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Gcd-closed sets and determinants of matrices associated with arithmetical functions

by

Shaofang Hong (Chengdu)

1. Introduction. Smith [11] proved that if f is an arithmetical function and $[f(i,j)]$ is the $n\times n$ matrix having f evaluated at the greatest common divisor of i and j as its i,j-entry, then

$$\det[f(i,j)] = (f*\mu)(1)(f*\mu)(2)\ldots(f*\mu)(n),$$

where μ is the Möbius function and $f*\mu$ is the Dirichlet convolution of f and μ. Apostol [2] extended Smith's result by showing that if f and g are arithmetical functions and if β is defined for positive integers t and r by $\beta(t,r)=\sum_{d\mid(t,r)} f(d)g(r/d)$, then $\det[\beta(i,j)] = [g(1)]^n f(1)\ldots f(n)$. He noted that as a consequence, $\det[C(i,j)] = n!$, where $C(t,r)$ is Ramanujan's sum. Paul McCarthy [8] generalized Smith's and Apostol's results to the class of even functions (mod r). He evaluated the determinants of $n\times n$ matrices of the form $[\beta(i,j)]$, where $\beta(t,r)$ is an even function of t (mod r). A complex-valued function $\beta(t,r)$ of the positive integral variables t and r is said to be an *even function of* t (mod r) if $\beta(t,r)=\beta((t,r),r)$ for all values of t. The functions considered by Smith and Apostol are even functions of t (mod r) for every r. Bourque and Ligh [3] evaluated the determinants of $n\times n$ matrices of the form $[\beta(x_i,x_j)]$, where the set $S=\{x_1,\ldots,x_n\}$ of distinct positive integers is *factor-closed* (i.e., S contains every divisor of x for any $x\in S$) and $\beta(t,r)$ is an even function of t (mod r).

Throughout this paper, let $S=\{x_1,\ldots,x_n\}$ be a set of distinct positive integers. The set S is said to be *gcd-closed* if $(x_i,x_j)\in S$ for $1\le i,j\le n$. Clearly, a factor-closed set is gcd-closed but not conversely. Let $f(t), g(t)$ and $h(t)$ be arithmetical functions. The $\Psi(t,r)$ is defined for all positive

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integers t and r as follows:

$$\Psi(t,r)=\sum_{d\mid(t,r)} f(d)g\left(\frac{t}{d}\right)h\left(\frac{r}{d}\right). \tag{1}$$

Define the class of arithmetical functions $L_S=\{l(t): l(d_1/d)=l(d_2/d)$ whenever $d\mid d_1$ for any $d_1,d_2\in S$ satisfying $d_1\mid d_2\}$. In 1993, Bourque and Ligh [3] evaluated the determinant of the $n\times n$ matrix $[\Psi(x_i,x_j)]$ if the set $S=\{x_1,\dots,x_n\}$ is factor-closed. In this paper we will evaluate the determinant of the $n\times n$ matrix $[\Psi(x_i,x_j)]$, where $S=\{x_1,\dots,x_n\}$ is gcd-closed and $g\in L_S$ or $h\in L_S$. As applications, we evaluate the determinants of $n\times n$ matrices of the form $[C(x_i,x_j)]$, where $S=\{x_1,\dots,x_n\}$ is gcd-closed, and $C(t,r)$ is Ramanujan's trigonometric sum. These results generalize Bourque and Ligh's results [3]. We also evaluate the determinant of $n\times n$ matrix $\left[\frac{g}{f*g}\left(\frac{x_i}{(x_i,x_j)}\right)\right]$, where f is completely multiplicative, $g(m)=\mu(m)h(m)$, h is multiplicative, $f(p)\ne 0$ and $f(p)\ne h(p)$ for all primes p, and $(f*g)(d)\ne 0$ for any positive integer d satisfying $d\mid x$, $x\in S$, and $S=\{x_1,\dots,x_n\}$ is gcd-closed.

2. Determinant of the matrix $[\Psi(x_i,x_j)]$. In the present section, we evaluate the determinant of the $n\times n$ matrix $\Psi(x_i,x_j)$, where $g\in L_S$ or $h\in L_S$ and $S=\{x_1,\dots,x_n\}$ is gcd-closed.

LEMMA 1 ([3]). *Let $T=\{y_1,\dots,y_m\}$ be a factor-closed set containing S. Then $[\Psi(x_i,x_j)]=G\Lambda H^{\mathrm{T}}$, where $\Lambda=\operatorname{diag}(f(y_1),\dots,f(y_m))$ and the $n\times m$ matrices G and H are defined by $G=[g(x_i/y_j)]$ and $H=[h(x_i/y_j)]$, respectively.* ■

LEMMA 2. *Let the set $S=\{x_1,\dots,x_n\}$ of distinct positive integers be gcd-closed. If $g\in L_S$ or $h\in L_S$ then there exist $n\times n$ lower triangular matrices M and N with diagonal elements 1 and an $n\times n$ lower triangular matrix P with diagonal elements*

$$\sum_{d\mid x_1} f(d)g\left(\frac{x_1}{d}\right)h\left(\frac{x_1}{d}\right),\sum_{\substack{d\mid x_2\\ d\nmid x_1}} f(d)g\left(\frac{x_2}{d}\right)h\left(\frac{x_2}{d}\right),\ \dots,\ \sum_{\substack{d\mid x_n\\ d\nmid x_l,x_l<x_n}} f(d)g\left(\frac{x_n}{d}\right)h\left(\frac{x_n}{d}\right),$$

such that $[\Psi(x_i,x_j)]=MPN^{\mathrm{T}}$.

Proof. Without loss of generality we may let $1\le x_1<\dots<x_n$. Let $S_k=\{d: d\in\mathbb{Z}^+,\ d\mid x_k,\ d\nmid x_t,\ t<k\}$, $1\le k\le n$. Clearly $S_{k_1}\cap S_{k_2}=\emptyset$ for $1\le k_1,k_2\le n$, $k_1\ne k_2$ and $S_1\cup\dots\cup S_n=\overline{S}$, where $\overline{S}$ is the minimal factor-closed set containing S (the factor closure of S). Let $S_k=\{y_{k,1},\dots,y_{k,p_k}\}$ $(1\le k\le n)$ and $m=p_1+\dots+p_n$ where $y_{k,1}<\dots<$

$y_{k,p_k} = x_k$. For $1 \le j \le m$, let

$$y_j = \begin{cases} y_{1,j} & \text{if } 1 \le j \le p_1, \\ y_{k,t} & \text{if } j = p_1 + \ldots + p_{k-1} + t \ (k \ge 2, 1 \le t \le p_k). \end{cases}$$

Thus $\overline{S} = \{y_1, \ldots, y_m\}$. Let the $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ be defined as follows:

$$a_{ij} = \begin{cases} g(x_i/y_j)\sqrt{f(y_j)} & \text{if } y_j \mid x_i, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$b_{ij} = \begin{cases} h(x_i/y_j)\sqrt{f(y_j)} & \text{if } y_j \mid x_i, \\ 0 & \text{otherwise}. \end{cases}$$

It follows immediately from Lemma 1 that

$$[\Psi(x_i, x_j)] = AB^{\mathrm{T}}. \tag{2}$$

Let $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ denote the systems of row vectors of A and B respectively. Let $\{\gamma_1, \ldots, \gamma_n\}$ and $\{\delta_1, \ldots, \delta_n\}$ denote the orthogonalization systems obtained from $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ respectively by using the Gram–Schmidt orthogonalization process (see [7]), then we have (where $\langle \beta, \beta \rangle$ denotes the inner product)

$$\begin{cases} \gamma_1 = \alpha_1, \\ \gamma_2 = \alpha_2 - \dfrac{\langle \alpha_2, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle}\gamma_1, \\ \quad \vdots \\ \gamma_n = \alpha_n - \dfrac{\langle \alpha_n, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle}\gamma_1 - \ldots - \dfrac{\langle \alpha_n, \gamma_{n-1} \rangle}{\langle \gamma_{n-1}, \gamma_{n-1} \rangle}\gamma_{n-1}, \end{cases}$$

and

$$\begin{cases} \delta_1 = \beta_1, \\ \delta_2 = \beta_2 - \dfrac{\langle \beta_2, \delta_1 \rangle}{\langle \delta_1, \delta_1 \rangle}\delta_1, \\ \quad \vdots \\ \delta_n = \beta_n - \dfrac{\langle \beta_n, \delta_1 \rangle}{\langle \delta_1, \delta_1 \rangle}\delta_1 - \ldots - \dfrac{\langle \beta_n, \delta_{n-1} \rangle}{\langle \delta_{n-1}, \delta_{n-1} \rangle}\delta_{n-1}. \end{cases}$$

Therefore

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \frac{\langle \alpha_2, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\langle \alpha_n, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle} & \frac{\langle \alpha_n, \gamma_2 \rangle}{\langle \gamma_2, \gamma_2 \rangle} & \ldots & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}, \tag{3}$$

and

$$(4)\qquad \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \frac{\langle \beta_2, \delta_1 \rangle}{\langle \delta_1, \delta_1 \rangle} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{\langle \beta_n, \delta_1 \rangle}{\langle \delta_1, \delta_1 \rangle} & \frac{\langle \beta_n, \delta_2 \rangle}{\langle \delta_2, \delta_2 \rangle} & \dots & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}.$$

Let M and N be the left matrices on the right-hand sides of equations (3) and (4) respectively. Then

$$(5)\qquad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} (\beta_1^{\mathrm{T}} \quad \beta_2^{\mathrm{T}} \quad \dots \quad \beta_n^{\mathrm{T}}) = M \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} (\delta_1^{\mathrm{T}} \quad \delta_2^{\mathrm{T}} \quad \dots \quad \delta_n^{\mathrm{T}}) N^{\mathrm{T}}.$$

It follows from (2) and (5) that

$$(6)\qquad [\Psi(x_i, x_j)] = M \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} (\delta_1^{\mathrm{T}} \quad \delta_2^{\mathrm{T}} \quad \dots \quad \delta_n^{\mathrm{T}}) N^{\mathrm{T}}.$$

Since $x_1 < \ldots < x_n$, it is easy to see that

$$(\alpha_1)^{(i)} = \begin{cases} g(x_1/y_{1,i})\sqrt{f(y_{1,i})} & \text{if } 1 \le i \le p_1, \\ 0 & \text{if } i > p_1, \end{cases}$$

and

$$(\beta_1)^{(i)} = \begin{cases} h(x_1/y_{1,i})\sqrt{f(y_{1,i})} & \text{if } 1 \le i \le p_1, \\ 0 & \text{if } i > p_1, \end{cases}$$

and for $k \ge 2, i > p_1 + \ldots + p_{k-1}$, we have

$$(\alpha_k)^{(i)} = \begin{cases} g(x_k/y_{k,t})\sqrt{f(y_{k,t})} & \text{if } i = p_1 + \ldots + p_{k-1} + t \ (1 \le t \le p_k), \\ 0 & \text{if } i > p_1 + \ldots + p_k, \end{cases}$$

and

$$(\beta_k)^{(i)} = \begin{cases} h(x_k/y_{k,t})\sqrt{f(y_{k,t})} & \text{if } i = p_1 + \ldots + p_{k-1} + t \ (1 \le t \le p_k), \\ 0 & \text{if } i > p_1 + \ldots + p_k. \end{cases}$$

Thus for $i = p_1 + \ldots + p_{k-1} + t$ $(k \ge 2, 1 \le t \le p_k)$, we have

$$(\gamma_k)^{(i)} = g\left(\frac{x_k}{y_{k,t}}\right)\sqrt{f(y_{k,t})} \quad \text{and} \quad (\delta_k)^{(i)} = h\left(\frac{x_k}{y_{k,t}}\right)\sqrt{f(y_{k,t})}.$$

To complete the proof of Lemma 2, we need the following:

LEMMA 3. *With the above notations, let $S = \{x_1, \dots, x_n\}$ be gcd-closed. If $g \in L_S$, then*

$$\gamma_1 = \left(g\left(\frac{x_1}{y_{1,1}}\right)\sqrt{f(y_{1,1})}, \dots, g\left(\frac{x_1}{y_{1,p_1}}\right)\sqrt{f(y_{1,p_1})}, 0, \dots, 0\right),$$

and for $k \ge 2$, we have

$$\gamma_k = \Bigg(\underbrace{0, \dots, 0}_{p_1+\dots+p_{k-1}}, g\left(\frac{x_k}{y_{k,1}}\right)\sqrt{f(y_{k,1})}, \dots, g\left(\frac{x_k}{y_{k,p_k}}\right)\sqrt{f(y_{k,p_k})}, 0, \dots, 0\Bigg).$$

Similarly, if $h \in L_S$, then

$$\delta_1 = \left(h\left(\frac{x_1}{y_{1,1}}\right)\sqrt{f(y_{1,1})}, \dots, h\left(\frac{x_1}{y_{1,p_1}}\right)\sqrt{f(y_{1,p_1})}, 0, \dots, 0\right),$$

and for $k \ge 2$, we have

$$\delta_k = \Bigg(\underbrace{0, \dots, 0}_{p_1+\dots+p_{k-1}}, h\left(\frac{x_k}{y_{k,1}}\right)\sqrt{f(y_{k,1})}, \dots, h\left(\frac{x_k}{y_{k,p_k}}\right)\sqrt{f(y_{k,p_k})}, 0, \dots, 0\Bigg).$$

Proof. Since the case $h \in L_S$ is similar to the case $g \in L_S$, we only consider the latter. We argue by induction on k. Clearly Lemma 3 is true for γ_1 (since $\gamma_1 = \alpha_1$). Since S is gcd-closed, $(x_2, x_1) = x_1$. Note that $g \in L_S$ implies

$$g\left(\frac{x_2}{y_{1,j}}\right) = g\left(\frac{x_1}{y_{1,j}}\right) \quad \text{for } 1 \le j \le p_1.$$

Thus

$$\alpha_2 = \left(g\left(\frac{x_1}{y_{1,1}}\right)\sqrt{f(y_{1,1})}, \dots, g\left(\frac{x_1}{y_{1,p_1}}\right)\sqrt{f(y_{1,p_1})}, g\left(\frac{x_2}{y_{2,1}}\right)\sqrt{f(y_{2,1})}, \dots,\right.$$
$$\left. g\left(\frac{x_2}{y_{2,p_2}}\right)\sqrt{f(y_{2,p_2})}, 0, \dots, 0\right).$$

Then $\langle \alpha_2, \gamma_1 \rangle = \langle \gamma_1, \gamma_1 \rangle$. Therefore

$$\begin{aligned}\gamma_2 &= \alpha_2 - \frac{\langle \alpha_2, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle}\gamma_1 = \alpha_2 - \gamma_1 \\ &= \Bigg(\underbrace{0, \dots, 0}_{p_1}, g\left(\frac{x_2}{y_{2,1}}\right)\sqrt{f(y_{2,1})}, \dots, g\left(\frac{x_2}{y_{2,p_2}}\right)\sqrt{f(y_{2,p_2})}, 0, \dots, 0\Bigg).\end{aligned}$$

So the assertion is true for γ_2. Suppose that it is true for γ_l, $1 \le l \le k-1$ $(k \ge 3)$. Now consider γ_k. Since $g \in L_S$, we have

$$\left(\alpha_k - \frac{\langle \alpha_k, \gamma_1 \rangle}{\langle \gamma_1, \gamma_1 \rangle}\gamma_1\right)^{(i)} = 0, \qquad 1 \le i \le p_1.$$

We claim that for each $e \in \{2, \ldots, k-1\}$ and each i with $p_1 + \ldots + p_{e-1} < i \le p_1 + \ldots + p_e$, we have

$$\left(\alpha_k - \frac{\langle \alpha_k, \gamma_e\rangle}{\langle \gamma_e, \gamma_e\rangle}\gamma_e\right)^{(i)} = 0.$$

In fact, if $(x_k, x_e) = x_e$, then $x_e \mid x_k$. Note that $g \in L_S$ implies $g(x_k/y_{e,i}) = g(x_e/y_{e,i})$ for $1 \le i \le p_e$. Thus $\langle \alpha_k, \gamma_e\rangle = \langle \gamma_e, \gamma_e\rangle$. Hence for each i with $p_1 + \ldots + p_{e-1} < i \le p_1 + \ldots + p_e$, we have

$$\left(\alpha_k - \frac{\langle \alpha_k, \gamma_e\rangle}{\langle \gamma_e, \gamma_e\rangle}\gamma_e\right)^{(i)} = (\alpha_k - \gamma_e)^{(i)} = 0.$$

If $(x_k, x_e) = x_r$ for some $1 \le r < e$, then $y_{e,i} \nmid x_k$ for all $1 \le i \le p_e$. Otherwise, there exists i, $1 \le i \le p_e$, such that $y_{e,i} \mid x_k$. So $y_{e,i} \mid x_r$. However, as $r < e$ we have $y_{e,i} \nmid x_r$. This is a contradiction. Thus for $p_1 + \ldots + p_{e-1} < i \le p_1 + \ldots + p_e$, $(\alpha_k)^{(i)} = 0$. So $\langle \alpha_k, \gamma_e\rangle = 0$. Hence for $p_1 + \ldots + p_{e-1} < i \le p_1 + \ldots + p_e$, we have

$$\left(\alpha_k - \frac{\langle \alpha_k, \gamma_e\rangle}{\langle \gamma_e, \gamma_e\rangle}\gamma_e\right)^{(i)} = (\alpha_k)^{(i)} = 0.$$

This completes the proof of the claim.

Thus it follows from the induction hypothesis and the claim that

$$\begin{aligned}\gamma_k &= \alpha_k - \frac{\langle \alpha_k, \gamma_1\rangle}{\langle \gamma_1, \gamma_1\rangle}\gamma_1 - \ldots - \frac{\langle \alpha_k, \gamma_{k-1}\rangle}{\langle \gamma_{k-1}, \gamma_{k-1}\rangle}\gamma_{k-1} \\ &= \Bigg(\underbrace{0, \ldots, 0}_{p_1+\ldots+p_{k-1}}, g\left(\frac{x_k}{y_{k,1}}\right)\sqrt{f(y_{k,1})}, \ldots, g\left(\frac{x_k}{y_{k,p_k}}\right)\sqrt{f(y_{k,p_k})}, 0, \ldots, 0\Bigg).\end{aligned}$$

The proof of Lemma 3 is complete. ∎

Now we continue to prove Lemma 2. Since $g \in L_S$ or $h \in L_S$, it follows from Lemma 3 that

$$(7)\quad \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} (\,\delta_1^{\mathrm{T}} \quad \delta_2^{\mathrm{T}} \quad \ldots \quad \delta_n^{\mathrm{T}}\,)$$

$$= \begin{pmatrix} \sum\limits_{j=1}^{p_1} f(y_{1,j})g(\frac{x_1}{y_{1,j}})h(\frac{x_1}{y_{1,j}}) & 0 & \ldots & 0 \\ * & \sum\limits_{j=1}^{p_2} f(y_{2,j})g(\frac{x_2}{y_{2,j}})h(\frac{x_2}{y_{2,j}}) & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & \sum\limits_{j=1}^{p_n} f(y_{n,j})g(\frac{x_n}{y_{n,j}})h(\frac{x_n}{y_{n,j}}) \end{pmatrix}$$

$$= \begin{pmatrix} \sum\limits_{d|x_1} f(d)g(\frac{x_1}{d})h(\frac{x_1}{d}) & 0 & \dots & 0 \\ * & \sum\limits_{\substack{d|x_2 \\ d\nmid x_1}} f(d)g(\frac{x_2}{d})h(\frac{x_2}{d}) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & \sum\limits_{\substack{d|x_n \\ d\nmid x_l,\, x_l<x_n}} f(d)g(\frac{x_n}{d})h(\frac{x_n}{d}) \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} \begin{pmatrix} \delta_1^{\mathrm{T}} & \delta_2^{\mathrm{T}} & \dots & \delta_n^{\mathrm{T}} \end{pmatrix}.$$

By (6) we have $[\Psi(x_i, x_j)] = MPN^{\mathrm{T}}$. Clearly the matrices M and N are lower triangular matrices with diagonal elements 1. By (7), P is a lower triangular matrix with diagonal elements

$$\sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} f(d)g\left(\frac{x_k}{d}\right)h\left(\frac{x_k}{d}\right), \quad k = 1, \dots, n.$$

This completes the proof of Lemma 2. ∎

Now we are ready to give the main result of this paper.

THEOREM 1. *Let $S = \{x_1, \dots, x_n\}$ be gcd-closed. If $g \in L_S$ or $h \in L_S$, then*

$$\det[\Psi(x_i, x_j)] = \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} f(d)g\left(\frac{x_k}{d}\right)h\left(\frac{x_k}{d}\right). \tag{8}$$

Proof. Since $S = \{x_1, \dots, x_n\}$ is gcd-closed and $g \in L_S$ or $h \in L_S$, by Lemma 2 there exist $n \times n$ lower triangular matrices M and N with diagonal elements 1 and an $n \times n$ lower triangular matrix P with diagonal elements

$$\sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} f(d)g\left(\frac{x_k}{d}\right)h\left(\frac{x_k}{d}\right), \quad k = 1, \dots, n,$$

such that $[\Psi(x_i, x_j)] = MPN^{\mathrm{T}}$. Thus

$$\det[\Psi(x_i, x_j)] = (\det M)(\det P)(\det N^{\mathrm{T}}).$$

Note that $\det M = \det N = 1$. So $\det N^{\mathrm{T}} = 1$. Note also that

$$\det P = \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} f(d)g\left(\frac{x_k}{d}\right)h\left(\frac{x_k}{d}\right).$$

It follows that (8) holds. ∎

REMARK 1. Let $S = \{x_1, \ldots, x_n\}$ be gcd-closed. If $g \in L_S$ or $h \in L_S$, then (8) gives a formula for $\det[\Psi(x_i, x_j)]$. If g, $h \notin L_S$, then we also expect to have a formula for $\det[\Psi(x_i, x_j)]$. This problem remains open.

3. Applications. In this section, we give some interesting applications of our main result.

THEOREM 2. *Let $S = \{x_1, \ldots, x_n\}$ be gcd-closed. If β is defined for positive integers t and r by*

$$\beta(t,r) = \sum_{d \mid (t,r)} f(d) g\left(\frac{r}{d}\right),$$

then

$$\det[\beta(x_i, x_j)] = \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_l,\, x_l < x_k}} f(d) g\left(\frac{x_k}{d}\right).$$

Proof. Let $h = \zeta$, where ζ is defined by $\zeta(d) = 1$ for all integers d. Clearly $\zeta \in L_S$. Then $\Psi(t,r) = \beta(t,r)$. Thus the result follows immediately from Theorem 1. ∎

REMARK 2. If $S = \{1, \ldots, n\}$, then Theorem 2 becomes Apostol's result [2].

COROLLARY 1. *If $S = \{x_1, \ldots, x_n\}$ is gcd-closed, then*

$$\det[C(x_i, x_j)] = \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_l,\, x_l < x_k}} d\mu\left(\frac{x_k}{d}\right).$$

Proof. Ramanujan's trigonometric sum $C(t,r)$ is defined by

$$C(t,r) = \sum_{\substack{k \ (\mathrm{mod}\ r) \\ (k,r)=1}} \exp\left(\frac{2\pi i t}{k}\right) = \sum_{d \mid (t,r)} d\mu\left(\frac{r}{d}\right).$$

So if we set $f(d) = d$ for all d, $g = \mu$, then this corollary follows from Theorem 2. ∎

Define the quotient function $\frac{f}{g}$ by

$$\frac{f}{g}(m) = \frac{f(m)}{g(m)} \quad \text{for positive integers } m.$$

LEMMA 4 ([1, Theorem 8.8]). *Let f be completely multiplicative. Let $g(m) = \mu(m)h(m)$, where h is multiplicative. Assume that $f(p) \neq 0$ and*

$f(p) \neq h(p)$ for all primes p. Then

$$\sum_{d|(t,k)} f(d)g\left(\frac{k}{d}\right) = F(k)\frac{g}{F}(N)$$

where $F = f * g$, and $N = k/(t,k)$. ■

THEOREM 3. *Let f be completely multiplicative. Let $g(m) = \mu(m)h(m)$, where h is multiplicative. Assume that $f(p) \neq 0$ and $f(p) \neq h(p)$ for all primes p. If $S = \{x_1, \ldots, x_n\}$ is gcd-closed and $(f * g)(d) \neq 0$ for any positive integer d satisfying $d \mid x$, $x \in S$, then*

$$\det\left[\frac{g}{f*g}\left(\frac{x_i}{(x_i,x_j)}\right)\right] = \prod_{k=1}^{n} \frac{1}{(f*g)(x_k)} \sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} f(d)g\left(\frac{x_k}{d}\right).$$

Proof. Let $s(k,t) = \sum_{d|(t,k)} f(d)g(k/d)$. From Lemma 4, one can deduce that

$$\begin{aligned}[s(x_i,x_j)] &= \left[(f*g)(x_i) \cdot \frac{g}{f*g}\left(\frac{x_i}{(x_i,x_j)}\right)\right] \\ &= \operatorname{diag}((f*g)(x_1), \ldots, (f*g)(x_n)) \cdot \left[\frac{g}{f*g}\left(\frac{x_i}{(x_i,x_j)}\right)\right].\end{aligned}$$

Thus we have

$$\det\left[\frac{g}{f*g}\left(\frac{x_i}{(x_i,x_j)}\right)\right] = \det[s(x_i,x_j)] \prod_{k=1}^{n} \frac{1}{(f*g)(x_k)}.$$

Therefore the result follows from the above equation and Theorem 2. ■

COROLLARY 2. *Let $S = \{x_1, \ldots, x_n\}$ be factor-closed, and let the arithmetical functions f and g be as in Theorem 3. Then*

$$\det\left[\frac{g}{f*g}\left(\frac{x_i}{(x_i,x_j)}\right)\right] = [g(1)]^n \prod_{k=1}^{n} \frac{f}{f*g}(x_k). \ ■$$

COROLLARY 3. *Let the arithmetical functions f and g be as in Theorem 3. Then*

$$\det\left[\frac{g}{f*g}\left(\frac{i}{(i,j)}\right)\right] = [g(1)]^n \prod_{k=1}^{n} \frac{f}{f*g}(k). \ ■$$

An arithmetical function $f(t)$ is said to be *quadratic* if it is the Dirichlet convolution of two completely multiplicative functions [9, 12]. In what follows we use Theorem 1 and the following result of Vaidyanathaswamy, concerning quadratic functions, to evaluate the determinants of $n \times n$ matrices of the form $[f(x_i x_j)]$, where $f(t)$ is a quadratic function.

LEMMA 5 (Vaidyanathaswamy [12]). *If $f = g * h$ where g and h are completely multiplicative functions, then f satisfies the identity*

$$f(t,r) = \sum_{d|(t,r)} f\left(\frac{t}{d}\right) f\left(\frac{r}{d}\right) g(d)h(d)\mu(d). \ \blacksquare$$

THEOREM 4. *Let $f = g * h$, where g and h are completely multiplicative. If $S = \{x_1, \ldots, x_n\}$ is gcd-closed and $f \in L_S$, then*

$$\det[f(x_i x_j)] = \prod_{k=1}^{n} \sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} g(d)h(d)\mu(d) \left[f\left(\frac{x_k}{d}\right)\right]^2.$$

Proof. This result follows from Lemma 5 and Theorem 1. ■

LEMMA 6 ([10]). *The arithmetical function f is a semi-multiplicative function if and only if for any positive integers m and n, $f(m)f(n) = f((m,n))f([m,n])$.* ■

LEMMA 7. *Let f be an arithmetical function. Then for any positive integer n,*

$$\sum_{d|n} (f * \mu)(d) = f(n).$$

Proof. Let the arithmetical function I be defined for any positive integer m as follows: $I(m) = [1/m]$, where $[x]$ denotes the greatest integer not greater than x. Since $\mu * \zeta = I$ (see [1]) and $f = f * I$, one has

$$\begin{aligned} f(n) &= (f * I)(n) = (f * (\mu * \zeta))(n) = ((f * \mu) * \zeta)(n) \\ &= \sum_{d|n} (f * \mu)(d)\zeta\left(\frac{n}{d}\right) = \sum_{d|n} (f * \mu)(d), \end{aligned}$$

as desired. The proof of Lemma 7 is complete. ■

THEOREM 5. *Let f be a semi-multiplicative function and $f[t,r]$ denote f evaluated at the least common multiple of t and r. If $S = \{x_1, \ldots, x_n\}$ is gcd-closed, then*

$$\det(f[x_i, x_j]) = \prod_{k=1}^{n} [f(x_k)]^2 \sum_{\substack{d|x_k \\ d\nmid x_l,\, x_l<x_k}} \left(\frac{1}{f} * \mu\right)(d). \tag{9}$$

Proof. Since f is semi-multiplicative, it follows from Lemma 6 that

$$(f[x_i, x_j]) = D(g(x_i, x_j))D,$$

where $g = 1/f$ and $D = \mathrm{diag}(f(x_1), \ldots, f(x_n))$. Thus

$$\det(f[x_i, x_j]) = \det[g(x_i, x_j)] \prod_{k=1}^{n} [f(x_k)]^2. \tag{10}$$

Let $g = h = \zeta$ and substitute $g * \mu$ for f in Theorem 1. By Lemma 7, one has $\Psi(x_i, x_j) = g(x_i, x_j)$. Thus it follows from Theorem 1 that

$$\det(g[x_i, x_j]) = \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_l,\, x_l < x_k}} (g * \mu)(d). \tag{11}$$

It then follows from (10) and (11) that (9) holds. ∎

Remark 3. If we set $f(d) = d$ for all integers d, then Theorem 5 reduces to Bourque and Ligh's result [4]. Bourque and Ligh [4] conjectured that the LCM matrix $([x_i, x_j])$ defined on a gcd-closed set $S = \{x_1, \ldots, x_n\}$ is nonsingular. We showed [5] that the Bourque–Ligh conjecture is true for a certain class of gcd-closed sets $S = \{x_1, \ldots, x_n\}$. We proved [6] that the Bourque–Ligh conjecture is true if $n \le 7$, but not true if $n \ge 8$. We believe that this result is true for general positive integer power LCM matrices. We conclude this paper by raising the following conjecture.

Conjecture. Let m be a given positive integer and n any positive integer. Then there is a positive integer $k(m)$, depending only on m, such that if $n \le k(m)$, then the power LCM matrix $([x_i, x_j]^m)$ defined on any gcd-closed set $S = \{x_1, \ldots, x_n\}$ is nonsingular. But for $n \ge k(m) + 1$, there exists a gcd-closed set $S = \{x_1, \ldots, x_n\}$ so that the power LCM matrix $([x_i, x_j]^m)$ defined on S is singular.

From [6], one knows that the above conjecture holds when $m = 1$. In fact, $k(1) = 7$. In a similar way to [6], one can show that for any integer $m \ge 2$, one has $k(m) \ge 7$.

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Mathematical College
Sichuan University
Chengdu 610064
P.R. China
E-mail: hongsf@263.net

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