Random Liouville functions and normal sets

by

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We define a random Liouville function $\lambda_Q$ which depends on a random set $Q$ of primes and prove that $A_Q = \{n \in \mathbb{N} \mid \lambda_Q(n) = -1\}$ is normal almost everywhere. This fact enables us to generate a family of normal sets such that the equation $xy = z$ is not solvable inside them. Additionally we prove that the equations $xy = z^2$, $x^2 + y^2 = \text{square}$, $x^2 - y^2 = \text{square}$ are solvable in any normal set, and for any equation $xy = cn^2$ ($c > 1$ is not a square) there exists a normal set $A_c$ such that the equation is not solvable inside $A_c$.

1. Introduction. With the familiar notion of normal numbers in mind, we shall call an infinite binary sequence normal if any binary word $\omega$ of length $|\omega|$ occurs in the sequence with the right frequency: $2^{-|\omega|}$. We have the natural bijection between infinite $\{0, 1\}$-sequences $\lambda$ and the subsets of the natural numbers $A_\lambda = \{i \mid \lambda_i = 1\}$. We now have

DEFINITION 1.1. A set $B \subset \mathbb{N}$ is called normal if the corresponding $\{0, 1\}$-sequence is normal.

In this note we shall be interested in normal sets and the possibility of solving diophantine equations in integers from a given, but arbitrary, normal set. We expect that there are many diophantine equations (or systems of equations) which, if they are solvable at all in integers, are solvable in integers from a given normal set. We call such equations $N$-regular, and we denote by DSN the family of $N$-regular equations (or systems of equations).

An equation, or a system of equations, is called partition-regular if for any finite partition of the natural numbers, the system is solvable within one of the cells of the partition. One of the earliest examples of a partition-regular equation is Schur’s equation: $x + y = z$. It is not hard to see that Schur’s equation is also $N$-regular. Rado in [6] classified all systems of linear diophantine equations that are partition regular. Rado’s theorem implies the

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familiar van der Waerden theorem on existence of arbitrarily long monochromatic arithmetic progressions in any finite coloring of the natural numbers.

Using Furstenberg’s theorem regarding Rado’s systems in [4], one can obtain the analogous result for N-regularity: namely, any Rado system of linear equations is in DSN.

From the foregoing, we have many linear equations in DSN. But little is known in the non-linear case. For example, it is an open question whether the Pythagorean equation \( x^2 + y^2 = z^2 \) is in DSN. The purpose of this note is to show that the equation \( xy = z \) is not in DSN. This equation is called the multiplicative Schur equation. It is an easy consequence of Schur’s additive theorem that his multiplicative equation is also partition-regular. In fact in any finite partition of \( \mathbb{N} \) one can find solutions to both the additive and the multiplicative equations in the same cell ([1]). Thus partition regularity does not imply N-regularity. To show that \( xy = z \) is not in DSN we will use a construction of random normal sets, based on a variant of the Liouville function \( \lambda(n) \) from number theory. Recall

**Definition 1.2.** Liouville’s function \( \lambda : \mathbb{N} \to \{-1, 1\} \) is defined by

\[
\lambda(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}) = (-1)^{e_1 + e_2 + \cdots + e_k}
\]

where \( p_1, \ldots, p_k \) are primes.

It is a well known and very deep question whether the set \( A = \{ n \in \mathbb{N} \mid \lambda(n) = -1 \} \) is normal (see [2] and [3]). It seems that at present we are far from resolving this outstanding problem. But just for clarity, if the answer to this question is positive, then the aforementioned set \( A \) gives us an example of a normal set with no solution to the equation \( xy = z \).

In the following we will use a random Liouville function \( \lambda_Q \) which is defined by a random choice of a subset \( Q \) inside \( P \) (the prime numbers) as follows:

\[
\lambda_Q(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}) = \lambda_Q(p_1)^{e_1} \lambda_Q(p_2)^{e_2} \cdots \lambda_Q(p_k)^{e_k}
\]

and

\[
\lambda_Q(p) = \begin{cases} 
-1, & p \in Q, \\
1, & p \notin Q.
\end{cases}
\]

By randomness of \( Q \) we mean that the choice of every prime number \( p \) is independent of the choice of any other prime numbers and \( \Pr(p \in Q) = 0.5 \) for any \( p \in P \).

One defines \( A_Q = \{ n \in \mathbb{N} \mid \lambda_Q(n) = -1 \} \). In Section 2 we prove

**Theorem 1.1.** For almost every \( Q \) the set \( A_Q \) is normal.

This theorem gives us an infinite family of normal sets such that the multiplicative Schur equation is not solvable in these sets.

In Section 3 we prove that the equations \( xy = z^2 \), \( x^2 + y^2 = \text{square} \) and \( u^2 - v^2 = \text{square} \) are in DSN.
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2. $A_Q$ is normal for a.e. $Q$. We start from an obvious claim about normality of $A_Q$.

Lemma 2.1. Let $Q \subset P$ be given. Then $A_Q$ is normal $\iff$ for any $k \in \mathbb{N} \cup \{0\}$ and any $i_1 < \cdots < i_k$ we have

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \lambda_Q(n)\lambda_Q(n+i_1)\cdots\lambda_Q(n+i_k) = 0.
$$

We proceed with the following statement which is readily proved:

Lemma 2.2. Let $\{a_n\}$ be a bounded sequence. Define $T_N = N^{-1} \sum_{n=1}^{N} a_n$. Then $T_N$ converges to a limit $t \iff$ there exists an increasing sequence $\{N_i\}$ of indices such that $N_i/N_{i+1} \to 1$ and $T_{N_i} \to t$ as $i \to \infty$.

The next step is to show

$$
\sum_{N=1}^{\infty} E\left(\left( \frac{1}{N^{40}} \sum_{n=1}^{N^{40}} \lambda_Q(n)\lambda_Q(n+i_1)\cdots\lambda_Q(n+i_k) \right)^2 \right) < \infty.
$$

Lemma 2.3. Let $T_N$ be as above. Then $E(T_N^2) \leq O(1/N^{0.05})$.

Proof. By linearity of expectation we get

$$
E(T_N^2) = \frac{1}{N^2} \sum_{x,y=1}^{N} E(\lambda_Q(x)\lambda_Q(x+i_1)\cdots\lambda_Q(x+i_k)\lambda_Q(y)\lambda_Q(y+i_1)\cdots\lambda_Q(y+i_k)).
$$

Note that for any $m \in \mathbb{N}$, $E(\lambda_Q(m)) = 0$ unless $m$ is a square in which case $E(\lambda_Q(m)) = 1$.

Set

$$
\phi(x) = \lambda_Q(x)\lambda_Q(x+i_1)\cdots\lambda_Q(x+i_k), \quad \xi(x) = x(x+i_1)\cdots(x+i_k).
$$

By distribution of $Q$ we get

$$
E(\phi(x)\phi(y)) = 1 \iff \xi(x)\xi(y) = m^2.
$$

Otherwise

$$
E(\phi(x)\phi(y)) = 0.
$$

Therefore, to obtain an upper bound on $E(T_N^2)$, we give an upper bound on the number of pairs $(x, y) \in [1, N] \times [1, N]$ which satisfy $\xi(x)\xi(y) = \text{square}$.

For a given $x \in [1, N]$ assume that $\xi(x) = c_x m^2$, where $c_x$ is a square-free number, say with prime factorization $c_x = p_{j_1}\cdots p_{j_l}$. Then we define
$h(x) = l$ (thus $h(x)$ is the number of primes in the prime factorization of the maximal square-free number which divides $x$). Denote by $D$ the set of all possible common divisors of the numbers $x, x + i_1, \ldots, x + i_k$ (i.e. positive integers which divide at least two of them). For a finite non-empty set $S$ of positive numbers we denote by $m(S)$ the product of all elements of $S$; for the empty set, we set $m(\emptyset) = 1$.

Note that $\xi(x)\xi(y) = \text{square}$ implies there exist $S_1 \subset D$ and $S_2 \subset \{p_{j_1}, \ldots, p_{j_2}\}$ such that $y = m(S_1)m(S_2)$ square.

Assume $|D| = r$ ($r$ depends only on the set $\{i_1, \ldots, i_k\}$ and does not depend on $x$). Then we obtain $\xi(x)\xi(y) = \text{square}$ for at most $2^r 2^{h(x)} \sqrt{N}$ $y$’s inside $[1, N]$. Thus

$$E(T_N^2) \leq \frac{1}{N^2} \left( \sum_{n=1}^{N} 2^r 2^{h(n)} \sqrt{N} \right) \leq \frac{c}{N^{1.5}} \sum_{n=1}^{N} 2^{h(n)}.$$

Therefore it remains to bound the expression $\sum_{n=1}^{N} 2^{h(n)}$.

Let $p = p_i$ be the smallest prime number such that $(k + 1)/\log_2 p \leq 0.45$. If $\xi(n)$ is not divisible by any of the primes $2, 3, \ldots, p$ then

$$h(n) \leq \log_p (n + i_k)^{k+1} = (k + 1) \frac{\log_2 (n + i_k)}{\log_2 p}.$$

This gives us

$$2^{h(n)} \leq (n + i_k)^{(k+1)/\log_2 p} \leq (n + i_k)^{0.45}.$$

But if $\xi(n)$ is arbitrary then $h(n)$ can increase by at most $i$, which means $2^{h(n)} \leq 2^i (n + i_k)^{0.45}$. Thus $\sum_{n=1}^{N} 2^{h(n)} \leq C_1 (N + i_k)^{1.45}$ and therefore we get

$$E(T_N^2) \leq C_2 \frac{1}{N^{0.05}}.$$

**Proof of Theorem 1.1.** From the last lemma we conclude that $\sum_{N=1}^{\infty} E(T_{N^{40}}^2) < \infty$. Thus $T_{N^{40}} \to 0$ almost surely. Lemma 2.2 implies that $T_N \to 0$ almost surely. And from Lemma 2.1 (with countably many conditions for $A_Q$ to be normal) it follows that for almost all $Q \subset P$ the sets $A_Q$ are normal.

We can now demonstrate the main result of this note.

**Theorem 2.1.** There exists a normal set $A \subset \mathbb{N}$ such that the multiplicative Schur equation is not solvable inside $A$.

**Proof.** We have already shown the existence of many $Q$ ($Q \subset P$) such that $A_Q$ is normal. By definition of $A_Q$, we have $xy \notin A_Q$ for any $x, y \in A_Q$.

**Corollary 2.1.** For any equation $xy = cn^k$ (where $c, k$ are natural numbers, $c$ is not a square and $k$ is even) we can find a normal set $A_{c,k} \subset \mathbb{N}$ such that for any $x, y \in A$ we have $xy \neq cn^k$ for every natural $n$. 

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Proof. We take \( A \) normal and such that \( \lambda_Q(c) = -1 \) (this happens with probability \( 1/2 \), and thus such sets exist). Then obviously we cannot solve the above equation inside \( A \).

3. Solvability of the equation \( xy = z^2 \) and related problems

**Theorem 3.1.** Let \( A \subset \mathbb{N} \) be a normal set. Then there exist \( x, y, z \in A \) \((x \neq y)\) such that \( xy = z^2 \).

**Proof.** For a set \( S \subset \mathbb{N} \) and \( a \in \mathbb{N} \) define \( S_a = \{ n \in \mathbb{N} \mid an \in S \} \). It is easily seen that if \( S \) is normal then so is each \( S_a \) (see [5]).

We denote by \( d(S) \) the density of a set \( S \), if it exists.

Let \( A \) be a normal set. Define \( R_n = A^{2n} \). For any \( n \) we have \( d(R_n) = 1/2 \).

Set \( \mu_N(S) = \frac{|S \cap \{1, \ldots, N\}|}{N} \) for any \( S \subset \mathbb{N} \) and any \( N \in \mathbb{N} \).

By Szemerédi’s theorem (finite version), for any \( \delta > 0 \) and \( l \in \mathbb{N} \) there exists \( N(l, \delta) \) such that for any \( N \geq N(l, \delta) \) and \( F \subset \{1, \ldots, N\} \) such that \( |F|/N \geq \delta \) the set \( F \) contains an arithmetic progression of length \( l \) (see [7]).

One chooses \( K \geq N(3, 1/3) \). Then there exists \( N_K \) such that \( \mu_{N_K}(R_i) \geq 1/3 \) for every \( 1 \leq i \leq K \).

We claim that there exists \( F \subset \{1, \ldots, K\} \) such that \( |F|/K \geq 1/3 \) and \( \mu_{N_K}(\bigcap_{j \in F} R_j) > 0 \). If not, let \( 1_{R_i} \) be the indicator function of the set \( R_i \) inside \( \{1, \ldots, N_K\} \). Then on the one hand,

\[
\int_{[1,N_K]} (1_{R_1} + \cdots + 1_{R_K}) \, d\mu_{N_K} \geq \sum_{j=1}^K \int_{[1,N_K]} 1_{R_j} \, d\mu_{N_K} \geq K/3.
\]

But on the other hand,

\[
\int_{[1,N_K]} (1_{R_1} + \cdots + 1_{R_K}) \, d\mu_{N_K} < K/3
\]

because \( 1_{R_1} + \cdots + 1_{R_K} < K/3 \).

Let \( F \) be as above. Then by the choice of \( K \) it follows that \( F \) necessarily contains an arithmetic progression of length 3. This means there exist \( a, b, c \in F \) such that \( a + c = 2b \). We have \( R_a \cap R_b \cap R_c \neq \emptyset \) and so there exists \( n \in \mathbb{N} \) such that \( x := n2^a \in A \), \( z := n2^b \in A \) and \( y := n2^c \in A \). Then

\[
xy = z^2.
\]

**Question.** Are the equations \( xy = c^2 z^2 \), where \( c > 0 \) is a natural number, always solvable inside an arbitrary normal set?

**Theorem 3.2.** Let \( A \subset \mathbb{N} \) be an arbitrary normal set. Then there exist \( x, y, u, v \in A \) such that \( x^2 + y^2 = \text{square} \) and \( u^2 - v^2 = \text{square} \).
Proof. Note that there exist \( a, b, c \in \mathbb{N} \) such that \( a^2 + b^2 = \text{square} \) and \( a^2 + c^2 = \text{square} \) and \( b^2 + c^2 = \text{square} \), for example \( a = 44, b = 117, c = 240 \).

Let \( A \subset \mathbb{N} \) be an arbitrary normal set. We look at the triple of sets \( A_a, A_b, A_c \) defined as in the proof of Theorem 3.1. Then \( d(A_a) = d(A_b) = d(A_c) = 1/2 \) and thus it cannot be true that the intersection of each pair from the triple is empty.

Without loss of generality, assume that \( A_a \cap A_b \neq \emptyset \). Thus there exists \( z \in A_a \cap A_b \) or equivalently \( za, zb \in A \). But \( a^2 + b^2 = \text{square} \) and therefore \( (za)^2 + (zb)^2 = \text{square} \).

The proof that the equation \( u^2 - v^2 = \text{square} \) is solvable in any normal set is similar. We use the fact that there exist \( a, b, c \in \mathbb{N} \) such that \( a < b < c \) and \( c^2 - b^2 = \text{square} \), \( c^2 - a^2 = \text{square} \) and \( b^2 - a^2 = \text{square} \), for example \( a = 153, b = 185, c = 697 \).

Questions. 1) For an arbitrary normal set \( A \) do there exist \( x, y, z \in A \) such that \( x^2 + y^2 = z^2 \)?

2) For an arbitrary normal set \( A \) do there exist \( x, y, z \in A \) such that \( x^2 - y^2 = z^2 \)?

References


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