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Sums for $U(2n, q^2)$ and their applications

by

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1. Introduction. Let v be a complex-valued function on the finite field $\mathbb{F}_q$, v' the lifting of v to $\mathbb{F}_{q^2}$ (cf. (2.1)), and let $u : \mathbb{F}_{q^2} \to \mathbb{C}$ be any function. Then we consider the sum

$$\sum_{w\in \mathrm{SU}(2n,q^2)} v'(\operatorname{tr} w), \tag{1.1}$$

where $\mathrm{SU}(2n, q^2)$ is a special unitary group over $\mathbb{F}_{q^2}$ (cf. (2.5), (2.6)) and $\operatorname{tr} w$ is the trace of w. Also, we investigate

$$\sum_{w\in U(2n,q^2)} u(\det w)v'(\operatorname{tr} w), \tag{1.2}$$

where $U(2n, q^2)$ is a unitary group over $\mathbb{F}_{q^2}$ (cf. (2.5)) and $\det w$ is the determinant of w.

In our previous papers about similar sums for classical groups over finite fields, u and v' are respectively a multiplicative character and a nontrivial additive character (or a lifting of a nontrivial additive character to the quadratic extension for unitary groups or a function closely related to a nontrivial additive character; cf. [4]–[15]).

However, as demonstrated in [3] and [16], all the computations can be carried out not only for characters but also for arbitrary functions. So we will work in the more general setting of arbitrary functions.

The main purpose of this paper is to find explicit expressions for the sums (1.1) and (1.2). It turns out that they are polynomials in q with coefficients involving certain simple sums.

Another purpose of this paper is to find a formula for the number $C(\alpha, \beta)$ of elements w in $U(2n, q^2)$ with $\det w = \alpha$, $\operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \operatorname{tr} w = \beta$, for any α in the

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kernel of the norm map $N_{\mathbb{F}_{q^2}/\mathbb{F}_q} : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ and any $\beta \in \mathbb{F}_q$. This follows easily from the explicit expression of (1.2) in (5.13) by specializing u and v to be the obvious functions. At the end of this paper, two tables of $C(\alpha, \beta)$'s for all possible values of α, β (an $(q+1) \times q$ matrix) are provided.

Finally, we state the main results of this paper. One is referred to the next section for some notations here.

Theorem A. *The sum* $\sum_{w \in U(2n,q^2)} u(\det w) v'(\operatorname{tr} w)$ *equals*

$$
\begin{aligned}
&(q^2-1)^{-1} q^{2n^2-n-1} \prod_{j=1}^{2n} (q^j - (-1)^j) S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v) \\
&\quad + q^{2n^2-n-2} \sum_{r=0}^{n} q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
&\quad \times \sum_{l=1}^{[(n-r+2)/2]} q^{2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1) \\
&\quad \times (MK_{n-r+2-2l}(v', \widetilde{u}_r; 1, 1 : q^2) - q^{-1}(q^2-1)^{n-r+1-2l} S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v)) \\
&\quad + \begin{cases}
q^{2n^2-1}((q^2-1)^{-1} S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) - u(1)) S_{\mathbb{F}_q}(v) \\
\qquad \times \displaystyle\sum_{\substack{0 \le r \le n \\ r \text{ even}}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \text{for } n \text{ even}, \\
q^{2n^2-1}((q^2-1)^{-1} S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) - u(-1)) S_{\mathbb{F}_q}(v) \\
\qquad \times \displaystyle\sum_{\substack{0 \le r \le n \\ r \text{ odd}}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \text{for } n \text{ odd}.
\end{cases}
\end{aligned}
$$

Here $\widetilde{u}_r, \widetilde{u}$ are as in (5.3) and (5.4) respectively, for $S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}), S_{\mathbb{F}_q}(v)$ one is referred to (2.20), $MK_m(v', \widetilde{u}_r; 1, 1 : q^2)$ is defined in (3.1) and (3.2), and the unspecified sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-r+1$ (it is 1 for $l = 1$ by convention).

THEOREM B. *Let $\alpha \in \mathrm{Ker}(N_{\mathbb{F}_{q^2}/\mathbb{F}_q} : \mathbb{F}_{q^2}^{\times} \to \mathbb{F}_q^{\times})$, $\beta \in \mathbb{F}_q$. Then the number $C(\alpha,\beta)$ of the elements $w \in U(2n,q^2)$ with $\det w = \alpha$, $\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathrm{tr}\, w = \beta$ is given by*

$$
\begin{aligned}
&(q+1)^{-1} q^{2n^2-n-1} \prod_{j=1}^{2n} (q^j - (-1)^j) \\
&\quad + q^{2n^2-n-2} \sum_{r=0}^{n} q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
&\quad \times \sum_{l=1}^{[(n-r+2)/2]} q^{2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1) \\
&\quad \times (\delta(n-r+2-2l, q^2; (-1)^r \alpha, \beta) - q^{-1}(q+1)^{-1}(q^2-1)^{n-r+2-2l}) \\
&\quad + \begin{cases}
q^{2n^2-1}((q+1)^{-1} - \delta_{\alpha,1}) \displaystyle\sum_{\substack{0 \le r \le n \\ r\ \text{even}}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \text{for } n \text{ even}, \\
q^{2n^2-1}((q+1)^{-1} - \delta_{\alpha,-1}) \displaystyle\sum_{\substack{0 \le r \le n \\ r\ \text{odd}}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \text{for } n \text{ odd}.
\end{cases}
\end{aligned}
$$

Here $\delta(m,q^2;(-1)^r\alpha,\beta)$ is as in (6.1) and (6.2), the unspecified sum is over all the integers $j_1,\ldots,j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-r+1$ (it is 1 for $l=1$ by convention), and $\delta_{\alpha,\pm 1}$ are the Kronecker delta (cf. (2.19)).

2. Preliminaries. In this section, we will fix some notations and collect from [8] some facts that will be used in what follows. Also, refer to [1] and [17] for some elementary facts below.

Let $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$ denote respectively the finite field with $q = p^d$ elements (p any prime, d a positive integer), and the quadratic extension of $\mathbb{F}_q$. Let $\tau : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ be the Frobenius automorphism given by

$$\alpha^\tau = \alpha^q.$$

Then, for $\alpha \in \mathbb{F}_{q^2}$,

$$\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \alpha = \alpha + \alpha^\tau, \qquad N_{\mathbb{F}_{q^2}/\mathbb{F}_q} \alpha = \alpha\alpha^\tau.$$

For a function $v : \mathbb{F}_q \to \mathbb{C}$, v' will be used to denote the function v "lifted to $\mathbb{F}_{q^2}$", i.e.,

$$v' = v \circ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}. \tag{2.1}$$

Also, for convenience we will denote the kernel of the norm map $N_{\mathbb{F}_{q^2}/\mathbb{F}_q} : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ by

$$KN_q = \mathrm{Ker}(N_{\mathbb{F}_{q^2}/\mathbb{F}_q} : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times). \tag{2.2}$$

In the following, $\mathrm{tr}\, A$ and $\det A$ denote respectively the trace of A and the determinant of A for a square matrix A, and, with "t" indicating the transpose, ${}^*B = {}^t(\beta_{ij}^\tau)$ for any matrix $B = (\beta_{ij})$ over $\mathbb{F}_{q^2}$. A square matrix B over $\mathbb{F}_{q^2}$ is called *Hermitian* if ${}^*B = B$. It is well known (cf. [2]), for each positive integer r, that the number h_r of all $r \times r$ nonsingular Hermitian matrices is given by

$$h_r = q^{\binom{r}{2}} \prod_{j=1}^{r} (q^j + (-1)^j). \tag{2.3}$$

Let $\mathrm{GL}(n, q)$ denote the group of all invertible $n \times n$ matrices with entries in $\mathbb{F}_q$. The order of $\mathrm{GL}(n, q)$ equals

$$g_n(q) = \prod_{j=0}^{n-1} (q^n - q^j) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1). \tag{2.4}$$

The unitary group $U(2n, q^2)$ is defined by

$$U(2n, q^2) = \{ w \in \mathrm{GL}(2n, q^2) \mid {}^*wJw = J \}, \tag{2.5}$$

where

$$J = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}.$$

Also,

$$\mathrm{SU}(2n, q^2) = \{ w \in U(2n, q^2) \mid \det w = 1 \}. \tag{2.6}$$

The composite of the matrix trace $\mathrm{tr} : U(2n, q^2) \to \mathbb{F}_{q^2}$ and the field trace $\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} : \mathbb{F}_{q^2} \to \mathbb{F}_q$ will be denoted by

$$t_{n,q} = \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \circ \mathrm{tr} : U(2n, q^2) \to \mathbb{F}_q. \tag{2.7}$$

$P(2n, q^2)$ denotes the maximal parabolic subgroup of $U(2n, q^2)$ defined by

$$P(2n, q^2) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n, q^2), B \text{ is } n \times n \\ \text{over } \mathbb{F}_{q^2} \text{ with } {}^*B + B = 0 \end{array} \right\}. \tag{2.8}$$

Also, we put

$$(2.9)\qquad Q = Q(2n, q^2) = \{w \in P(2n, q^2) \mid \det w = 1\}$$

$$= \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n, q^2), \\ \det A \in \mathbb{F}_q^\times, {}^*B + B = 0 \end{array} \right\},$$

$$(2.10)\qquad Q^- = Q^-(2n, q^2) = \{w \in P(2n, q^2) \mid \det w = -1\}$$

$$= \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n, q^2), \\ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det A) = 0, \\ {}^*B + B = 0 \end{array} \right\}.$$

Note here that, for $A \in \mathrm{GL}(n, q^2)$,

$$(2.11)\qquad \det A \in \mathbb{F}_q^\times \;\Leftrightarrow\; (\det A)^{q-1} = 1,$$

$$\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det A) = 0 \;\Leftrightarrow\; (\det A)^{q-1} = -1.$$

In [8], it was noted that, starting from the Bruhat decomposition

$$U(2n, q^2) = \coprod_{r=0}^{n} P\sigma_r P,$$

one can obtain the following decompositions:

$$(2.12)\qquad U(2n, q^2) = \coprod_{r=0}^{n} P\sigma_r(B_r \backslash Q),$$

$$(2.13)\qquad \mathrm{SU}(2n, q^2) = \Big(\coprod_{\substack{0 \le r \le n \\ r\,\mathrm{even}}} Q\sigma_r(B_r \backslash Q) \Big) \amalg \Big(\coprod_{\substack{0 \le r \le n \\ r\,\mathrm{odd}}} Q^-\sigma_r(B_r \backslash Q) \Big),$$

where

$$(2.14)\qquad B_r = B_r(q^2) = \{w \in Q(2n, q^2) \mid \sigma_r w \sigma_r^{-1} \in P(2n, q^2)\},$$

$$(2.15)\qquad \sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix}.$$

For integers n, r with $0 \le r \le n$, the *q-binomial coefficients* are defined by

$$(2.16)\qquad \begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1} \frac{q^{n-j} - 1}{q^{r-j} - 1}.$$

Then, as was noted in [8],

$$(2.17)\qquad |B_r(q^2) \backslash Q(2n, q^2)| = q^{r^2} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2}.$$

Also, from the q-binomial theorem (cf. [8, (2.13)]), one can see that

$$\sum_{r=0}^{n} |B_r \backslash Q| = \sum_{r=0}^{n} q^{r^2} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} = \prod_{j=1}^{n} (q^{2j-1}+1). \tag{2.18}$$

$[y]$ denotes the greatest integer $\le y$, for a real number y. For $\alpha, \beta \in \mathbb{F}_{q^2}$, we will use the Kronecker delta, so that

$$\delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{2.19}$$

For a complex-valued function h defined on a finite set X and a subset Y of X, $S_Y(h)$ will be used to denote

$$S_Y(h) = \sum_{\alpha \in Y} h(\alpha). \tag{2.20}$$

3. Some propositions. For any $f, h : \mathbb{F}_q \to \mathbb{C}$, $\alpha, \beta \in \mathbb{F}_q$, and $m \in \mathbb{Z}_{\ge 0}$, define, for $m > 0$,

$$\begin{aligned} &MK_m(f, h; \alpha, \beta : q) \\ &\quad = \sum_{\alpha_1, \ldots, \alpha_m \in \mathbb{F}_q^\times} h(\alpha_1 \ldots \alpha_m) f(\alpha\alpha_1 + \beta\alpha_1^{-1} + \ldots + \alpha\alpha_m + \beta\alpha_m^{-1}), \end{aligned} \tag{3.1}$$

and

$$MK_0(f, h; \alpha, \beta : q) = h(1) f(0). \tag{3.2}$$

We first recall the following proposition from [8, Theorem 4.2], which will be needed in proving Proposition 3.2.

Proposition 3.1. *For integers $n \ge 1$, $\alpha, \beta, \gamma \in \mathbb{F}_q^\times$, and λ a nontrivial additive character of $\mathbb{F}_q$, the sum*

$$\begin{aligned} &S_n(\gamma; \alpha, \beta) \\ &\quad = \sum_{w \in \mathrm{SL}(n,q)} \lambda\left(\alpha \,\mathrm{tr} \begin{bmatrix} 1_{n-1} & 0 \\ 0 & \gamma \end{bmatrix} w + \beta \,\mathrm{tr} \left(\begin{bmatrix} 1_{n-1} & 0 \\ 0 & \gamma \end{bmatrix} w \right)^{-1} \right) \end{aligned} \tag{3.3}$$

is given by

$$\begin{aligned} &S_n(\gamma; \alpha, \beta) \\ &\quad = q^{(n-2)(n+1)/2} \\ &\qquad \times \sum_{l=1}^{[(n+2)/2]} q^l BK_{n+1-2l}(\lambda; \alpha; \alpha(-\alpha\beta^{-1})^{l-1}\gamma; \beta; \beta(-\alpha^{-1}\beta)^{l-1}\gamma^{-1} : q) \\ &\qquad \times \sum \prod_{\nu=1}^{l-1} (q^{j_\nu - 2\nu} - 1), \end{aligned} \tag{3.4}$$

where $BK_n(\lambda;a;b;c;d:q)$ is the bihyperkloosterman sum defined, for $n>0$, by

$$\text{(3.5)}\quad BK_n(\lambda;a;b;c;d:q) = \sum_{\alpha_1,\ldots,\alpha_n\in\mathbb{F}_q^\times} \lambda\Big(a\sum_{j=1}^n \alpha_j + b\prod_{j=1}^n \alpha_j^{-1} + c\sum_{j=1}^n \alpha_j^{-1} + d\prod_{j=1}^n \alpha_j\Big)$$

and by

$$\text{(3.6)}\quad BK_0(\lambda;a;b;c;d:q) = \lambda(b+d),$$

and the inner sum runs over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-1\le j_{l-1}\le j_{l-2}\le\ldots\le j_1\le n+1$ (it is 1 for $l=1$ by convention).

Here when $n=2k$ is even and $l=[(n+2)/2]=k+1$, we understand that

$$\text{(3.7)}\quad \begin{aligned} &BK_{n+1-2l}(\lambda;\alpha;\alpha(-\alpha\beta^{-1})^{l-1}\gamma;\beta;\beta(-\alpha^{-1}\beta)^{l-1}\gamma^{-1}:q)\\ &\quad= BK_{-1}(\lambda;\alpha;\alpha(-\alpha\beta^{-1})^{l-1}\gamma;\beta;\beta(-\alpha^{-1}\beta)^{l-1}\gamma^{-1}:q)\\ &\quad= \begin{cases} 1 & \text{for } \gamma=(-\alpha^{-1}\beta)^k,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

For any functions $f,h:\mathbb{F}_q\to\mathbb{C}$, $\alpha,\beta\in\mathbb{F}_q^\times$, we want to find an explicit expression for the sum

$$\text{(3.8)}\quad \sum_{w\in \mathrm{GL}(n,q)} f(\det w)h(\alpha\,\mathrm{tr}\,w+\beta\,\mathrm{tr}\,w^{-1}).$$

When f is identically 1, this has been considered in [16]. As we need only the case of $\alpha=\beta=1$, we will just consider that case.

With λ as in Proposition 3.1,

$$\begin{aligned}
&\sum_{w\in\mathrm{GL}(n,q)} f(\det w)\,h(\mathrm{tr}\,w+\mathrm{tr}\,w^{-1})\\
&= \sum_{\gamma\in\mathbb{F}_q^\times} f(\gamma)\sum_{w\in\mathrm{SL}(n,q)} h\left(\mathrm{tr}\begin{bmatrix}1_{n-1} & 0\\ 0 & \gamma\end{bmatrix}w+\mathrm{tr}\left(\begin{bmatrix}1_{n-1} & 0\\ 0 & \gamma\end{bmatrix}w\right)^{-1}\right)\\
&= q^{-1}\sum_{\gamma\in\mathbb{F}_q^\times}\sum_{\varepsilon\in\mathbb{F}_q}\sum_{\zeta\in\mathbb{F}_q}\sum_{w\in\mathrm{SL}(n,q)} f(\gamma)\\
&\quad\times\lambda\left(\zeta\left(\mathrm{tr}\begin{bmatrix}1_{n-1} & 0\\ 0 & \gamma\end{bmatrix}w+\mathrm{tr}\left(\begin{bmatrix}1_{n-1} & 0\\ 0 & \gamma\end{bmatrix}w\right)^{-1}-\varepsilon\right)\right)h(\varepsilon)\\
&= q^{-1}\sum_{\gamma\in\mathbb{F}_q^\times}\sum_{\varepsilon\in\mathbb{F}_q}\sum_{\zeta\in\mathbb{F}_q^\times} f(\gamma)h(\varepsilon)S_n(\gamma;\zeta,\zeta)\lambda(-\zeta\varepsilon)\\
&\quad + q^{-1}(q-1)^{-1}g_n(q)S_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h)
\end{aligned}$$

(cf. (3.3)). Using the expression of $S_n(\gamma;\zeta,\zeta)$ in (3.4), this can be written as

$$q^{(n-2)(n+1)/2}\sum_{l=1}^{[(n+2)/2]} q^l \sum \prod_{\nu=1}^{l-1}(q^{j_\nu-2\nu}-1)$$
$$\times q^{-1}\sum_{\gamma\in\mathbb{F}_q^\times}\sum_{\varepsilon\in\mathbb{F}_q}\sum_{\zeta\in\mathbb{F}_q^\times} BK_{n+1-2l}(\lambda;\zeta;(-1)^{l-1}\zeta\gamma;\zeta;(-1)^{l-1}\zeta\gamma^{-1}:q)$$
$$\times \lambda(-\zeta\varepsilon)f(\gamma)h(\varepsilon)+q^{-1}(q-1)^{-1}g_n(q)S_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h).$$

If $t=n+1-2l>0$, then, reversing the above steps, we get

$$\text{(3.9)}\quad q^{-1}\sum_{\gamma\in\mathbb{F}_q^\times}\sum_{\varepsilon\in\mathbb{F}_q}\sum_{\zeta\in\mathbb{F}_q^\times} BK_{n+1-2l}(\lambda;\zeta;(-1)^{l-1}\zeta\gamma;\zeta;(-1)^{l-1}\zeta\gamma^{-1}:q)$$
$$\times \lambda(-\zeta\varepsilon)f(\gamma)h(\varepsilon)$$
$$= q^{-1}\sum_{\gamma\in\mathbb{F}_q^\times}\sum_{\varepsilon\in\mathbb{F}_q}\sum_{\zeta\in\mathbb{F}_q}\sum_{\alpha_1,\dots,\alpha_t\in\mathbb{F}_q^\times}\lambda\Big(\zeta\Big(\sum_{j=1}^t\alpha_j+(-1)^{l-1}\gamma\prod_{j=1}^t\alpha_j^{-1}$$
$$+\sum_{j=1}^t\alpha_j^{-1}+(-1)^{l-1}\gamma^{-1}\prod_{j=1}^t\alpha_j-\varepsilon\Big)\Big)f(\gamma)h(\varepsilon)$$
$$-q^{-1}(q-1)^tS_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h)$$
$$=\sum_{\gamma\in\mathbb{F}_q^\times}\sum_{\alpha_1,\dots,\alpha_t\in\mathbb{F}_q^\times} f(\gamma)h\Big(\sum_{j=1}^t\alpha_j+(-1)^{l-1}\gamma\prod_{j=1}^t\alpha_j^{-1}$$
$$+\sum_{j=1}^t\alpha_j^{-1}+(-1)^{l-1}\gamma^{-1}\prod_{j=1}^t\alpha_j\Big)-q^{-1}(q-1)^tS_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h)$$
$$=\sum_{\alpha_1,\dots,\alpha_{t+1}\in\mathbb{F}_q^\times} f((-1)^{l-1}\alpha_1\dots\alpha_{t+1})h(\alpha_1+\alpha_1^{-1}+\dots+\alpha_{t+1}+\alpha_{t+1}^{-1})$$
$$\Big(\text{by putting } \alpha_{t+1}=(-1)^{l-1}\gamma\prod_{j=1}^t\alpha_j^{-1}\Big)$$
$$\text{(3.10)}\quad = MK_{t+1}(h,f_{l-1};1,1:q)-q^{-1}(q-1)^tS_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h)$$

where we put

$$\text{(3.11)}\quad f_{l-1}(\alpha)=f((-1)^{l-1}\alpha).$$

One can check that, even for $t = n+1-2l = 0$, we get the same expression as in (3.10) for (3.9). However, if n is even and $l = [(n+2)/2] = n/2+1$, so that $t = n+1-2l = -1$, then, from (3.7), we see that (3.9) is

$$f((-1)^{n/2})(h(0) - q^{-1}S_{\mathbb{F}_q}(h)). \tag{3.12}$$

On the other hand, for $t = n+1-2l = -1$, the expression in (3.10) is

$$f((-1)^{n/2})h(0) - q^{-1}(q-1)^{-1}S_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h) \tag{3.13}$$

(cf. (3.2)).

From these considerations and taking into account the difference between (3.12) and (3.13), we finally get the following proposition.

PROPOSITION 3.2. *Let $f, h : \mathbb{F}_q \to \mathbb{C}$ be any functions. Then the sum in* (3.8) (*for $\alpha = \beta = 1$*)

$$\sum_{w\in \mathrm{GL}(n,q)} f(\det w)h(\operatorname{tr} w + \operatorname{tr} w^{-1})$$

is given by

$$\begin{aligned} q^{(n-2)(n+1)/2} &\sum_{l=1}^{[(n+2)/2]} q^l \sum \prod_{\nu=1}^{l-1}(q^{j_\nu - 2\nu} - 1) \\ &\times (MK_{n+2-2l}(h, f_{l-1}; 1, 1 : q) - q^{-1}(q-1)^{n+1-2l}S_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h)) \\ &+ q^{-1}(q-1)^{-1}g_n(q)S_{\mathbb{F}_q^\times}(f)S_{\mathbb{F}_q}(h) \\ &+ \begin{cases} q^{n^2/2-1}\prod_{\nu=1}^{n/2}(q^{n+1-2\nu}-1)S_{\mathbb{F}_q}(h) \\ \qquad \times((q-1)^{-1}S_{\mathbb{F}_q^\times}(f) - f((-1)^{n/2})) & \text{for } n \text{ even}, \\ 0 & \text{for } n \text{ odd}, \end{cases} \end{aligned} \tag{3.14}$$

where the unspecified sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n+1$ (*it is* 1 *for $l=1$ by convention*), *$MK_m(h, f_{l-1}; 1, 1 : q)$ is as in* (3.1) *and* (3.2), *one is referred to* (2.20) *for $S_{\mathbb{F}_q^\times}(f)$ and $S_{\mathbb{F}_q}(h)$, and $g_n(q)$ is as in* (2.4).

The following "incomplete" sums in (3.15) can be obtained by using similar ideas to the derivation of Proposition 3.2. The details will be left to the reader. In the next proposition, it is understood that either $+1$ or -1 is always assumed whenever they appear at the same time.

PROPOSITION 3.3. *Let $v : \mathbb{F}_q \to \mathbb{C}$ be any function with $v' = v \circ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$. Then*

$$\sum_{\substack{w\in \mathrm{GL}(n,q^2)\\ (\det w)^{q-1}=\pm 1}} v'(\mathrm{tr}\, w + \mathrm{tr}\, w^{-1}) \tag{3.15}$$

$$= q^{(n-2)(n+1)} \sum_{l=1}^{[(n+2)/2]} q^{2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1)$$

$$\times \Big(\sum v'\Big(\sum_{j=1}^{n+2-2l} \alpha_j + \sum_{j=1}^{n+2-2l} \alpha_j^{-1} \Big)$$

$$- q^{-1}(q+1)^{-1}(q^2-1)^{n+2-2l} S_{\mathbb{F}_q}(v) \Big)$$

$$+ q^{-1}(q+1)^{-1} g_n(q^2) S_{\mathbb{F}_q}(v)$$

$$+ \begin{cases} q^{n^2-1} \prod_{\nu=1}^{n/2} (q^{2n+2-4\nu} - 1)((q+1)^{-1} - \delta_{1,\pm 1}) S_{\mathbb{F}_q}(v) & \text{for } n \text{ even}, \\ 0 \quad \text{for } n \text{ odd}. \end{cases}$$

Here the unspecified sum involving the product notation is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n+1$ (it is 1 for $l = 1$ by convention), the innermost sum is over all $\alpha_1, \ldots, \alpha_{n+2-2l} \in \mathbb{F}_{q^2}^\times$ satisfying $(\alpha_1 \ldots \alpha_{n+2-2l})^{q-1} = \pm 1$, and one is referred to (2.20), (2.4) *respectively for $S_{\mathbb{F}_q}(v)$, $g_n(q^2)$.*

4. $\mathrm{SU}(2n, q^2)$ **case.** In this section, we will consider the sum in (1.1)

$$\sum_{w \in \mathrm{SU}(2n,q^2)} v'(\mathrm{tr}\, w)$$

for any function $v : \mathbb{F}_q \to \mathbb{C}$ with $v' = v \circ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$, and find an explicit expression for this by using the decomposition in (2.13).

The sum in (1.1) can be written, using (2.13), as

$$\sum_{\substack{0\le r\le n\\ r \text{ even}}} |B_r \backslash Q| \sum_{w\in Q} v'(\mathrm{tr}\, w\sigma_r) \tag{4.1}$$

$$+ \sum_{\substack{0\le r\le n\\ r \text{ odd}}} |B_r \backslash Q| \sum_{w\in Q^-} v'(\mathrm{tr}\, w\sigma_r), \tag{4.2}$$

where $B_r = B_r(q^2)$, $Q = Q(2n, q^2)$, $Q^- = Q^-(2n, q^2)$, σ_r are respectively as in (2.14), (2.9), (2.10), (2.15). Here one has to observe that, for each $y \in Q$,

$$\sum_{w \in Q} v'(\operatorname{tr} w\sigma_r y) = \sum_{w \in Q} v'(\operatorname{tr} yw\sigma_r) = \sum_{w \in Q} v'(\operatorname{tr} w\sigma_r),$$

and $yQ^- = Q^-$.

Write $w \in Q$ (cf. (2.9)) as

$$w = \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix}, \tag{4.3}$$

with

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad {}^*A^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ -{}^*B_{12} & B_{22} \end{bmatrix}, \tag{4.4}$$

$$B_{11} + {}^*B_{11} = 0, \quad B_{22} + {}^*B_{22} = 0. \tag{4.5}$$

Here A_{11}, A_{12}, A_{21}, A_{22} are respectively of sizes $r \times r$, $r \times (n-r)$, $(n-r) \times r$, $(n-r) \times (n-r)$, and similarly for ${}^*A^{-1}$ and B. Then, for any r $(0 \le r \le n)$, the inner sum in (4.1) is

$$\sum_{w \in Q} v'(\operatorname{tr} w\sigma_r) \tag{4.6}$$

$$= \sum v'(\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^*B_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22}), \tag{4.7}$$

where the sum is over $A \in \mathrm{GL}(n, q^2)$ with $\det A \in \mathbb{F}_q^\times$, B_{11}, B_{12}, B_{22} subject to the conditions in (4.5).

Consider the sum in (4.7) first for the case $1 \le r \le n-1$, so that A_{12} does appear. We divide the sum into four subsums

$$\sum_{A_{12} \ne 0} \cdots + \sum_{\substack{A_{12}=0 \\ A_{11} \in \mathrm{I}}} \cdots + \sum_{\substack{A_{12}=0 \\ A_{11} \in \mathrm{II}}} \cdots + \sum_{\substack{A_{12}=0 \\ A_{11} \text{ Hermitian}}} \cdots, \tag{4.8}$$

where

$$\begin{aligned} \mathrm{I} &= \{A_{11} = (a_{ij}) \in \mathrm{GL}(r, q^2) \mid a_{ji} \ne a_{ij}^\tau \\ &\qquad\qquad \text{for some } i, j \text{ with } 1 \le i < j \le r\}, \\ \mathrm{II} &= \{A_{11} = (a_{ij}) \in \mathrm{GL}(r, q^2) \mid a_{ji} = a_{ij}^\tau \text{ for all } i, j \text{ with} \\ &\qquad\qquad 1 \le i < j \le r, \text{ and } a_{ii} \notin \mathbb{F}_q \text{ for some } i\ (1 \le i \le r)\}. \end{aligned} \tag{4.9}$$

Note here that I and II above are disjoint and that

$$\mathrm{I} \cup \mathrm{II} = \mathrm{GL}(r, q^2) - \{A \in \mathrm{GL}(r, q^2) \mid {}^*A = A\}. \tag{4.10}$$

The first sum in (4.8) is

$$q^{(n-r)^2} \sum_{A, B_{11}} \sum_{B_{12}} v'(\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^*B_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22}), \tag{4.11}$$

where A is with $A_{12} \neq 0$, $\det A \in \mathbb{F}_q^\times$, and B_{11} is with $B_{11} + {}^*B_{11} = 0$. Fix such A, B_{11}. Write $A_{12} = (a_{ij})$, $B_{12} = (b_{ij})$. Then $a_{kl} \neq 0$ for some k, l ($1 \leq k \leq r$, $1 \leq l \leq n-r$), so that the inner sum in (4.11) is

$$\sum_{\substack{\text{all } b_{ij} \text{ with} \\ (i,j)\neq(k,l)}} \sum_{b_{kl}} v'(-a_{kl}b_{kl}^\tau + \ldots) = q^{2r(n-r)-1} S_{\mathbb{F}_q}(v). \tag{4.12}$$

Here one is referred to (2.20) for $S_{\mathbb{F}_q}(v)$, and one must observe that the inner sum in (4.12) is

$$\sum_{\gamma \in \mathbb{F}_{q^2}} v'(\gamma) = q S_{\mathbb{F}_q}(v). \tag{4.13}$$

Put $B_{11} = (b_{ij})$. Then the first condition in (4.5) is equivalent to

$$\begin{aligned} \operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} b_{ii} &= 0 \quad \text{for } 1 \leq i \leq r, \\ b_{ij} + b_{ji}^\tau &= 0 \quad \text{for } 1 \leq i < j \leq r. \end{aligned} \tag{4.14}$$

In particular,

$$|\{B_{11} \mid B_{11} + {}^*B_{11} = 0\}| = q^{r^2}. \tag{4.15}$$

Combining (4.11), (4.12), (4.15), and noting

$$\begin{aligned} &|\{A \in \mathrm{GL}(n, q^2) \mid A_{12} \neq 0, \det A \in \mathbb{F}_q^\times\}| \\ &\qquad = (q+1)^{-1}(g_n(q^2) - q^{2r(n-r)} g_r(q^2) g_{n-r}(q^2)) \end{aligned}$$

(cf. (2.4)), we see that the first sum in (4.8) equals

$$(q+1)^{-1} q^{n^2-1} (g_n(q^2) - q^{2r(n-r)} g_r(q^2) g_{n-r}(q^2)) S_{\mathbb{F}_q}(v). \tag{4.16}$$

The subsum of (4.7) with $A_{12} = 0$ is

$$\begin{aligned} &\sum_{A_{21}, B_{12}, B_{22}} \sum_{A_{11}, A_{22}, B_{11}} v'(\operatorname{tr} A_{11} B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) \\ &\qquad = q^{(n-r)^2 + 4r(n-r)} \sum_{A_{11}, A_{22}, B_{11}} v'(\operatorname{tr} A_{11} B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}). \end{aligned} \tag{4.17}$$

Let $A_{11} = (a_{ij})$, $B_{11} = (b_{ij})$. Then, from (4.14), one observes that

$$\operatorname{tr} A_{11} B_{11} = \sum_{i=1}^{r} a_{ii} b_{ii} + \sum_{1 \leq i < j \leq r} (a_{ji} - a_{ij}^\tau) b_{ij}. \tag{4.18}$$

The subsum of the sum in (4.17) with $A_{11} \in \mathrm{I}$ (cf. (4.9)) is

$$\sum_{A_{11} \in \mathrm{I},\, A_{22}} \sum_{B_{11}} v'(\operatorname{tr} A_{11} B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}). \tag{4.19}$$

Fix $A_{11} = (a_{ij}) \in \mathrm{I}$, A_{22}. Then $a_{ts} \neq a_{st}^\tau$, for some s, t with $1 \leq s < t \leq r$. By the same argument as in the case of (4.11) and in view of (4.14) and

(4.18), we see that the inner sum in (4.19) is

$$q^{r^2-1} S_{\mathbb{F}_q}(v). \tag{4.20}$$

Combining (4.17), (4.19), and (4.20) shows that the second sum in (4.8) is

$$q^{n^2+2rn-2r^2-1} S_{\mathbb{F}_q}(v) \sum 1, \tag{4.21}$$

where the sum is over A_{11}, A_{22} with $(\det A_{11})(\det A_{22}) \in \mathbb{F}_q^\times$, $A_{11} \in \mathrm{I}$.

The subsum of the sum in (4.17) with $A_{11} \in \mathrm{II}$ (cf. (4.9)) is

$$\sum_{A_{11} \in \mathrm{II},\, A_{22}} \sum_{B_{11}} v'(\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}). \tag{4.22}$$

Fix $A_{11} = (a_{ij}) \in \mathrm{II}$, A_{22}. Then $a_{ss} \notin \mathbb{F}_q$, for some s $(1 \le s \le r)$. In view of (4.14) and (4.18) and with $B_{11} = (b_{ij})$, the inner sum in (4.22) is

$$\begin{aligned} (4.23)\quad q^{r^2-1} \sum v'(a_{ss}b_{ss} + \ldots) &= q^{r^2-1} \sum_{\alpha \in \mathbb{F}_q} v(\alpha \operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(a_{ss}\eta) + \operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\ldots)) \\ &= q^{r^2-1} S_{\mathbb{F}_q}(v). \end{aligned}$$

Here the unspecified sum in (4.23) is over b_{ss} with $\operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} b_{ss} = 0$, η is a fixed nonzero element in $\mathbb{F}_{q^2}$ with $\operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \eta = 0$, and $a_{ss} \notin \mathbb{F}_q$ implies that $\operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(a_{ss}\eta) \neq 0$. Combining (4.17), (4.22), and (4.23), we see that the third sum in (4.8) equals

$$q^{n^2+2rn-2r^2-1} S_{\mathbb{F}_q}(v) \sum 1, \tag{4.24}$$

where the sum is over A_{11}, A_{22} with $(\det A_{11})(\det A_{22}) \in \mathbb{F}_q^\times$, $A_{11} \in \mathrm{II}$.

Adding up (4.21) and (4.24), we see that the sum of the second and third sums in (4.8) is

$$q^{n^2+2rn-2r^2-1} S_{\mathbb{F}_q}(v) \sum 1, \tag{4.25}$$

where the sum is over A_{11}, A_{22} with $(\det A_{11})(\det A_{22}) \in \mathbb{F}_q^\times$, and A_{11} is not Hermitian (cf. (4.10)). Now, (4.25) is easily seen to be equal to

$$(q+1)^{-1} q^{n^2+2rn-2r^2-1} g_{n-r}(q^2)(g_r(q^2) - h_r) S_{\mathbb{F}_q}(v), \tag{4.26}$$

where h_r is as in (2.3).

One observes that, for A_{11} Hermitian and B_{11} with $B_{11} + {}^*B_{11} = 0$,

$$v'(\operatorname{tr} A_{11}B_{11} + \operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) = v'(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1})$$

(cf. (4.14), (4.18)). So the subsum of the sum in (4.17) with A_{11} Hermitian is

$$q^{r^2} \sum_{A_{11}, A_{22}} v'(\operatorname{tr} A_{22} + \operatorname{tr} A_{22}^{-1}) = q^{r^2} h_r \sum_{\substack{w \in \mathrm{GL}(n-r, q^2) \\ \det w \in \mathbb{F}_q^\times}} v'(\operatorname{tr} w + \operatorname{tr} w^{-1}). \tag{4.27}$$

So, combining (4.17) and (4.27), we see that the last sum in (4.8) equals

$$q^{n^2+2rn-2r^2} h_r \sum_{\substack{w\in \mathrm{GL}(n-r,q^2)\\ \det w\in \mathbb{F}_q^\times}} v'(\operatorname{tr} w + \operatorname{tr} w^{-1}). \tag{4.28}$$

Adding up (4.16), (4.26), and (4.28), we have shown that, for $1 \le r \le n-1$,

$$\begin{aligned}\sum_{w\in Q} v'(\operatorname{tr} w\sigma_r) &\\ = (q+1)^{-1} q^{n^2-1}(g_n(q^2) - q^{2rn-2r^2} g_{n-r}(q^2)h_r) S_{\mathbb{F}_q}(v)&\\ + q^{n^2+2rn-2r^2} h_r \sum_{\substack{w\in \mathrm{GL}(n-r,q^2)\\ \det w\in \mathbb{F}_q^\times}} v'(\operatorname{tr} w + \operatorname{tr} w^{-1}).&\end{aligned} \tag{4.29}$$

For $r = 0$, one can check that the sum in (4.6) is given by the same expression as in (4.29) with the convention $h_0 = 1$. In view of (2.3), this convention is natural. On the other hand, for $r = n$, one shows, using a similar argument to the above $1 \le r \le n-1$ case, that the sum in (4.6) is given by

$$\sum_{w\in Q} v'(\operatorname{tr} w\sigma_n) = q^{n^2-1}((q+1)^{-1} g_n(q^2) - h_n) S_{\mathbb{F}_q}(v) + q^{n^2} h_n v(0). \tag{4.30}$$

The details are left to the reader.

From (4.29) and (4.30), the sum in (4.1) can be expressed as

$$\begin{aligned}&\sum_{\substack{0\le r\le n\\ r\text{ even}}} |B_r\backslash Q| \sum_{w\in Q} v'(\operatorname{tr} w\sigma_r)\\ &= \sum_{\substack{0\le r\le n-1\\ r\text{ even}}} |B_r\backslash Q| \Big\{ (q+1)^{-1} q^{n^2-1}(g_n(q^2) - q^{2rn-2r^2} g_{n-r}(q^2)h_r) S_{\mathbb{F}_q}(v)\\ &\quad + q^{n^2+2rn-2r^2} h_r \sum_{\substack{w\in \mathrm{GL}(n-r,q^2)\\ \det w\in \mathbb{F}_q^\times}} v'(\operatorname{tr} w + \operatorname{tr} w^{-1}) \Big\}\\ &\quad + \begin{cases} |B_n\backslash Q| \{ q^{n^2-1}((q+1)^{-1} g_n(q^2) - h_n) S_{\mathbb{F}_q}(v) + q^{n^2} h_n v(0)\} & \text{for } n \text{ even},\\ 0 \quad \text{for } n \text{ odd}. \end{cases}\end{aligned} \tag{4.31}$$

Next, for any r $(0 \le r \le n)$ we consider the inner sum of the sum in (4.2). Write $w \in Q^-$ (cf. (2.10)) as

$$w = \begin{bmatrix} A & 0 \\ 0 & {}^*A^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix}.$$

Here A satisfies the condition $\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det A) = 0$, i.e., $(\det A)^{q-1} = -1$, whereas $A \in Q$ satisfies $\det A \in \mathbb{F}_q^\times$, i.e., $(\det A)^{q-1} = 1$ (cf. (2.11)).

With this change in mind and glancing through the above argument, one can see that, for $0 \le r \le n-1$,

$$\sum_{w \in Q^-} v'(\mathrm{tr}\, w\sigma_r) \tag{4.32}$$

$$= (q+1)^{-1} q^{n^2-1}(g_n(q^2) - q^{2rn-2r^2} g_{n-r}(q^2) h_r) S_{\mathbb{F}_q}(v) + q^{n^2+2rn-2r^2} h_r \sum_{\substack{w \in \mathrm{GL}(n-r,q^2) \\ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det w)=0}} v'(\mathrm{tr}\, w + \mathrm{tr}\, w^{-1}). \tag{4.33}$$

On the other hand, for $r = n$, the sum in (4.32) is given by

$$\sum_{w \in Q^-} v'(\mathrm{tr}\, w\sigma_n) = q^{n^2-1}((q+1)^{-1} g_n(q^2) - h_n \delta_{1,-1}) S_{\mathbb{F}_q}(v) + q^{n^2} h_n v(0) \delta_{1,-1}. \tag{4.34}$$

Here $\delta_{1,-1}$ is the Kronecker delta so that

$$\delta_{1,-1} = \begin{cases} 1 & \text{if char}\, \mathbb{F}_q = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.35}$$

The Kronecker delta appears here, since, for a nonsingular Hermitian matrix A over $\mathbb{F}_{q^2}$, we have

$$\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det A) = (\det A)\, \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q} 1 = 0 \iff \text{char}\, \mathbb{F}_q = 2.$$

From (4.33) and (4.34), the sum in (4.2) can now be expressed as

$$\begin{aligned} &\sum_{\substack{0 \le r \le n \\ r \text{ odd}}} |B_r \backslash Q| \sum_{w \in Q^-} v'(\mathrm{tr}\, w\sigma_r) \\ &= \sum_{\substack{0 \le r \le n-1 \\ r \text{ odd}}} |B_r \backslash Q| \\ &\quad \times \Big\{ (q+1)^{-1} q^{n^2-1}(g_n(q^2) - q^{2rn-2r^2} g_{n-r}(q^2) h_r) S_{\mathbb{F}_q}(v) \\ &\quad + q^{n^2+2rn-2r^2} h_r \sum_{\substack{w \in \mathrm{GL}(n-r,q^2) \\ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\det w)=0}} v'(\mathrm{tr}\, w + \mathrm{tr}\, w^{-1}) \Big\} \\ &\quad + \begin{cases} 0 & \text{for } n \text{ even}, \\ |B_n \backslash Q| \{ q^{n^2-1}((q+1)^{-1} g_n(q^2) - h_n \delta_{1,-1}) S_{\mathbb{F}_q}(v) \\ \qquad + q^{n^2} h_n v(0) \delta_{1,-1} \} & \text{for } n \text{ odd}. \end{cases} \end{aligned} \tag{4.36}$$

By (4.31) and (4.36), the sum in (1.1) can be written as

$$\sum_{w\in \mathrm{SU}(2n,q^2)} v'(\mathrm{tr}\, w) \tag{4.37}$$

$$= \sum_{0\le r\le n} |B_r\backslash Q|\Big\{(q+1)^{-1}q^{n^2-1}(g_n(q^2)-q^{2rn-2r^2}g_{n-r}(q^2)h_r)S_{\mathbb{F}_q}(v)$$

$$+ q^{n^2+2rn-2r^2}h_r \sum_{\substack{w\in \mathrm{GL}(n-r,q^2)\\ (\det w)^{q-1}=(-1)^r}} v'(\mathrm{tr}\, w + \mathrm{tr}\, w^{-1})\Big\}$$

$$+ \begin{cases} |B_n\backslash Q|q^{n^2-1}h_n((q+1)^{-1}-1)S_{\mathbb{F}_q}(v) & \text{for } n \text{ even},\\ |B_n\backslash Q|q^{n^2-1}h_n((q+1)^{-1}-\delta_{1,-1})S_{\mathbb{F}_q}(v) & \text{for } n \text{ odd}\end{cases}$$

(cf. (2.11)). Here in (4.37) we adopt the convention that

$$\sum_{\substack{w\in \mathrm{GL}(0,q^2)\\ (\det w)^{q-1}=1}} v'(\mathrm{tr}\, w + \mathrm{tr}\, w^{-1}) = v(0), \tag{4.38}$$

$$\sum_{\substack{w\in \mathrm{GL}(0,q^2)\\ (\det w)^{q-1}=-1}} v'(\mathrm{tr}\, w + \mathrm{tr}\, w^{-1}) = v(0)\delta_{1,-1}. \tag{4.39}$$

Finally, from (2.3), (2.4), (2.17), (3.15), and (4.37), we get the following main theorem of this section.

THEOREM 4.1. *For any function* $v:\mathbb{F}_q\to\mathbb{C}$ *with* $v' = v\circ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$, *the sum over* $\mathrm{SU}(2n,q^2)$

$$\sum_{w\in \mathrm{SU}(2n,q^2)} v'(\mathrm{tr}\, w)$$

is given by

$$(q+1)^{-1}q^{2n^2-n-1}\prod_{j=1}^{2n}(q^j-(-1)^j)S_{\mathbb{F}_q}(v) \tag{4.40}$$

$$+ q^{2n^2-n-2}\sum_{r=0}^{n} q^{\binom{r+1}{2}}\begin{bmatrix} n\\ r\end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j)$$

$$\times \sum_{l=1}^{[(n-r+2)/2]} q^{2l}\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1)$$

$$\times \Big(\sum v'\Big(\sum_{j=1}^{n-r+2-2l}\alpha_j + \sum_{j=1}^{n-r+2-2l}\alpha_j^{-1}\Big)$$

$$
- q^{-1}(q+1)^{-1}(q^2-1)^{n-r+2-2l}S_{\mathbb{F}_q}(v)\Big)
$$

$$
+\begin{cases}
q^{2n^2-1}((q+1)^{-1}-1)S_{\mathbb{F}_q}(v) \\
\qquad \times \displaystyle\sum_{\substack{0\le r\le n \\ r\ even}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r}(q^j+(-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2}(q^{2n-2r+2-4\nu}-1) \quad \textit{for } n \textit{ even}, \\
q^{2n^2-1}((q+1)^{-1}-\delta_{1,-1})S_{\mathbb{F}_q}(v) \\
\qquad \times \displaystyle\sum_{\substack{0\le r\le n \\ r\ odd}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r}(q^j+(-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2}(q^{2n-2r+2-4\nu}-1) \quad \textit{for } n \textit{ odd}.
\end{cases}
$$

Here $S_{\mathbb{F}_q}(v)$ and $\delta_{1,-1}$ are as in (2.20) and (4.35) respectively (cf. (2.19)), the first unspecified sum is over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-1\le j_{l-1}\le\ldots\le j_1\le n-r+1$, and the second one is over $\alpha_1,\ldots,\alpha_{n-r+2-2l}\in\mathbb{F}_{q^2}^{\times}$ with $(\alpha_1\ldots\alpha_{n-r+2-2l})^{q-1}=(-1)^r$. Also, when $m=0$, our conventions here for

$$
\sum_{\substack{\alpha_1,\ldots,\alpha_m \\ (\alpha_1\ldots\alpha_m)^{q-1}=1}} v'\Big(\sum_{j=1}^{m}\alpha_j+\sum_{j=1}^{m}\alpha_j^{-1}\Big),
$$

$$
\sum_{\substack{\alpha_1,\ldots,\alpha_m \\ (\alpha_1\ldots\alpha_m)^{q-1}=-1}} v'\Big(\sum_{j=1}^{m}\alpha_j+\sum_{j=1}^{m}\alpha_j^{-1}\Big)
$$

are $v(0)$ and $v(0)\delta_{1,-1}$ respectively (cf. (4.38), (4.39)).

5. $U(2n,q^2)$ **case.** Here we will consider the sum in (1.2)

$$
\sum_{w\in U(2n,q^2)} u(\det w)v'(\operatorname{tr} w)
$$

for any functions $u:\mathbb{F}_{q^2}\to\mathbb{C}$, $v:\mathbb{F}_q\to\mathbb{C}$ with $v'=v\circ\operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$, and find an explicit expression for this by using the decomposition in (2.12).

The sum in (1.2) can be written, using (2.12), as

$$\sum_{r=0}^{n} |B_r \backslash Q| \sum_{w \in P} u((-1)^r \det w) v'(\operatorname{tr} w\sigma_r). \tag{5.1}$$

Write $w \in P$ as in (4.3) with A, ${}^*A^{-1}$, B as in (4.4) and (4.5). Note here that, in contrast to the Q and Q^- of the $\mathrm{SU}(2n, q^2)$ case, we do not have any restriction on A. Then the inner sum in (5.1) is

$$\sum_{A,B} u\left((-1)^r \frac{\det A}{(\det A)^\tau}\right) v'(\operatorname{tr} A_{11}B_{11} - \operatorname{tr} A_{12}{}^*B_{12} + \operatorname{tr} A_{22} + \operatorname{tr} E_{22}). \tag{5.2}$$

Here the sum is over $A \in \mathrm{GL}(n, q^2)$, B_{11}, B_{12}, B_{22} subject to the conditions in (4.5).

Before we move on, we will introduce the following notation that will be needed later. For $u : \mathbb{F}_{q^2} \to \mathbb{C}$, $r \in \mathbb{Z}$, $\widetilde{u}_r : \mathbb{F}_{q^2} \to \mathbb{C}$ is the function defined by

$$\widetilde{u}_r(\alpha) = u((-1)^r \alpha^{q-1}). \tag{5.3}$$

In particular, we put

$$\widetilde{u}(\alpha) = \widetilde{u}_0(\alpha) = u(\alpha^{q-1}). \tag{5.4}$$

It is easy to see that the sum $S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}_r)$ (cf. (2.20)) is independent of r, so that

$$S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}_r) = S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) \quad \text{for any } r \in \mathbb{Z}. \tag{5.5}$$

Consider now the sum in (5.2) first for the case $1 \le r \le n-1$, so that A_{12} does appear. Divide the sum in (5.2) just as in (4.8) (cf. (4.9)):

$$\sum_{A_{12} \ne 0} \cdots + \sum_{\substack{A_{12}=0 \\ A_{11} \in \mathrm{I}}} \cdots + \sum_{\substack{A_{12}=0 \\ A_{11} \in \mathrm{II}}} \cdots + \sum_{\substack{A_{12}=0 \\ A_{11} \text{ Hermitian}}} \cdots \tag{5.6}$$

The first sum in (5.6) is $(q-1)^{-1} S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})$ times the corresponding sum for the $\mathrm{SU}(2n, q^2)$ case in (4.16), i.e., it is

$$(q^2-1)^{-1} q^{n^2-1}(g_n(q^2) - q^{2r(n-r)} g_r(q^2) g_{n-r}(q^2)) S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v) \tag{5.7}$$

(cf. (5.3)–(5.5)). Similarly, the sum of the second and third subsums in (5.6) is

$$(q^2-1)^{-1} q^{n^2+2rn-2r^2-1} g_{n-r}(q^2)(g_r(q^2) - h_r) S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v), \tag{5.8}$$

which is again $(q-1)^{-1} S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})$ times the corresponding sum in (4.26). On the other hand, the last sum in (5.6) is easily seen to be equal to

$$q^{n^2+2rn-2r^2} h_r \sum_{w \in \mathrm{GL}(n-r, q^2)} \widetilde{u}_r(\det w) v'(\operatorname{tr} w + \operatorname{tr} w^{-1}). \tag{5.9}$$

The arguments are completely analogous to the corresponding ones for the $\mathrm{SU}(2n, q^2)$ case.

Adding up (5.7)–(5.9), we have shown that, for $1 \leq r \leq n-1$,

$$\sum_{w \in P} u((-1)^r \det w) v'(\operatorname{tr} w \sigma_r) \tag{5.10}$$

$$= (q^2-1)^{-1} q^{n^2-1} (g_n(q^2) - q^{2rn-2r^2} g_{n-r}(q^2) h_r) S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v) + q^{n^2+2rn-2r^2} h_r \sum_{w \in \mathrm{GL}(n-r,q^2)} \widetilde{u}_r(\det w) v'(\operatorname{tr} w + \operatorname{tr} w^{-1}). \tag{5.11}$$

If $r = 0$, then (5.10) is given by the same expression as in (5.11) with the convention $h_0 = 1$. On the other hand, if $r = n$, then, using a similar argument to the above $1 \leq r \leq n-1$ case, we see that the sum in (5.10) is given by

$$q^{n^2-1}((q^2-1)^{-1} g_n(q^2) S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) - h_n u((-1)^n)) S_{\mathbb{F}_q}(v) + q^{n^2} h_n u((-1)^n) v(0). \tag{5.12}$$

From (5.1), (5.11), and (5.12), we see that the sum in (1.2) equals

$$\begin{aligned}
&\sum_{w \in U(2n,q^2)} u(\det w) v'(\operatorname{tr} w) \\
&= (q^2-1)^{-1} q^{n^2-1} g_n(q^2) S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v) \sum_{r=0}^{n} |B_r \backslash Q| \\
&\quad - (q^2-1)^{-1} q^{n^2-1} S_{\mathbb{F}_{q^2}^\times}(\widetilde{u}) S_{\mathbb{F}_q}(v) \sum_{r=0}^{n-1} |B_r \backslash Q| q^{2rn-2r^2} g_{n-r}(q^2) h_r \\
&\quad + q^{n^2} \sum_{r=0}^{n-1} |B_r \backslash Q| q^{2rn-2r^2} h_r \sum_{w \in \mathrm{GL}(n-r,q^2)} \widetilde{u}_r(\det w) v'(\operatorname{tr} w + \operatorname{tr} w^{-1}) \\
&\quad + |B_n \backslash Q| q^{n^2-1} h_n u((-1)^n)(q v(0) - S_{\mathbb{F}_q}(v)).
\end{aligned}$$

Now, from (2.3), (2.4), (2.17), (2.18), and (3.14), we get the following main theorem of this section. Here one has to observe that $(\widetilde{u}_r)_{l-1} = \widetilde{u}_{r+(l-1)(q-1)} = \widetilde{u}_r$ (cf. (3.11), (5.3)).

Theorem 5.1. *For any function $u : \mathbb{F}_{q^2} \to \mathbb{C}$, and any function $v : \mathbb{F}_q \to \mathbb{C}$ with $v' = v \circ \operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}$, the sum over $U(2n, q^2)$*

$$\sum_{w \in U(2n,q^2)} u(\det w) v'(\operatorname{tr} w)$$

is given by

$$(5.13)\quad (q^2-1)^{-1}q^{2n^2-n-1}\prod_{j=1}^{2n}(q^j-(-1)^j)S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})S_{\mathbb{F}_q}(v)$$

$$+\,q^{2n^2-n-2}\sum_{r=0}^{n}q^{\binom{r+1}{2}}\begin{bmatrix}n\\r\end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j)$$

$$\times\sum_{l=1}^{[(n-r+2)/2]}q^{2l}\sum\prod_{\nu=1}^{l-1}(q^{2j_\nu-4\nu}-1)$$

$$\times\,(MK_{n-r+2-2l}(v',\widetilde{u}_r;1,1:q^2)-q^{-1}(q^2-1)^{n-r+1-2l}S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})S_{\mathbb{F}_q}(v))$$

$$+\begin{cases} q^{2n^2-1}((q^2-1)^{-1}S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})-u(1))S_{\mathbb{F}_q}(v) \\ \qquad\times\displaystyle\sum_{\substack{0\le r\le n\\ r\ even}}q^{\binom{r}{2}}\begin{bmatrix}n\\r\end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j) \\ \qquad\times\displaystyle\prod_{\nu=1}^{(n-r)/2}(q^{2n-2r+2-4\nu}-1)\quad \textit{for } n \textit{ even}, \\ q^{2n^2-1}((q^2-1)^{-1}S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})-u(-1))S_{\mathbb{F}_q}(v) \\ \qquad\times\displaystyle\sum_{\substack{0\le r\le n\\ r\ odd}}q^{\binom{r}{2}}\begin{bmatrix}n\\r\end{bmatrix}_{q^2}\prod_{j=1}^{r}(q^j+(-1)^j) \\ \qquad\times\displaystyle\prod_{\nu=1}^{(n-r)/2}(q^{2n-2r+2-4\nu}-1)\quad \textit{for } n \textit{ odd}. \end{cases}$$

Here one is referred to (5.3), (5.4) for $\widetilde{u}_r$, $\widetilde{u}$ and to (2.20) for $S_{\mathbb{F}_{q^2}^\times}(\widetilde{u})$, $S_{\mathbb{F}_q}(v)$, the unspecified sum is over all integers $j_1,\ldots,j_{l-1}$ satisfying $2l-1\le j_{l-1}\le\ldots\le j_1\le n-r+1$ (it is 1 for $l=1$ by convention), and $MK_m(v',\widetilde{u}_r;1,1:q^2)$ is as in (3.1).

6. Applications to certain enumerations. As applications of the results in Sections 4 and 5, we will derive some counting formulas related to $U(2n,q^2)$ and $\mathrm{SU}(2n,q^2)$.

For each $\alpha\in KN_q$ (cf. (2.2)), $\beta\in\mathbb{F}_q$, and $m\in\mathbb{Z}_{\ge 0}$, we define, for $m\ge 1$,

$$(6.1)\quad \delta(m,q^2;\alpha,\beta)=|\{(\alpha_1,\ldots,\alpha_m)\in(\mathbb{F}_{q^2}^\times)^m\mid(\alpha_1\ldots\alpha_m)^{q-1}=\alpha,\ \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\alpha_1+\alpha_1^{-1}+\ldots+\alpha_m+\alpha_m^{-1})=\beta\}|$$

and

$$\delta(0, q^2; \alpha, \beta) = \delta_{\alpha,1}\delta_{\beta,0} = \begin{cases} 1 & \text{if } \alpha = 1 \text{ and } \beta = 0, \\ 0 & \text{otherwise} \end{cases} \tag{6.2}$$

(cf. (2.19)). Then, with the choices of $u : \mathbb{F}_{q^2} \to \mathbb{C}$, $v : \mathbb{F}_q \to \mathbb{C}$ as

$$u(y) = \delta_{y,\alpha}, \qquad v(y) = \delta_{y,\beta} \tag{6.3}$$

and any $m \in \mathbb{Z}_{\geq 0}$,

$$MK_m(v', \widetilde{u}_r; 1, 1 : q^2) = \delta(m, q^2; (-1)^r\alpha, \beta). \tag{6.4}$$

Also, with u, v as in (6.3), we have

$$S_{\mathbb{F}_{q^2}^{\times}}(\widetilde{u}) = q - 1, \qquad S_{\mathbb{F}_q}(v) = 1. \tag{6.5}$$

The following theorem is now an easy consequence of Theorem 5.1 with the choices of u and v as in (6.3), in view of the observations made in (6.4) and (6.5).

THEOREM 6.1. *For each $\alpha \in KN_q$* (cf. (2.2)) *and $\beta \in \mathbb{F}_q$, we put*

$$C(\alpha, \beta) = |\{w \in U(2n, q^2) \mid \det w = \alpha,\ t_{n,q}(w) = \beta\}| \tag{6.6}$$

(cf. (2.7)). *Then $C(\alpha, \beta)$ is given by*

$$\begin{aligned}
(6.7)\quad & (q+1)^{-1} q^{2n^2-n-1} \prod_{j=1}^{2n} (q^j - (-1)^j) \\
& + q^{2n^2-n-2} \sum_{r=0}^{n} q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
& \times \sum_{l=1}^{[(n-r+2)/2]} q^{2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1) \\
& \times (\delta(n-r+2-2l, q^2; (-1)^r\alpha, \beta) - q^{-1}(q+1)^{-1}(q^2-1)^{n-r+2-2l}) \\
& + \begin{cases} q^{2n^2-1}((q+1)^{-1} - \delta_{\alpha,1}) \displaystyle\sum_{\substack{0 \leq r \leq n \\ r\ even}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\ \qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \textit{for } n \textit{ even}, \\ q^{2n^2-1}((q+1)^{-1} - \delta_{\alpha,-1}) \displaystyle\sum_{\substack{0 \leq r \leq n \\ r\ odd}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\ \qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \textit{for } n \textit{ odd}. \end{cases}
\end{aligned}$$

Here the unspecified sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l - 1 \le j_{l-1} \le \ldots \le j_1 \le n - r + 1$ (it is 1 for $l = 1$ by convention), and $\delta(m, q^2; (-1)^r\alpha, \beta)$ is as in (6.1) and (6.2).

For $\beta \in \mathbb{F}_q$ and $m \in \mathbb{Z}_{\ge 0}$, we define, for $m \ge 1$,

$$\delta(m, q^2; \beta) = |\{(\alpha_1, \ldots, \alpha_m) \in (\mathbb{F}_{q^2}^{\times})^m \mid \operatorname{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(\alpha_1 + \alpha_1^{-1} + \ldots + \alpha_m + \alpha_m^{-1}) = \beta\}| \tag{6.8}$$

and

$$\delta(0, q^2; \beta) = \delta_{\beta,0}. \tag{6.9}$$

Then it is easy to see that, for any $m \in \mathbb{Z}_{\ge 0}$,

$$\sum_{\alpha \in KN_q} \delta(m, q^2; \alpha, \beta) = \delta(m, q^2; \beta) \tag{6.10}$$

(cf. (6.1), (6.2)).

From the expression of $C(\alpha, \beta)$ in (6.7) and using (6.10), we now obtain that of $C(\beta)$ in (6.11) below. Note here that, for each $\beta \in \mathbb{F}_q$,

$$C(\beta) = \sum_{\alpha \in KN_q} C(\alpha, \beta).$$

COROLLARY 6.2. *For each* $\beta \in \mathbb{F}_q$, *we put*

$$C(\beta) = |\{w \in U(2n, q^2) \mid t_{n,q}(w) = \beta\}| \tag{6.11}$$

(cf. (2.7)). *Then*

$$\begin{aligned} C(\beta) &= q^{2n^2-n-1} \prod_{j=1}^{2n} (q^j - (-1)^j) \\ &\quad + q^{2n^2-n-2} \sum_{r=0}^{n} q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\ &\quad \times \sum_{l=1}^{[(n-r+2)/2]} q^{2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1) \\ &\quad \times (\delta(n - r + 2 - 2l, q^2; \beta) - q^{-1}(q^2 - 1)^{n-r+2-2l}), \end{aligned}$$

where the unspecified sum runs over all integers $j_1, \ldots, j_{l-1}$ *satisfying* $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-r+1$ (*it is* 1 *for* $l = 1$ *by convention*), *and* $\delta(m, q^2; \beta)$ *is as in* (6.8) *and* (6.9).

The following formula for $D(\beta)$ (cf. (6.12)) can be obtained from (6.7) by simply observing that $D(\beta) = C(1, \beta)$. Alternatively, it follows from (4.40) by specializing v to be the obvious function.

COROLLARY 6.3. *For each $\beta \in \mathbb{F}_q$, put*

$$D(\beta) = |\{w \in \mathrm{SU}(2n,q^2) \mid t_{n,q}(w) = \beta\}|. \tag{6.12}$$

Then

$$\begin{aligned}
D(\beta) = {} & (q+1)^{-1} q^{2n^2-n-1} \prod_{j=1}^{2n} (q^j - (-1)^j) \\
& + q^{2n^2-n-2} \sum_{r=0}^{n} q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
& \times \sum_{l=1}^{[(n-r+2)/2]} q^{2l} \sum \prod_{\nu=1}^{l-1} (q^{2j_\nu - 4\nu} - 1) \\
& \times (\delta(n-r+2-2l, q^2; (-1)^r, \beta) - q^{-1}(q+1)^{-1}(q^2-1)^{n-r+2-2l}) \\
& + \begin{cases}
q^{2n^2-1}((q+1)^{-1} - 1) \displaystyle\sum_{\substack{0 \le r \le n \\ r \text{ even}}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \text{for } n \text{ even}, \\
q^{2n^2-1}((q+1)^{-1} - \delta_{1,-1}) \displaystyle\sum_{\substack{0 \le r \le n \\ r \text{ odd}}} q^{\binom{r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q^2} \prod_{j=1}^{r} (q^j + (-1)^j) \\
\qquad \times \displaystyle\prod_{\nu=1}^{(n-r)/2} (q^{2n-2r+2-4\nu} - 1) \quad \text{for } n \text{ odd}.
\end{cases}
\end{aligned}$$

Here the unspecified sum is over all integers $j_1, \ldots, j_{l-1}$ satisfying $2l-1 \le j_{l-1} \le \ldots \le j_1 \le n-r+1$ (it is 1 for $l=1$ by convention), and $\delta(m, q^2; (-1)^r, \beta)$ is as in (6.1) and (6.2).

REMARK. Here we illustrate our formula in (6.7) for the cases of $n=4$, $q=3$ and $n=3$, $q=4$. To produce the following tables the formula was encoded into a Mathematica program by Mr. Lee whom I wish to thank. Below, $\mathbb{F}_9 = \mathbb{F}_3(\theta)$ with $\theta^2+1=0$, and $\mathbb{F}_{16} = \mathbb{F}_2(\omega)$ with $\omega^4+\omega^3+1=0$.

The symmetries in the tables are not surprising. For $w \in U(2n,q^2)$, $\det w^{-1} = (\det w)^{-1}$, and $t_{n,q} w^{-1} = t_{n,q} w$ (cf. (2.5), (2.7)). So

$$C(\alpha,\beta) = C(\alpha^{-1},\beta). \tag{6.13}$$

Also, for $\gamma \in KN_q \cap \mathbb{F}_q^\times$ (cf. (2.2)), $w \in U(2n,q^2)$, $\gamma w \in U(2n,q^2)$, $\det \gamma w = \gamma^{2n} \det w$, and $t_{n,q}(\gamma w) = \gamma t_{n,q}(w)$. This implies that

$$C(\alpha,\beta) = C(\gamma^{2n}\alpha, \gamma\beta) \quad \text{for } \gamma \in KN_q \cap \mathbb{F}_q^\times. \tag{6.14}$$

Now, (6.13) and (6.14) explain the symmetries in the tables.

Tables for $C(\alpha, \beta)$

$U(8, 9)$

$C(\alpha, \beta)$	$\beta = 0$	$\beta = 1$	$\beta = 2$
$\alpha = 1$	348404946102203499680204203542	348404866248608262213916355829	348404866248608262213916355829
$\alpha = 2$	348404888313371102211168336288	348404895143024460948434289456	348404895143024460948434289456
$\alpha = \theta$	348404902715982873530028187728	348404887941718575289004363736	348404887941718575289004363736
$\alpha = 2\theta$	348404902715982873530028187728	348404887941718575289004363736	348404887941718575289004363736

$U(6, 16)$

$C(\alpha, \beta)$	$\beta = 0$	$\beta = 1$	$\beta = \omega^5$	$\beta = \omega^{10}$
$\alpha = 1$	280145279251629735936	280127317656521932800	280127345861572165632	280127345861572165632
$\alpha = \omega^{12}$	280132068688141484032	280131089634240233472	280131089629945266176	280133040678969016320
$\alpha = \omega^3$	280132068688141484032	280131089634240233472	280131089629945266176	280133040678969016320
$\alpha = \omega^9$	280132068688141484032	280131089634240233472	280133040678969016320	280131089629945266176
$\alpha = \omega^6$	280132068688141484032	280131089634240233472	280133040678969016320	280131089629945266176

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