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On an extension of a theorem of Schur

by

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1. Introduction. A well-known theorem of I. Schur ([3], [4]) states that "If the positive integers less than $n!e$ are partitioned into n classes in any way, then $x+y=z$ can be solved in integers (not necessarily distinct) within one class".

For example, in the case $n=2$, the above theorem may be used for $\{1,2,3,4,5\}$. Let us consider the partition $A_1=\{1,2\}$ and $A_2=\{3,4,5\}$. It is clear that $x+y=z$ has a solution within A_1. But such a partition does not give a solution if we restrict to triplets (x,y,z) of pairwise distinct integers. Note that if $x+y=z$ with $x,y,z\neq 0$, only x and y may be equal.

Sierpiński [4] has proved that a solution in distinct integers is certain if we replace the upper bound $[n!e]$ by $2^{[n!e]}$, and Irving ([1], [2]) improved this result with the bound $\left[\frac{1}{2}(2n+1)e\cdot n!\right]+2$ (where, as usual, $[x]$ denotes the greatest integer less than or equal to x).

The purpose of this paper is to prove the following theorem:

THEOREM. *Let $n\geq 2$ be an integer. If the set $A=\{1,2,\ldots,[n!\cdot ne]+1\}$ is divided into n classes in any way, then at least one of the classes contains two different numbers and their sum.*

2. Preliminaries. Let $n\geq 2$ be an integer. We define the finite sequence (α_k) for $k=1,\ldots,n+1$ by

$$\alpha_k = k + n\cdot(k-1)!\sum_{i=0}^{k-2}\frac{1}{i!},$$

with the convention $\sum_{i=p}^{q}=0$ when $p>q$. In particular $\alpha_1=1$. Moreover, it is clear that $\alpha_k\in\mathbb{N}^*$ for each k.

LEMMA 1. *We have $\alpha_{n+1}=[n!\cdot ne]+1$.*

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Proof. We have

$$\alpha_{n+1} = n + 1 + n \cdot n! \sum_{i=0}^{n-1} \frac{1}{i!} = 1 + n \cdot n! \sum_{i=0}^{n} \frac{1}{i!},$$

that is,

$$\alpha_{n+1} = 1 + n \cdot n! e - n \cdot n! R_n \quad \text{with} \quad R_n = \sum_{i=n+1}^{\infty} \frac{1}{i!}.$$

But

$$R_n = \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\right) < \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)^2} + \ldots\right)$$
$$= \frac{1}{n \cdot n!}.$$

Thus $\alpha_{n+1} < 1 + n \cdot n! e < \alpha_{n+1} + 1$. Since α_{n+1} is an integer, we deduce that $\alpha_{n+1} = [n! \cdot ne] + 1$.

LEMMA 2. *For each $k \geq 1$, we have $\alpha_{k+1} = k(\alpha_k + n - k + 1) + 1$.*

Proof. Let $k \geq 1$. We have

$$\frac{\alpha_{k+1} - 1}{k} = 1 + n \cdot (k-1)! \sum_{i=0}^{k-1} \frac{1}{i!} = 1 + n + n \cdot (k-1)! \sum_{i=0}^{k-2} \frac{1}{i!}.$$

Thus $(\alpha_{k+1} - 1)/k = 1 + n + \alpha_k - k$. The conclusion follows.

3. Proof of the Theorem. Take partition of $A = \{1, \ldots, \alpha_{n+1}\}$ into n pairwise disjoint classes. For $E \subset A$ and $a \in A$, define $\Delta_a(E) = \{x - a \mid x \in E \text{ and } x > a\}$. Then we consider the following algorithm:

(1) Define S to be the set of all the classes of the decomposition.

(2) Choose one of the classes in S with the maximum number of elements. Denote it by A_1. Set $S = S - \{A_1\}$.

(3) Define $E_1 = A_1, a_1 = \min E_1, F_1 = \Delta_{a_1}(E_1) - \{a_1\}$.

(4) If $F_1 \cap A_1 \neq \emptyset$ then stop. Otherwise, set $p = 1$ and continue.

(5) Choose an element of S which contains the maximum number of elements from F_p. Denote it by A_{p+1}. Set $S = S - \{A_{p+1}\}$.

(6) Define

$$E_{p+1} = A_{p+1} \cap F_p, \quad a_{p+1} = \min E_{p+1},$$

$$F_{p+1} = \Delta_{a_{p+1}}(E_{p+1}) - \left\{ \sum_{i=0}^{j} a_{p+1-i} \mid j = 0, 1, \ldots, p \right\}.$$

(7) If $F_{p+1} \cap (\bigcup_{i=1}^{p} A_i) \neq \emptyset$ then stop. Otherwise, continue.

(8) Set $p = p + 1$. Go to (5).

Claim 1. *For each $p \in \{1, \ldots, n\}$, if E_p is constructed by the algorithm then*:

- *E_p contains at least $\alpha_{n-p+1} + p + 1$ elements.*
- *F_p is constructed, and it contains at least α_{n-p+1} elements.*

Proof. By induction on p. For $p = 1$, from the pigeon-hole principle, A_1 contains at least

$$\frac{\alpha_{n+1}}{n} = \frac{1}{n} + n! \sum_{i=0}^{n} \frac{1}{i!}$$

elements. Thus, $E_1 = A_1$ contains at least

$$1 + n! \sum_{i=0}^{n} \frac{1}{i!} = \alpha_n + 2$$

elements (from Lemma 2). Moreover, $\alpha_n + 2 \geq 2$, thus a_1 is well defined. And F_1 is constructed: we form the differences in (3) (which decreases the number of elements by 1), and delete a_1 (if necessary). Thus F_1 contains at least α_n elements.

Let p be a fixed integer with $1 \leq p < n$. Suppose that the conclusion holds for the value p and that E_{p+1} is constructed by the algorithm. Then, by the induction hypothesis, F_p contains at least α_{n-p+1} elements, none of which belongs to $\bigcup_{i=1}^{p} A_i$ (otherwise the algorithm would have stopped from (7), and E_{p+1} would not have been constructed). From the pigeon-hole principle and Lemma 2, A_{p+1} contains at least

$$\frac{\alpha_{n-p+1}}{n-p} = \alpha_{n-p} + p + 1 + \frac{1}{n-p}$$

elements from F_p. It follows that E_{p+1} contains at least $\alpha_{n-p}+p+2$ elements. Moreover, $\alpha_{n-p}+p+2 \geq 2$, thus a_{p+1} is well defined, and F_{p+1} is constructed: we form the differences and make the deletions (if necessary, and not more than $p + 1$). Then F_{p+1} contains at least α_{n-p} elements, which ends the induction, and completes the proof of the claim.

Remark. The claim ensures that there will be no problem at step (6) of the algorithm.

Case 1: *The algorithm stops at* (4). Then there exists $b \in F_1 \cap A_1$. Thus $b = a_i - a_1$ for some $a_i \in A_1$, and $b \neq a_1$ since $a_1 \notin F_1$. Thus $b + a_1 = a_i$ in A_1, and the conclusion of the Theorem holds.

Case 2: *The algorithm stops at* (7) *with $p < n$.* First note that it did not stop before.

Claim 2. *For each $k \in \{1, \ldots, p\}$, the number $X_k = a_p + a_{p-1} + \ldots + a_k$ belongs to E_k.*

Proof. By (descending) induction. For $k = p$, we directly have $X_p = a_p \in E_p$. Let $k \in \{2, \ldots, p\}$ be fixed. Suppose that $X_k \in E_k$. Since $E_k \subset F_{k-1} \subset \Delta_{a_{k-1}}(E_{k-1})$, there exists $x \in E_{k-1}$ such that $X_k = x - a_{k-1}$. Thus $X_{k-1} = X_k + a_{k-1} = x$ belongs to E_{k-1}. This ends the induction and completes the proof.

Since the algorithm has stopped at (7), there exists $k \in \{1, \ldots, p\}$ such that $F_p \cap A_k \neq \emptyset$. Then $b = b_{i_1} - a_p \in F_p \cap A_k$ for some $b_{i_1} \in E_p$.

If $k = p$, we have $b \in F_p$. Then, by construction, $b \neq a_p$. Thus $b_{i_1} = b + a_p$ in A_p, and the conclusion of the Theorem holds.

If $k < p$, then $b_{i_1} \in E_p \subset F_{p-1}$. It follows that

$$\begin{aligned}
b_{i_1} &= b_{i_2} - a_{p-1} && \text{for some } b_{i_2} \in E_{p-1} \subset F_{p-2},\\
b_{i_2} &= b_{i_3} - a_{p-2} && \text{for some } b_{i_3} \in E_{p-2} \subset F_{p-3},\\
&\ldots\\
b_{i_{p-k}} &= b_{i_{p-k+1}} - a_k && \text{for some } b_{i_{p-k+1}} \in E_k \subset A_k.
\end{aligned}$$

Summing, we obtain

$$b = b_{i_{p-k+1}} - \sum_{i=k}^{p} a_i = b_{i_{p-k+1}} - X_k.$$

But, from Claim 2, we have $X_k \in E_k \subset A_k$. And, since $b \in F_p$, we deduce that $b \neq X_k$ (see step (6)). It follows that $b + X_k = b_{i_{p-k+1}}$ in A_k, and the conclusion of the Theorem holds.

CASE 3: *The algorithm does not stop until* $p = n$ *in* (8). Then A_n, E_n, F_n are constructed by the algorithm. From Claim 1, F_n contains at least $\alpha_1 = 1$ elements. Thus $F_n \neq \emptyset$ and

$$F_n \cap \Big(\bigcup_{i=1}^{n} A_i\Big) = F_n \cap A \neq \emptyset.$$

We deduce that the algorithm stops at (7). The reasoning used in Case 2 may be repeated word for word, and the conclusion of the Theorem holds. Thus the proof is complete.

4. Remarks. Following Sierpiński [4], given a natural number k, denote by $n(k)$ the least natural number n with the following property: if the numbers $1, \ldots, n$ are divided into k classes, then at least one class contains two different numbers together with their sum.

Then according to Walker [6]

$$n(1) = 3, \quad n(2) = 9, \quad n(3) = 24, \quad n(4) = 67, \quad n(5) = 197.$$

Also, see [5], p. 440,

$$n(k) \geq 1 + 315^{(k-1)/5}.$$

Thus

$$315^{(k-1)/5} + 1 \leq n(k) \leq [k! \cdot ke] + 1.$$

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