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On the average of the sum-of-p-prime-divisors function

by

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1. Introduction and statement of the results. The study of the error term in the average of the sum-of-divisors function is one of the classical problems (see Walfisz [11], Gronwall [6], Wigert [12] and some more recent works by Pétermann [8], [9]) in analytic number theory.

Recently, Adhikari and Coppola [3] proved some Ω-results for the error term in the average of the sum-of-odd-divisors function using a method due to Pétermann [8], [9]. It should be mentioned that the basic method in these papers is averaging over arithmetical progressions which could be traced back to a paper of Hardy and Littlewood. But almost all of the later users (in [2]–[4], [7]–[10] and several other papers) of the method have picked up the idea from the paper [5] of Erdős and Shapiro. The underlying philosophy of the Erdős–Shapiro paper has been described in [1].

In the present paper, we use ideas from another paper of Pétermann [10] for the corresponding problem for the function $\sigma_{(p)}(n)$ for a prime p where $\sigma_{(p)}(n)$ is defined to be the sum of the divisors of n which are relatively coprime to p, that is,

$$\sigma_{(p)}(n) = \sum_{\substack{d\mid n\\ d\not\equiv 0\ (p)}} d.$$

We call $\sigma_{(p)}(n)$ the *sum-of-p-prime-divisors function.*

More precisely, defining the error terms $R_p(x)$ by

$$R_p(x) := \sum_{n\le x} \sigma_{(p)}(n) - \frac{\pi^2 x^2}{12}\left(1-\frac{1}{p}\right), \tag{1}$$

we prove the following.

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Theorem 1. *We have*

$$R_p(x) = \Omega_{\pm}(x \log\log x).$$

The present paper is organized as follows: first, in Section 2, we give the necessary lemmas; then, in Section 3, we prove our theorem.

As usual, $[x]$ indicates the integer part of the real (positive) number x and $\{x\} = x - [x]$ its fractional part.

Acknowledgements. The authors would like to thank the referee for his/her suggestions, including the one which leads to two-sided omega results in our theorem as compared to the one-sided one in our earlier manuscript. Also, the authors agree with the referee that the method should be referred to as "Erdős–Shapiro" method since the above mentioned paper of Erdős and Shapiro contains in germ mostly everything concerning the technique for this type of functions.

2. Lemmas. We define $R'_p(x)$ by

$$R'_p(x) := \sum_{n \le x} \frac{\sigma_{(p)}(n)}{n} - \frac{\pi^2 x}{6}\left(1 - \frac{1}{p}\right). \tag{2}$$

Lemma 1. *For each natural number n we have*

$$\frac{\sigma_{(p)}(n)}{n} = \sum_{d \mid n} \frac{\alpha_p(d)}{d},$$

where

$$\alpha_p(d) = \begin{cases} 1 & \text{if } p \nmid d, \\ -(p-1) & \text{otherwise.} \end{cases}$$

Proof. Let $n = p^r Q$, $(p, Q) = 1$. Let $d_1, \ldots, d_k$ be the distinct divisors of Q. Then $Q/d_1, \ldots, Q/d_k$ is a permutation of the divisors of Q. Now for a particular d_i, we have

$$\frac{1}{d_i}\left(1 - \frac{p-1}{p} - \frac{p-1}{p^2} - \ldots - \frac{p-1}{p^r}\right) = \frac{1}{d_i p^r}.$$

In the above equation, the left hand side is

$$\frac{1}{d_i} - \frac{p-1}{pd_i} - \frac{p-1}{p^2 d_i} - \ldots - \frac{p-1}{p^r d_i}$$

while the right hand side can be written as

$$\frac{Q/d_i}{n}.$$

Now if we sum over i the left hand side gives

$$\sum_{d \mid n} \frac{\alpha_p(d)}{d}$$

and the right hand side gives

$$\frac{1}{n}\sum_i Q/d_i = \frac{1}{n}\sum_i d_i = \frac{1}{n}\sum_{\substack{d|n\\ d\not\equiv 0\ (p)}} d = \frac{1}{n}\sigma_{(p)}(n).$$

LEMMA 2. *We have*

$$\sum_{n\le x}\frac{\alpha_p(n)}{n} = \log p + O\left(\frac{1}{x}\right).$$

Proof. We have

$$\begin{aligned}\sum_{n\le x}\frac{\alpha_p(n)}{n} &= \sum_{\substack{n\le x\\ p\nmid n}}\frac{1}{n} + \sum_{\substack{n\le x\\ p|n}}\frac{-(p-1)}{n} = \sum_{n\le x}\frac{1}{n} - p\sum_{\substack{n\le x\\ p|n}}\frac{1}{n}\\ &= \log x - \log(x/p) + O(1/x) = \log p + O(1/x).\end{aligned}$$

LEMMA 3.

$$R'_p(x) = -x\sum_{d>x}\frac{\alpha_p(d)}{d^2} - \sum_{d\le x}\frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\}.$$

Proof. We have

$$\begin{aligned}\sum_{d=1}^{\infty}\frac{\alpha_p(d)}{d^2} &= \sum_{\substack{d=1\\ p\nmid n}}^{\infty}\frac{1}{d^2} - (p-1)\sum_{\substack{d=1\\ p|n}}^{\infty}\frac{1}{d^2}\\ &= \sum_{d=1}^{\infty}\frac{1}{d^2} - \frac{1}{p}\sum_{d=1}^{\infty}\frac{1}{d^2} = \frac{\pi^2}{6}\left(1-\frac{1}{p}\right).\end{aligned}$$

Therefore, from (2) and Lemma 1, we have

$$R'_p(x) = \sum_{n\le x}\sum_{d|n}\frac{\alpha_p(d)}{d} - x\sum_{d=1}^{\infty}\frac{\alpha_p(d)}{d^2}.$$

The right hand side is

$$\sum_{d\le x}\frac{\alpha_p(d)}{d}\left[\frac{x}{d}\right] - x\sum_{d=1}^{\infty}\frac{\alpha_p(d)}{d^2} = -x\sum_{d>x}\frac{\alpha_p(d)}{d^2} - \sum_{d\le x}\frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\}.$$

LEMMA 4 (Montgomery [7]). *If* b, r (>0) *are integers such that* $(b,r)=1$ *and* β *is a real number*, *then*

$$\sum_{n=1}^{r}\left\{\frac{nb}{r}+\beta\right\} = \frac{r-1}{2} + \{r\beta\}.$$

LEMMA 5 (Montgomery [7]). *With notations as in the last lemma, for any positive integer N, we have*

$$\sum_{n=1}^{N}\left\{\frac{nb}{r}+\beta\right\}=\frac{N}{r}\{r\beta\}+\frac{N}{2}\left(\frac{r-1}{r}\right)+O(r).$$

LEMMA 6.

$$\frac{R_p(x)}{x}-R'_p(x)=O(1).$$

Proof. From (1) and (2) we have

$$\frac{R_p(x)}{x}-R'_p(x)=\frac{1}{x}\sum_{n\le x}\sigma_{(p)}(n)-\sum_{n\le x}\frac{\sigma_{(p)}(n)}{n}+\frac{\pi^2 x}{12}\left(1-\frac{1}{p}\right).$$

Using Lemma 1, the right hand side can be seen to be equal to

$$\frac{x}{2}\sum_{d>x}\frac{\alpha_p(d)}{d^2}+\frac{1}{2}\sum_{d\le x}\frac{\alpha_p(d)}{d}-\frac{1}{2x}\sum_{d\le x}\alpha_p(d)\left(\left\{\frac{x}{d}\right\}-\left\{\frac{x}{d}\right\}^2\right).$$

Since $\sum_{d=1}^{\infty}\alpha_p(d)/d$ is convergent and $|\alpha_p(d)|\le p-1$, the lemma follows.

LEMMA 7.

$$R'_p(x)=-\sum_{d\le y}\frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\}+O(1)$$

uniformly for $x\ge 2$, $y\ge\sqrt{x}$.

Proof. From Lemma 3,

$$R'_p(x)=-x\sum_{d>x}\frac{\alpha_p(d)}{d^2}-\sum_{d\le x}\frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\}.$$

Since

$$\sum_{d>x}\frac{\alpha_p(d)}{d^2}=O\left(\frac{1}{x}\right),$$

we only have to show that

$$\sum_{y<d\le x}\frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\}=O(1)$$

for $\sqrt{x}\le y\le x$. We choose K such that $1\le K\le x/y$. In the range $x/K<d\le x/(K-1)$, $\{x/d\}$ is monotone. Hence by Lemma 2,

$$\sum_{x/K<d\le x/(K-1)}\frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\}=O\left(\frac{K}{x}\right).$$

Summing up over K such that $2 \le K \le x/y$, we obtain

$$\sum_{y\le d\le x} \frac{\alpha_p(d)}{d}\left\{\frac{x}{d}\right\} = O\left(\frac{1}{x}\sum_{1\le K\le x/y} K\right) = O\left(\frac{1}{x}\cdot\frac{x^2}{y^2}\right) = O(1),$$

as $y \ge \sqrt{x}$.

3. Proof of Theorem 1. For $q \sim \sqrt{N}$, $\beta \le q$ and $y = (N+1)q/\sqrt{N} \sim N$, from Lemma 7, we have

$$\begin{aligned}\sum_{n=1}^{N} R_p'(nq+\beta) &= -\sum_{n=1}^{N}\sum_{d\le y}\frac{\alpha_p(d)}{d}\left\{\frac{nq+\beta}{d}\right\} + O(N)\\ &= -\sum_{d\le y}\frac{\alpha_p(d)}{d}\sum_{n=1}^{N}\left\{\frac{nq+\beta}{d}\right\} + O(N)\\ &= -\sum_{d\le y}\frac{\alpha_p(d)}{d}\sum_{n=1}^{N}\left\{\frac{q/(d,q)}{d/(d,q)}n + \frac{\beta}{d}\right\} + O(N)\\ &= -\sum_{d\le y}\frac{\alpha_p(d)}{d}\cdot\frac{N}{d}(d,q)\left(\left\{\frac{\beta}{(d,q)}\right\} - \frac{1}{2}\right) + O(N),\end{aligned}$$

using Lemma 5 in the last step.

We should remark that in Theorem 1 of [10], the identity has been established for a large class of functions. Our lemmas show that our function falls in that class and we get such an identity from that theorem as well.

Now, following the application (d) in [10] (with a slight modification as in [4], which is helpful because $\alpha_p(d)$ is negative or positive according as p divides d or not), we make our choices of q and β. More precisely, we take

$$q = \frac{m!}{p^r}, \quad \text{where} \quad p^r \,\|\, m!.$$

For the choice $\beta = 0$,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} R_p'(nq+\beta) &= \sum_{\substack{d\le q^2\\ p\mid d}} -\frac{(p-1)(d,q)}{2d^2} + \sum_{\substack{d\le q^2\\ p\nmid d}}\frac{(d,q)}{2d^2} + O(1)\\ &= \sum_{\substack{d\le q^2\\ p\mid d}} -\frac{p(d,q)}{2d^2} + \sum_{d\le q^2}\frac{(d,q)}{2d^2} + O(1)\\ &= \left(1-\frac{1}{2p}\right)\sum_{l\le q^2/p}\frac{(l,q)}{2l^2} + \sum_{q^2/p<d\le q^2}\frac{(d,q)}{2d^2} + O(1).\end{aligned}$$

In the last line, the first sum is greater than $C \log m$ for some positive number C and the second sum is $O(1)$. This gives $R'_p(x) = \Omega_+(\log\log x)$. Similarly, for the choice $\beta = q - 1$, we obtain $R'_p(x) = \Omega_-(\log\log x)$.

Therefore, from Lemma 6 we get

$$R_p(x) = \Omega_\pm(x \log\log x).$$

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