On an extension of a theorem of Schur

by

PIERRE BORNSZTEIN (Antony)

1. Introduction. A well-known theorem of I. Schur ([3], [4]) states that “If the positive integers less than \( n! e \) are partitioned into \( n \) classes in any way, then \( x + y = z \) can be solved in integers (not necessarily distinct) within one class”.

For example, in the case \( n = 2 \), the above theorem may be used for \( \{1, 2, 3, 4, 5\} \). Let us consider the partition \( A_1 = \{1, 2\} \) and \( A_2 = \{3, 4, 5\} \). It is clear that \( x + y = z \) has a solution within \( A_1 \). But such a partition does not give a solution if we restrict to triplets \((x, y, z)\) of pairwise distinct integers. Note that if \( x + y = z \) with \( x, y, z \neq 0 \), only \( x \) and \( y \) may be equal.

Sierpiński [4] has proved that a solution in distinct integers is certain if we replace the upper bound \([n!e]\) by \(2^{[n!e]}\), and Irving ([1], [2]) improved this result with the bound \( \left[ \frac{1}{2}(2n + 1)e \cdot n! \right] + 2 \) (where, as usual, \([x]\) denotes the greatest integer less than or equal to \( x \)).

The purpose of this paper is to prove the following theorem:

**Theorem.** Let \( n \geq 2 \) be an integer. If the set \( A = \{1, 2, \ldots, [n! \cdot ne] + 1\} \) is divided into \( n \) classes in any way, then at least one of the classes contains two different numbers and their sum.

2. Preliminaries. Let \( n \geq 2 \) be an integer. We define the finite sequence \((\alpha_k)\) for \( k = 1, \ldots, n + 1 \) by

\[
\alpha_k = k + n \cdot (k - 1)! \sum_{i=0}^{k-2} \frac{1}{i!},
\]

with the convention \( \sum_{i=p}^{q} = 0 \) when \( p > q \). In particular \( \alpha_1 = 1 \). Moreover, it is clear that \( \alpha_k \in \mathbb{N}^* \) for each \( k \).

**Lemma 1.** We have \( \alpha_{n+1} = [n! \cdot ne] + 1 \).

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Proof. We have
\[ \alpha_{n+1} = n + 1 + n \cdot n! \sum_{i=0}^{n-1} \frac{1}{i!} = 1 + n \cdot n! \sum_{i=0}^{n} \frac{1}{i!}, \]
that is,
\[ \alpha_{n+1} = 1 + n \cdot n! e - n \cdot n! R_n \]
with \( R_n = \sum_{i=n+1}^{\infty} \frac{1}{i!} \).
But
\[ R_n = \frac{1}{n!} \left( \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \cdots \right) < \frac{1}{n!} \left( \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right) \]
so\[ R_n = \frac{1}{n \cdot n!}. \]
Thus \( \alpha_{n+1} < 1 + n \cdot n! e < \alpha_{n+1} + 1 \). Since \( \alpha_{n+1} \) is an integer, we deduce that \( \alpha_{n+1} = \lfloor n! \cdot ne \rfloor + 1 \).

Lemma 2. For each \( k \geq 1 \), we have \( \alpha_{k+1} = k(\alpha_k + n - k + 1) + 1 \).
Proof. Let \( k \geq 1 \). We have
\[ \frac{\alpha_{k+1} - 1}{k} = 1 + n \cdot (k - 1)! \sum_{i=0}^{k-1} \frac{1}{i!} = 1 + n + n \cdot (k - 1)! \sum_{i=0}^{k-2} \frac{1}{i!}. \]
Thus \( \frac{\alpha_{k+1} - 1}{k} = 1 + n + \alpha_k - k \). The conclusion follows.

3. Proof of the Theorem. Take partition of \( A = \{1, \ldots, \alpha_{n+1}\} \) into \( n \) pairwise disjoint classes. For \( E \subseteq A \) and \( a \in A \), define \( \Delta_a(E) = \{ x - a \mid x \in E \text{ and } x > a \} \). Then we consider the following algorithm:

(1) Define \( S \) to be the set of all the classes of the decomposition.
(2) Choose one of the classes in \( S \) with the maximum number of elements.
Denote it by \( A_1 \). Set \( S = S - \{A_1\} \).
(3) Define \( E_1 = A_1, a_1 = \min E_1, F_1 = \Delta_{a_1}(E_1) - \{a_1\} \).
(4) If \( F_1 \cap A_1 \neq \emptyset \) then stop. Otherwise, set \( p = 1 \) and continue.
(5) Choose an element of \( S \) which contains the maximum number of elements from \( F_p \). Denote it by \( A_{p+1} \). Set \( S = S - \{A_{p+1}\} \).
(6) Define
\[ E_{p+1} = A_{p+1} \cap F_p, \quad a_{p+1} = \min E_{p+1}, \]
\[ F_{p+1} = \Delta_{a_{p+1}}(E_{p+1}) - \left\{ \sum_{i=0}^{j} a_{p+1-i} \mid i = 0, 1, \ldots, p \right\}. \]
(7) If \( F_{p+1} \cap (\bigcup_{i=1}^{p} A_i) \neq \emptyset \) then stop. Otherwise, continue.
(8) Set \( p = p + 1 \). Go to (5).
Claim 1. For each \( p \in \{1, \ldots, n\} \), if \( E_p \) is constructed by the algorithm then:

- \( E_p \) contains at least \( \alpha_{n-p+1} + p + 1 \) elements.
- \( F_p \) is constructed, and it contains at least \( \alpha_{n-p+1} \) elements.

Proof. By induction on \( p \). For \( p = 1 \), from the pigeon-hole principle, \( A_1 \) contains at least

\[
\frac{\alpha_{n+1}}{n} = \frac{1}{n} + n! \sum_{i=0}^{n} \frac{1}{i!}
\]

elements. Thus, \( E_1 = A_1 \) contains at least

\[
1 + n! \sum_{i=0}^{n} \frac{1}{i!} = \alpha_n + 2
\]

elements (from Lemma 2). Moreover, \( \alpha_n + 2 \geq 2 \), thus \( a_1 \) is well defined. And \( F_1 \) is constructed: we form the differences in (3) (which decreases the number of elements by 1), and delete \( a_1 \) (if necessary). Thus \( F_1 \) contains at least \( \alpha_n \) elements.

Let \( p \) be a fixed integer with \( 1 \leq p < n \). Suppose that the conclusion holds for the value \( p \) and that \( E_{p+1} \) is constructed by the algorithm. Then, by the induction hypothesis, \( F_p \) contains at least \( \alpha_{n-p+1} \) elements, none of which belongs to \( \bigcup_{i=1}^{p} A_i \) (otherwise the algorithm would have stopped from (7), and \( E_{p+1} \) would not have been constructed). From the pigeon-hole principle and Lemma 2, \( A_{p+1} \) contains at least

\[
\frac{\alpha_{n-p+1}}{n-p} = \alpha_{n-p} + p + 1 + \frac{1}{n-p}
\]

elements from \( F_p \). It follows that \( E_{p+1} \) contains at least \( \alpha_{n-p} + p + 2 \) elements. Moreover, \( \alpha_{n-p} + p + 2 \geq 2 \), thus \( a_{p+1} \) is well defined, and \( F_{p+1} \) is constructed: we form the differences and make the deletions (if necessary, and not more than \( p + 1 \)). Then \( F_{p+1} \) contains at least \( \alpha_{n-p} \) elements, which ends the induction, and completes the proof of the claim.

Remark. The claim ensures that there will be no problem at step (6) of the algorithm.

Case 1: The algorithm stops at (4). Then there exists \( b \in F_1 \cap A_1 \). Thus \( b = a_i - a_1 \) for some \( a_i \in A_1 \), and \( b \neq a_1 \) since \( a_1 \notin F_1 \). Thus \( b + a_1 = a_i \) in \( A_1 \), and the conclusion of the Theorem holds.

Case 2: The algorithm stops at (7) with \( p < n \). First note that it did not stop before.

Claim 2. For each \( k \in \{1, \ldots, p\} \), the number \( X_k = a_p + a_{p-1} + \ldots + a_k \) belongs to \( E_k \).
Proof. By (descending) induction. For \( k = p \), we directly have \( X_p = a_p \in E_p \). Let \( k \in \{2, \ldots, p\} \) be fixed. Suppose that \( X_k \in E_k \). Since \( E_k \subset F_{k-1} \subset \Delta a_{k-1} (E_{k-1}) \), there exists \( x \in E_{k-1} \) such that \( X_k = x - a_{k-1} \). Thus \( X_{k-1} = X_k + a_{k-1} = x \) belongs to \( E_{k-1} \). This ends the induction and completes the proof.

Since the algorithm has stopped at (7), there exists \( k \in \{1, \ldots, p\} \) such that \( F_p \cap A_k \neq \emptyset \). Then \( b = b_{i_1} - a_p \in F_p \cap A_k \) for some \( b_{i_1} \in E_p \).

If \( k = p \), we have \( b \in F_p \). Then, by construction, \( b \neq a_p \). Thus \( b_{i_1} = b + a_p \) in \( A_p \), and the conclusion of the Theorem holds.

If \( k < p \), then \( b_{i_1} \in E_p \subset F_{p-1} \). It follows that

\[
\begin{align*}
b_{i_1} &= b_{i_2} - a_{p-1} & & \text{for some } b_{i_2} \in E_{p-1} \subset F_{p-2}, \\
b_{i_2} &= b_{i_3} - a_{p-2} & & \text{for some } b_{i_3} \in E_{p-2} \subset F_{p-3}, \\
\vdots \\
b_{i_{p-k}} &= b_{i_{p-k+1}} - a_k & & \text{for some } b_{i_{p-k+1}} \in E_k \subset A_k.
\end{align*}
\]

Summing, we obtain

\[
b = b_{i_{p-k+1}} - \sum_{i=k}^{p} a_i = b_{i_{p-k+1}} - X_k.
\]

But, from Claim 2, we have \( X_k \in E_k \subset A_k \). And, since \( b \in F_p \), we deduce that \( b \neq X_k \) (see step (6)). It follows that \( b + X_k = b_{i_{p-k+1}} \) in \( A_k \), and the conclusion of the Theorem holds.

Case 3: The algorithm does not stop until \( p = n \) in (8). Then \( A_n, E_n, F_n \) are constructed by the algorithm. From Claim 1, \( F_n \) contains at least \( \alpha_1 = 1 \) elements. Thus \( F_n \neq \emptyset \) and

\[
F_n \cap \left( \bigcup_{i=1}^{n} A_i \right) = F_n \cap A \neq \emptyset.
\]

We deduce that the algorithm stops at (7). The reasoning used in Case 2 may be repeated word for word, and the conclusion of the Theorem holds. Thus the proof is complete.

4. Remarks. Following Sierpiński [4], given a natural number \( k \), denote by \( n(k) \) the least natural number \( n \) with the following property: if the numbers 1, \ldots, \( n \) are divided into \( k \) classes, then at least one class contains two different numbers together with their sum.

Then according to Walker [6]

\[
n(1) = 3, \quad n(2) = 9, \quad n(3) = 24, \quad n(4) = 67, \quad n(5) = 197.
\]

Also, see [5], p. 440,

\[
n(k) \geq 1 + 315^{(k-1)/5}.
\]
Thus
\[315^{(k-1)/5} + 1 \leq n(k) \leq [k! \cdot ke] + 1.\]

References


50 rue Prosper Legouté
92160 Antony, France
E-mail: pbornszt@club-internet.fr

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