$$

We write

$$A=\{a_m : m=1,2,\ldots\},\qquad a_1<a_2<\ldots$$

Then from (4.11) we can easily show that

$$\lim_{m\to\infty}\frac{a_m}{m}=1.$$

By the definition of the set A, there exists a sequence $\{\lambda_m\}$ such that

$$(a_m-1/4)\beta<\lambda_m\le(a_m+1/4)\beta\quad\text{and}\quad|\varrho(\lambda_m)|\le\exp(-\lambda_m).$$

Then

$$\lim_{m\to\infty}\frac{\lambda_m}{m}=\beta\quad\text{and}\quad\limsup_{m\to\infty}\frac{\log|\varrho(\lambda_m)|}{\lambda_m}\le-1.$$

Now by Lemma 7 we have

$$\limsup_{r\to\infty}\frac{\log|\varrho(r)|}{r}\le-1. \tag{4.12}$$

Assume $\varrho(z)\not\equiv0$. We can write

$$\varrho(s)=\int_{\mathbb{C}}e^{sz}\,d\nu(s),$$

where the measure ν is defined by $\nu(A)=\mu h^{-1}(-A), A\in\mathcal{B}(\mathbb{C})$, so its support is included in $\{s\in\mathbb{C}:-1<\sigma<-1/2\}$. Hence, by Lemma 5, we get

$$\limsup_{r\to\infty}\frac{\log|\varrho(r)|}{r}>-1,$$

which contradicts (4.12). Therefore we conclude that $\varrho(z)\equiv0$, which is the assertion of Lemma 6. The proof of Lemma 2 is now complete.

5. The support of the measure P_φ. Now we can deduce our theorem from Lemma 2 in much the same way as described in Section 6.5 of [12]. In this section we determine the support of the measure P_φ defined in Section 2. Let

$$S_N=\{f\in H(D_N): f(s)\ne0 \text{ for any } s\in D_N,\ \text{or } f(s)\equiv0\}.$$

LEMMA 8. *The support of the measure P_φ is the set S_N.*

In order to deduce this lemma from Lemma 2, we need two more lemmas:

LEMMA 9. *Let $\{f_n(s)\}$ be a sequence of functions analytic on D_N such that $f_n(s)\to f(s)$ (as $n\to\infty$) uniformly on D_N. Suppose $f(s)\not\equiv0$. Then* *an interior point* s_0 *of* D_N *is a zero of* $f(s)$ *if and only if there exists a sequence* $\{s_n\}$ *in* D_N *such that* $s_n \to s_0$ *(as* $n \to \infty$*) and* $f_n(s_n) = 0$ *for* $n > n_0 = n_0(s_0)$*.*

This is the Hurwitz theorem (see Section 3.45 of Titchmarsh [20]). The next lemma is Theorem 1.7.10 of [12]. Denote by $S(\xi)$ the support of the random element ξ.

LEMMA 10. *Let* $\{\xi_m\}$ *be a sequence of independent* $H(D_N)$*-valued random elements such that the series*

$$\sum_{m=1}^{\infty} \xi_m \tag{5.1}$$

converges almost surely. Then the support of the sum (5.1) *is the closure of the set of all* $f \in H(D_N)$ *which may be written as a convergent series*

$$f = \sum_{m=1}^{\infty} f_m, \qquad f_m \in S(\xi_m).$$

Proof of Lemma 8. By the definition $\{\omega(p)\}$ is a sequence of independent random variables defined on $(\Omega, \mathcal{B}(\Omega), m_H)$, and the support of each $\omega(p)$ is the unit circle γ. Hence

$$\left\{ \log\left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} + \log\left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1} \right\}$$

is a sequence of independent $H(D_N)$-valued random elements, and the set

$$\left\{ f \in H(D_N) : f(s) = -\log\left(1 - \frac{\alpha(p)a}{p^s}\right) + \log\left(1 - \frac{\beta(p)a}{p^s}\right),\ a \in \gamma \right\}$$

is the support of each element. Consequently, by Lemma 10, the support of the $H(D_N)$-valued random element

$$\log\varphi(s,\omega,F) = -\sum_p \left\{ \log\left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right) + \log\left(1 - \frac{\beta(p)\omega(p)}{p^s}\right) \right\}$$

is the closure of the set of all convergent series $\sum_p f_p(s; a_p)$. By Lemma 2 the latter set is dense in $H(D_N)$.

The map $\exp : H(D_N) \to H(D_N)$ is continuous, sending $\log\varphi(s,\omega,F)$ to $\varphi(s,\omega,F)$, and sending $H(D_N)$ onto $S_N \setminus \{0\}$. Therefore the support of $\varphi(s,\omega,F)$ contains the set $S_N \setminus \{0\}$. By the definition the support is a closed set (see Definition 1.2.13 of [12]), and by Lemma 9 we have $\overline{S_N \setminus \{0\}} = S_N$. Thus

$$S(\varphi) \supseteq S_N. \tag{5.2}$$

On the other hand,

$$\left(1-\frac{\alpha(p)\omega(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)\omega(p)}{p^s}\right)^{-1}, \quad s\in D_N,\ \omega\in\Omega,$$

is non-zero for all primes p. Hence $\varphi(s,\omega,F)$ is an almost surely convergent product of non-vanishing factors. Again by Lemma 9 we see that $\varphi(s,\omega,F)\in S_N$ almost surely. Thus $S(\varphi)\subseteq S_N$. This and (5.2) give the assertion of Lemma 8.

6. Completion of the proof of the theorem. Let K be a compact subset of D with connected complement. Then we can find $N>0$ such that $K\subset D_N$. Let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K.

First we assume that $f(s)$ has a non-vanishing analytic continuation to $H(D_N)$. Denote by G the set of functions $g\in H(D_N)$ for which

$$\sup_{s\in K}|g(s)-f(s)|<\varepsilon.$$

The set G is open, hence by Lemma 1 we have

$$\liminf_{T\to\infty}\nu_T(\sup_{s\in K}|\varphi(s+i\tau,F)-f(s)|<\varepsilon)\geq P_\varphi(G). \tag{6.1}$$

Obviously $f\in S_N$, hence by Lemma 8 it is contained in the support of the random element $\varphi(s,\omega,F)$. Since G is a neighbourhood of f, we have $P_\varphi(G)>0$. This together with (6.1) implies the assertion of the theorem in this case.

Now consider the general case. First we quote

Lemma 11. *Let K be a compact subset of $\mathbb{C}$ whose complement is connected. Then any continuous function $f(s)$ on K which is analytic in the interior of K is approximable uniformly on K by polynomials in s.*

This is the Mergelyan theorem, and the proof can be found, for example, in Walsh [22].

Since $f(s)\neq 0$ on K, by Lemma 11 we can find a polynomial $p(s)$ such that $p(s)\neq 0$ on K and

$$\sup_{s\in K}|f(s)-p(s)|<\varepsilon/4. \tag{6.2}$$

Since $p(s)$ has only finitely many zeros, we can find a region G_1 such that $K\subset G_1$ and $p(s)\neq 0$ on G_1. We choose $\log p(s)$ to be analytic in the interior of G_1. Applying Lemma 11 to $\log p(s)$, we find another polynomial $q(s)$ such that

$$\sup_{s\in K}|p(s)-e^{q(s)}|<\varepsilon/4.$$

From this and (6.2) it follows that

$$\sup_{s\in K} |f(s) - e^{q(s)}| < \varepsilon/2. \tag{6.3}$$

Since $e^{q(s)} \neq 0$ for all s, we can use the result of the case already proved, which yields

$$\liminf_{T\to\infty} \nu_T(\sup_{s\in K} |\varphi(s+i\tau, F) - e^{q(s)}| < \varepsilon/2) > 0.$$

Together with (6.3), this completes the proof of the theorem.

References

[1] B. Bagchi, *The statistical behaviour and universality properties of the Riemann zeta-function and other allied Dirichlet series*, Ph.D. thesis, Indian Statistical Institute, Calcutta, 1981.

[2] —, *A joint universality theorem for Dirichlet L-functions*, Math. Z. 181 (1982), 319–334.

[3] P. Deligne, *La conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. 43 (1974), 273–307.

[4] S. M. Gonek, *Analytic properties of zeta and L-functions*, Ph.D. thesis, University of Michigan, 1979.

[5] A. Good, *On the distribution of the values of Riemann's zeta-function*, Acta Arith. 38 (1981), 347–388.

[6] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I*, Math. Ann. 114 (1937), 1–28; *II*, ibid. 316–351.

[7] A. Kačėnas and A. Laurinčikas, *On Dirichlet series related to certain cusp forms*, Liet. Mat. Rink. 38 (1998), 82–97 (in Russian) = Lithuanian Math. J. 38 (1998), 64–76.

[8] A. Laurinčikas, *Distribution des valeurs de certaines séries de Dirichlet*, C. R. Acad. Sci. Paris 289 (1979), 43–45.

[9] —, *Distribution of values of generating Dirichlet series of multiplicative functions*, Liet. Mat. Rink. 22 (1982), 101–111 (in Russian) = Lithuanian Math. J. 22 (1982), 56–63.

[10] —, *The universality theorem*, Liet. Mat. Rink. 23 (1983), 53–62 (in Russian) = Lithuanian Math. J. 23 (1983), 283–289.

[11] —, *The universality theorem II*, Liet. Mat. Rink. 24 (1984), 113–121 (in Russian) = Lithuanian Math. J. 24 (1984), 143–149.

[12] —, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, 1996.

[13] —, *On limit distribution of the Matsumoto zeta-function II*, Liet. Mat. Rink. 36 (1996), 464–485 (in Russian) = Lithuanian Math. J. 36 (1996), 371–387.

[14] —, *The universality of the Lerch zeta-function*, Liet. Mat. Rink. 37 (1997), 367–375 (in Russian) = Lithuanian Math. J. 37 (1997), 275–280.

[15] —, *On the Matsumoto zeta-function*, Acta Arith. 84 (1998), 1–16.

[16] K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

[17] R. A. Rankin, *An Ω-result for the coefficients of cusp forms*, Math. Ann. 203 (1973), 239–250.

[18] A. Reich, *Universelle Wertverteilung von Eulerprodukten*, Nachr. Akad. Wiss. Göttingen II Math.-Phys. Kl. 1977, no. 1, 1–17.
[19] —, *Zur Universalität und Hypertranszendenz der Dedekindschen Zetafunktion*, Abh. Braunschweig. Wiss. Ges. 33 (1982), 197–203.
[20] E. C. Titchmarsh, *The Theory of Functions*, Oxford Univ. Press, Oxford, 1939.
[21] S. M. Voronin, *A theorem on the "universality" of the Riemann zeta-function*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 475–486 (in Russian) = Math. USSR-Izv. 9 (1975), 443–453.
[22] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, Amer. Math. Soc. Colloq. Publ. 20, 1960.

Department of Mathematics
Vilnius University
Naugarduko, 24
2006 Vilnius, Lithuania
E-mail: antanas.laurincikas@maf.vu.lt

Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya, 464-8602, Japan
E-mail: kohjimat@math.nagoya-u.ac.jp

Received on 14.5.1999
and in revised form on 12.12.2000 (3603)