ACTA ARITHMETICA
150.4 (2011)

On first layers of $\mathbb{Z}_p$-extensions II

by

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1. Introduction. In this paper, we let p be a fixed odd prime number and let $\mathbb{Z}_p$ be the ring of p-adic integers. For a number field k, let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k and let k_n be its unique subfield of degree p^n over k. We write $\Gamma = G(k_\infty/k)$ for the Galois group k_∞ over k. Let Cl'_n be the p-Sylow subgroup of the quotient of the ideal class group of k_n by the subgroup generated by the primes lying over p. Following Kuz'min, we define the *Tate module* $\mathrm{T}_p(k)$ of k to be the inverse limit of Cl'_n with respect to the norm maps,

$$\mathrm{T}_p(k) = \varprojlim_n \mathrm{Cl}'_n.$$

Note that Kuz'min's Tate module $\mathrm{T}_p(k)$ is slightly different from the usual Tate module which is defined as the inverse limit of the p-primary parts of the ideal class groups of k_n. Let Ψ_k be the group of all elements of k whose pth roots generate a field which can be embedded into a $\mathbb{Z}/p^n\mathbb{Z}$-extension of k for all $n \geq 0$. Write $N_{K/k}$ for the norm map of an extension K/k. Let Θ_k be the group of all elements of k whose pth roots generate a field which can be embedded into a $\mathbb{Z}_p$-extension of k. Let

$$k^{\mathrm{univ}} = \bigcap_{n\geq 1} N_{k_n/k} k_n^\times$$

be the group of universal norm elements for k_∞/k. Let $\varprojlim_{n\geq 1} k_n^\times$ be the inverse limit with respect to the norm maps. In this paper, we call $(\alpha_n) \in \varprojlim_{n\geq 1} k_n^\times$ a *norm compatible sequence*, and call $\alpha \in k$ a *norm compatible element* if there is a norm compatible sequence $(\alpha_n)_{n\geq 1}$ such that $\alpha = N_{k_1/k}\alpha_1$. Finally, let k^{comp} consist of all norm compatible elements of k, i.e.,

$$k^{\mathrm{comp}} = \{\alpha \in k^\times \mid N_{k_1/k}\alpha_1 = \alpha \text{ for some } (\alpha_n) \in \varprojlim_{n\geq 1} k_n^\times\}.$$

2010 *Mathematics Subject Classification*: 11R18, 11R20, 11R23, 11R27, 11S25.
Key words and phrases: $\mathbb{Z}_p$-extension, Tate module, Gross conjecture, norm compatible.

DOI: 10.4064/aa150-4-4

The main result of this paper relates the group Θ_k to $\mathrm{T}_p(k)$ using Ψ_k, k^{univ} and k^{comp}.

THEOREM 1.1. *Let p be an odd prime and let $k_\infty = \bigcup_n k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of a number field k containing a primitive pth root of unity. If $\mathrm{T}_p(k_n)^{\Gamma_n}$ is finite for all $n \geq 0$, then there exists an exact sequence*

$$1 \to \frac{\Theta_k \cap k^{\mathrm{univ}}}{k^{\mathrm{comp}}} \to \mathrm{T}_p(k)^{\Gamma} \to \Psi_k/\Theta_k \to \Psi_k/k^{\mathrm{univ}}\Theta_k \to 1.$$

Note that for a number field k, the finiteness of $\mathrm{T}_p(k)^{\Gamma}$ is equivalent to what Kolster calls the Gross conjecture for k (see Proposition 1.2 of [6]).

COROLLARY 1.2. *Let p be an odd prime and let $k_\infty = \bigcup_n k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of a number field k containing a primitive pth root of unity. If the Gross conjecture is true for $\{k_n\}_n$, then there exists an exact sequence*

$$1 \to \frac{\Theta_k \cap k^{\mathrm{univ}}}{k^{\mathrm{comp}}} \to \mathrm{T}_p(k)^{\Gamma} \to \Psi_k/\Theta_k \to \Psi_k/k^{\mathrm{univ}}\Theta_k \to 1.$$

When k is abelian, we have the following corollary.

COROLLARY 1.3. *Let p be an odd prime and let k be an abelian field containing a primitive pth root of unity. Then there exists an exact sequence*

$$1 \to \frac{\Theta_k \cap k^{\mathrm{univ}}}{k^{\mathrm{comp}}} \to \mathrm{T}_p(k)^{\Gamma} \to \Psi_k/\Theta_k \to \Psi_k/k^{\mathrm{univ}}\Theta_k \to 1.$$

Using Lemma 3.2 (below) and the proof of Theorem 1.1 one can show that if k is abelian and there is only one prime of k lying over p, then $\mathrm{T}_p(k)^{\Gamma} = 1$.

2. Kuz'min's results and norm compatible elements. We start with the following definition for a number field k and a group H.

DEFINITION 2.1. For an extension field K of k and a group H, we will say K is *H-extendable over k* if there is an extension field $F \supset K$ such that F/k is Galois and the Galois group $G(F/k)$ is isomorphic to H.

As defined in the introduction, Θ_k denotes the group of all elements α in $k^\times$ such that $k(\alpha^{1/p})$ is $\mathbb{Z}_p$-extendable over k. We will use the same objects and notation as [1]. For each $n > 0$, let ζ_{p^n} denote a primitive p^nth root of unity in a fixed algebraic closure k^{alg} of k such that $\zeta_{p^n}^p = \zeta_{p^{n-1}}$ for each $n > 1$, and let $\mu_{p^n} = \langle \zeta_{p^n} \rangle$ denote the group of all p^nth roots of unity in k^{alg}. For each integer $n \geq 1$, let $\tau(\zeta_{p^n}) = \zeta_{p^n}^{1+p}$. Thus τ defines a topological generator for $G(k_\infty/k)$ of the cyclotomic $\mathbb{Z}_p$-extension of k. Let

$$\Lambda = \mathbb{Z}_p[[T]] \cong \varprojlim \mathbb{Z}_p[G(k_n/k)]$$

be the Iwasawa algebra, i.e., the inverse limit of $\mathbb{Z}_p[G(k_n/k)]$ for which the generator τ satisfies $1+T=\tau$. Let $\tilde{k}_\infty = \varprojlim_{n\ge 1} k_n^\times/(k_n^\times)^T(k^\times)^p$ denote the inverse limit with respect to the norm maps $N_n : k_n \to k$ and let $\tilde{\tilde{k}}_\infty = \varprojlim_{n\ge 1} k_n^\times/(k_n^\times)^T(k_n^\times)^p$ be defined in the same way. Let $\tilde{\pi}$ (resp. $\tilde{\tilde{\pi}}$) be the natural projection from $\tilde{k}_\infty$ (resp. $\tilde{\tilde{k}}_\infty$) into $k^\times/(k^\times)^p$,

$$\tilde{\pi}((a_n)_{n\ge 1}) = N_1(b) \bmod (k^\times)^p$$

where $b \in k_1$ denotes any lifting of a_1. There are upper and lower bounds for the group $\Theta_k/(k^\times)^p$ in terms of the above inverse limits (Theorem A of [9]).

Theorem 2.2. *Let p be an odd prime and let k be a number field such that $k = k(\zeta_p) \ne k(\zeta_{p^2})$. Then*

$$\tilde{\pi}(\tilde{k}_\infty) \subset \Theta_k/(k^\times)^p \subset \tilde{\tilde{\pi}}(\tilde{\tilde{k}}_\infty).$$

We have the exact sequence

$$1 \to \varprojlim_n (k_n^\times)^T(k^\times)^p \to \varprojlim_n k_n^\times \to \tilde{k}_\infty$$

induced from the short exact sequence

$$1 \to (k_n^\times)^T(k^\times)^p \to k_n^\times \to k_n^\times/(k_n^\times)^T(k^\times)^p \to 1$$

by taking inverse limits with respect to the norm maps. As in the introduction, we define the natural projection map π from $\tilde{k}_\infty$ into $k^\times/(k^\times)^p$, i.e., $\pi((a_n \bmod (k_n^\times)^T(k^\times)^p)) = N_n(a_n) \bmod k^p$. Note that $N_n(a_n)$ is independent of n and $k^{\mathrm{comp}} = \pi(\varprojlim_n k_n^\times) \subset \tilde{k}^{\mathrm{comp}} = \pi(\tilde{k}_\infty)$. Theorem 2.2 shows that $\Theta_k \supset k^{\mathrm{comp}}(k^\times)^p$. Notice that the proof of Theorem 2.2 of [9] shows easily that this containment holds even without the assumption $k(\zeta_p) \ne k(\zeta_{p^2})$.

Corollary 2.3. *Let p be an odd prime number and let k contain ζ_p. Then $\Theta_k \supset k^{\mathrm{comp}}(k^\times)^p$.*

For the local field k_v of k at a finite place v, let $k_{v,\infty}$ be the cyclotomic $\mathbb{Z}_p$-extension of k_v. We also write $k_v^{\mathrm{comp}} = \pi(\varprojlim_n k_{v,n}^\times)$ where $k_{v,n}$ is the subfield of $k_{v,\infty}$ of degree p^n over k_v. Write k^{loc} for the set of all elements which are locally norm compatible,

$$k^{\mathrm{loc}} = \{\alpha \in k^\times \mid \exists (\alpha_{v,n}) \in \varprojlim_n k_{v,n} \text{ such that } N_1\alpha_{v,1} = \alpha \text{ for all finite places } v\}.$$

It follows from the definition above that $k^{\mathrm{loc}} = k^\times \bigcap_v k_v^{\mathrm{comp}}$ for all finite places of k. Then by a well known property of local compactness, we have $\bigcap_n N_{k_{v,n}/k_v} k_{v,n} = k_v^{\mathrm{comp}}$. For completeness, we briefly explain this. Again, we write N_n for the local norm $N_{k_{v,n}/k_v}$. For $r \ge m > 0$ and $\alpha \in \bigcap_n N_n k_{v,n}$, let $X_r(\alpha) = N_{r,m} k_{v,r}^\times \cap N_m^{-1}\alpha$ where $N_m = N_{k_m/k}, N_{r,m} = N_{k_r/k_m}$ is the corresponding norm map and $N_m^{-1}\alpha = \{b \in k_m^\times \mid N_m(b) = \alpha\}$. Since $\alpha \in \bigcap_n N_n k_{v,n}$, $X_r(\alpha)$ is non-empty and compact. The family $X_r(\alpha)$ has the finite intersection property as $r \geq m$ varies because for a finite set of numbers $n_g > \cdots > n_1 > m$, $X_{n_i}(\alpha)$ is a decreasing chain. It follows that there is $\beta_m \in \bigcap_{r\geq m} N_{r,m} k_{v,r}$ such that $N_m \beta_m = \alpha$. In this way, one can construct a norm compatible sequence whose zeroth term is α. Hence, k^{loc} is equal to the set of elements of local universal norms, i.e., $k^{\mathrm{loc}} = k^\times \bigcap_{v,n} N_n k_{v,n}$ for all finite places of k (cf. example on page 526 of [1]). Since k_∞/k is unramified at primes prime to p, and k^{loc} are p-units, it follows that $k^{\mathrm{loc}} = U_k(p) \bigcap_{v|p,n} N_n k_{v,n}$. Let $k^{\mathrm{univ}} = \bigcap_n N_n k_n$ be the group of universal norms of k_∞/k. Using Hasse's norm theorem (see Theorem A of [1]), it is possible to rephrase some results of §1 of [1] in the following lemmas which will be useful in our theorem.

Lemma 2.4. *Let p be an odd prime number. Then $k^{\mathrm{loc}} = k^{\mathrm{univ}}$.*

Proof. By Hasse's norm theorem, $\alpha \in N_n k_n = k^{\mathrm{univ}}$ if and only if $\alpha \in N_{k_{v',n}/k_v} k_{v',n}$ for all places v and $v' \mid v$. Since p is an odd prime, if v is an infinite place, then $k_{v',n} = k_v$. Thus we have $\alpha \in N_n k_n$ if and only if $\alpha \in N_{k_{v',n}/k_v} k_{v',n}$ for all finite places $v' \mid v$. For a finite place $v' \mid v$, we have, for some $m \geq n \geq r$, $k_{v',n} = k_v \mathbb{Q}_m = k_{v,r}$ where $\mathbb{Q}_m$ is the intermediate field of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ of degree p^m. It follows that $\alpha \in N_n k_n$ for all n if and only if $\alpha \in \bigcap_n N_n k_{v,n}$ for all finite places v if and only if $\alpha \in k^{\mathrm{loc}}$. ∎

Lemma 2.5. *Let p be an odd prime number. Then $k(\alpha^{1/p})/k$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n if and only if $k_v(\alpha^{1/p})/k_v$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n and all finite places v of k.*

Proof. This follows from Hasse's norm theorem, Corollaire on page 524 of [1] and $k_{v,r} = k_{v',n}$ for some $n \geq r$ and $v' \mid v$. ∎

Lemma 2.6. *$k_v(\alpha^{1/p})/k_v$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n if and only if $k_v(\alpha^{1/p})/k_v$ is $\mathbb{Z}_p$-extendable. Moreover, $k_v(\alpha^{1/p})/k_v$ is $\mathbb{Z}_p$-extendable if and only if $\alpha \in k_v^{\mathrm{comp}}(k_v^\times)^p$.*

Proof. This follows from the example on page 526 of [1] and the above argument of local compactness. ∎

Let Ψ_k denote the set of all $\alpha \in k^\times$ such that $k(\alpha^{1/p})$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n. Lemmas 2.5 and 2.6 lead to the following corollary.

Corollary 2.7. *$\Psi_k = \bigcap_n (N_n k_n^\times (k^\times)^p) = k^\times \bigcap_v (k_v^{\mathrm{comp}}(k_v^\times)^p)$ for all finite places v of k.*

For a $\mathbb{Z}_p[G]$-module M, we denote by $\mathrm{tor}(M) = \mathrm{tor}_{\mathbb{Z}_p}(M)$ the $\mathbb{Z}_p$-torsion submodule and denote by $\mathrm{Fr}(M) = \mathrm{Fr}_{\mathbb{Z}_p}(M)$ the quotient module $M/\mathrm{tor}(M)$. Notice also that in the following theorem, the assumption $k(\zeta_p) \neq k(\zeta_{p^2})$ is not necessary.

THEOREM 2.8. *Let p be an odd prime and let $k_\infty = \bigcup_n k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of a number field k containing a primitive pth root of unity. If $\mathrm{T}_p(k_n)^{\Gamma_n}$ is finite for all $n \geq 0$, then there exists an exact sequence*

$$1 \to \frac{\Theta_k \cap k^{\mathrm{univ}}}{k^{\mathrm{comp}}} \to \mathrm{T}_p(k)^\Gamma \to \Psi_k/\Theta_k \to \Psi_k/k^{\mathrm{univ}}\Theta_k \to 1.$$

Proof. We start with a proposition which is due to Kuz'min (cf. Proposition 7.5 of [7]). In §3 of [10] we modified the proposition and its proof for our purpose by using global class field theory.

PROPOSITION 2.9. *Let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k. Then*

$$\mathrm{T}_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}}}.$$

Under the assumption that $\mathrm{T}_p(k_n)^{\Gamma_n}$ is finite for all $n \geq 0$, we claim that there is an isomorphism of abelian groups

$$\mathrm{T}_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{comp}}}.$$

Firstly, we know from Lemma 2.4 and Proposition 2.9 that

$$\mathrm{T}_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{(U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}}}.$$

Using a compactness argument which was explained after Corollary 2.3, the isomorphism above reduces to

$$\mathrm{T}_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{(U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{univ}}}.$$

Since $(U_k(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p)^{\mathrm{univ}} = \bigcap_n N_n(U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p) = \bigcap_n (N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p)$ and

$$N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p \supset (U_k(p)^{\mathrm{univ}})^{p^n} \otimes \mathbb{Z}_p$$

it follows that $\mathrm{T}_p(k)^\Gamma$ is finite if and only if the decreasing chain of modules $\{N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p\}$ must stop, that is, there is $n = n(k)$ such that $N_s U_s(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p = N_m U_m(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p$ for $m \geq s \geq n$. Since the index $(N_s U_s(p)^{\mathrm{univ}} : N_m U_m(p)^{\mathrm{univ}})$ is p-primary, the condition leads to

$$N_s U_s(p)^{\mathrm{univ}} = N_m U_m(p)^{\mathrm{univ}} \quad \text{for } m \geq s \geq n.$$

Hence, the condition above is a necessary and sufficient condition for $\mathrm{T}_p(k)^\Gamma$ to be finite. Therefore, by replacing the ground field k by k_n, the hypothesis for k_n implies that there is a function $x : \mathbb{N} \cup \{0\} \to \mathbb{N}$ such that for each $n \in \mathbb{N} \cup \{0\}$, $x(n) > n$ and

$$N_{x^{r+1}(n),x^r(n)} U^{\mathrm{univ}}_{x^{r+1}(n)} = N_{x^{r+1}(n),x^r(n)} N_{x^{r+2}(n),x^{r+1}(n)} U^{\mathrm{univ}}_{x^{r+2}(n)}$$

where $x^r = x \circ \cdots \circ x$ denotes the rth composite of x with $x^0(n) = n$. Let $\alpha \in (U_k^{\mathrm{univ}})^{\mathrm{univ}}$. Then by taking $n = 0$ above, we have $\alpha \in N_{x(0),0} U_{x(0)}^{\mathrm{univ}} = N_{x(0)} U_{x(0)}^{\mathrm{univ}}$ and for each $\alpha_n \in N_{x^{n+1}(0),x^n(0)} U_{x^{n+1}(0)}^{\mathrm{univ}}$, we can find $\alpha_{n+1} \in N_{x^{n+2}(0),x^{n+1}(0)} U_{x^{n+2}(0)}^{\mathrm{univ}}$ such that

$$\alpha_n = N_{x^{n+1}(0),x^n(0)} \alpha_{n+1}.$$

This gives rise to a norm compatible sequence $\{\alpha_n\}_{n \in \mathbb{N} \cup \{0\}}$ with $\alpha_0 = \alpha$. This shows that

$$(U_k^{\mathrm{univ}})^{\mathrm{univ}} = \bigcap_n N_n U_n(p)^{\mathrm{univ}} = U_k(p)^{\mathrm{comp}}$$

and

$$\mathrm{T}_p(k)^{\Gamma} \cong \frac{U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{N_{x(0)}(U_{x(0)}(p)^{\mathrm{univ}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p} = \frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{comp}}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Again, since $(U_k(p)^{\mathrm{univ}})^{p^{x(0)}} \subset N_{x(0)}(U_{x(0)}(p)^{\mathrm{univ}})$, it follows that $\frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{comp}}}$ is a p-primary group and hence

$$\mathrm{T}_p(k)^{\Gamma} \cong \frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{comp}}} = \frac{k^{\mathrm{univ}}}{k^{\mathrm{comp}}}.$$

This together with Corollary 2.3 completes the proof of Theorem 2.8. ∎

The assumption of finiteness of $\mathrm{T}_p(k_n)^{\Gamma_n}$ is equivalent to the Gross conjecture. This is explained in §1 of [6] and references therein.

COROLLARY 2.10. *Let p be an odd prime and let $k_\infty = \bigcup_n k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of a number field k containing a primitive pth root of unity. If the Gross conjecture is true for $\{k_n\}_n$, then there exists an exact sequence*

$$1 \to \frac{\Theta_k \cap k^{\mathrm{univ}}}{k^{\mathrm{comp}}} \to \mathrm{T}_p(k)^{\Gamma} \to \Psi_k/\Theta_k \to \Psi_k/k^{\mathrm{univ}}\Theta_k \to 1.$$

For an abelian field k, we have the following corollary.

COROLLARY 2.11. *Let p be an odd prime and let k be an abelian field containing a primitive pth root of unity. Then there exists an exact sequence*

$$1 \to \frac{\Theta_k \cap k^{\mathrm{univ}}}{k^{\mathrm{comp}}} \to \mathrm{T}_p(k)^{\Gamma} \to \Psi_k/\Theta_k \to \Psi_k/k^{\mathrm{univ}}\Theta_k \to 1.$$

Proof. Since k is abelian, the composite field $k_n = k\mathbb{Q}_n$ is also abelian for all $n \geq 0$. Since an abelian field satisfies the Gross conjecture (cf. §3 of Ch. II in [5]), the exact sequence above follows from Theorem 2.8. ∎

The commutativity of "comp" and tensor product is a strong condition. Lemma 3.2 shows that this condition is satisfied for an abelian field k when there is only one prime of k dividing p.

LEMMA 2.12. *For all $r \geq 0$, if $\bigcap_{n\geq r} N_n(U_n(p)\otimes_{\mathbb{Z}}\mathbb{Z}_p) = (\bigcap_{n\geq r} N_n U_n(p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, then*

$$(U_r(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}} = U_r(p)^{\mathrm{comp}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Proof. By a compactness argument, the following identity holds:

$$(U_r(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}} = \bigcap_{n\geq r} N_n(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

By assumption, we have $\bigcap_{n=r}^{\infty}(N_n(U_n(p))\otimes_{\mathbb{Z}}\mathbb{Z}_p) = (\bigcap_{n=r}^{\infty} N_n U_n(p))\otimes_{\mathbb{Z}}\mathbb{Z}_p = U_r(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. For $m \geq n \geq r$, the norm map $N_{m,n}$ obviously maps $(U_m(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}}$ onto $(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{comp}}$. This leads to the surjection

$$N_{m,n} : U_m(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \twoheadrightarrow U_n(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Hence, it follows from

$$U_n(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = N_{m,n}(U_m(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = (N_{m,n}U_m(p)^{\mathrm{univ}}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

that

$$1 = \frac{U_n(p)^{\mathrm{univ}}}{N_{m,n}U_m(p)^{\mathrm{univ}}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

However, we know the following inclusion $U_m(p)^{\mathrm{univ}} \supset U_n(p)^{\mathrm{univ}}$. Since

$$U_n(p)^{\mathrm{univ}} \supset N_{m,n}U_m(p)^{\mathrm{univ}} \supset (U_n(p)^{\mathrm{univ}})^{p^{m-n}}$$

it follows that

$$U_n(p)^{\mathrm{univ}} = N_{m,n}U_m(p)^{\mathrm{univ}}.$$

By applying the above equality for all successive pairs $m \geq n$ $(\geq r)$, we can construct a norm compatible sequence from a universal norm element of $U_r(p)^{\mathrm{univ}}$. This leads to $U_n(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{comp}}$ and hence

$$(U_r(p)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\mathrm{comp}} = (U_r(p)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\mathrm{univ}} = U_r(p)^{\mathrm{univ}}\otimes_{\mathbb{Z}}\mathbb{Z}_p = U_r(p)^{\mathrm{comp}}\otimes_{\mathbb{Z}}\mathbb{Z}_p.$$

This completes the proof. ∎

REMARK. We remark that our arguments can be carried out inside p-adic fields when Leopoldt's conjecture holds. Let $U_\infty(p)[p] = \varprojlim_n (U_n(p) \otimes \mathbb{Z}_p)$ denote the inverse limit of $U_n(p) \otimes \mathbb{Z}_p$ with respect to the norm maps. Modulo Leopoldt's conjecture, we have the inclusion $U_n(p) \otimes \mathbb{Z}_p \hookrightarrow \prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}[p]$, where $k_{n,\mathfrak{p}}[p]$ is the profinite p-completion of $k_{n,\mathfrak{p}}$. By taking inverse limits, we have $U_\infty(p)[p] \hookrightarrow \prod_{\mathfrak{p}|p} \varprojlim_n (k_{n,\mathfrak{p}}[p])$. Let $\overline{M}$ denote the image of a Galois module M under this inclusion. For instance, $\overline{U}_n(p)$ denotes the image of $U_n(p)[p]$ in $\prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}[p]$ which is isomorphic to $U_n(p)[p]$ etc. Lemma 2.12 becomes

$$\overline{\pi(\varprojlim_n U_n(p))} = \pi(\varprojlim_n \overline{U}_n(p)).$$

Typical examples of fields k such that k_n satisfies Leopoldt's conjecture are the subfields of K where K is either an abelian extension of $\mathbb{Q}$ or of an imaginary quadratic field.

For the corollary to follow, we may assume the following condition which loses essentially no generality. We write $k = F(\zeta_p)$ with $F/\mathbb{Q}$ unramified at p and write σ_p for the Frobenius element of p. Then $k_n = F(\zeta_{p^{n+1}})$. For each prime $\mathfrak{p}_{n+1}$ in k_{n+1} and $\mathfrak{p}_n$ in k_n with $\mathfrak{p}_{n+1} \,|\, \mathfrak{p}_n$ lying over p, we can identify $G(k_{n+1}/k_n)$ with its decomposition group $G(k_{n+1,\mathfrak{p}_{n+1}}/k_{n,\mathfrak{p}_n})$ since each prime lying over p is totally ramified at k_{n+1}/k_n, where $k_{n,\mathfrak{p}_n}$ denotes the completion of k_n at the prime $\mathfrak{p}_n$ and similarly for $k_{n,\mathfrak{p}_{n+1}}$. Hence we have the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & U_{n+1}(p) & \longrightarrow & \prod_{\mathfrak{p}'|p} k^{\times}_{n+1,\mathfrak{p}'} = (k_{n+1} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times} \\ & & {\scriptstyle N_{n+1,n}}\Big\downarrow & & {\scriptstyle \prod_{\mathfrak{p}'|p} N_{n+1,n}}\Big\downarrow \\ 0 & \longrightarrow & U_n(p) & \longrightarrow & \prod_{\mathfrak{p}|p} k^{\times}_{n,\mathfrak{p}} = (k_n \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times} \end{array}$$

where the horizontal arrow denotes the diagonal embedding $U_n(p) \hookrightarrow \prod_{\mathfrak{p}|p} k^{\times}_{n,\mathfrak{p}}$. By taking inverse limits, we have

$$\varprojlim U_n(p) \hookrightarrow \varprojlim \prod_{\mathfrak{p}|p} k^{\times}_{n,\mathfrak{p}} = \varprojlim \, (k_n \otimes_{\mathbb{Q}} \mathbb{Q}_p)^{\times}$$

since the projective limit is a left exact functor. On the other hand, we have similar settings over the profinite p-completions to which we can associate Coleman's power series. Let $\overline{U}_n(p)$ be the topological closure of the image of $U_n(p)$ under the diagonal map. Modulo Leopoldt's conjecture, $\overline{U}_n(p)$ is equal to the image of the induced injection map

$$1 \to U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}[p]^{\times}.$$

By taking inverse limits, we have

$$\varprojlim U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow \varprojlim \overline{U}_n(p) \subset \prod_{\mathfrak{p}|p} \varprojlim k_{n,\mathfrak{p}}[p]^{\times}.$$

Let $\mathcal{O}_F$ denote the ring of integers of F. Then $\mathcal{O}_F$ is identified with the diagonal embedding via

$$\mathcal{O}_F \hookrightarrow \widehat{\mathcal{O}}_F = \prod_{\mathfrak{p}|p} \mathcal{O}_{F_{\mathfrak{p}}} = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

where $\mathcal{O}_{F_{\mathfrak{p}}}$ is the ring of integers of the completion $F_{\mathfrak{p}}$ of F at a prime $\mathfrak{p}$ lying over p. Lemma 2.12 leads to the following corollary of Coleman's theorem for the case of the p-adic completion of norm compatible p-units.

COROLLARY 2.13. *Under the hypothesis of Lemma 2.12, let (β_n) be an element of $\varprojlim(U_n(p)[p])$, where the inverse limit is taken with respect to the norm maps. Then there is a unique Laurent series $\tilde{f} \in \widehat{\mathcal{O}}_F((X)) \otimes \mathbb{Z}_p$ such that*

$$\tilde{f}(\zeta_{p^{n+1}} - 1) = \beta_n^{\sigma_p^n} \quad \text{and} \quad \mathcal{N}\tilde{f} = \tilde{f}^{\sigma_p}$$

where $\mathcal{N}$ denotes Coleman's norm operator.

Proof. Recall the following theorem of Coleman as stated in Theorem 4.1 of [11], which follows from Theorem 16 and Corollary 17 of [2]:

THEOREM (Coleman). *Let (α_n) be an element of $\varprojlim\,(k_n \otimes \mathbb{Q}_p)^\times$, where the inverse limit is taken with respect to the norm maps. Then there is a unique Laurent series f in $\widehat{\mathcal{O}}_F((X))$ such that*

$$f(\zeta_{p^{n+1}} - 1) = \alpha_n^{\sigma_p^n} \quad \text{and} \quad \mathcal{N}f = f^{\sigma_p}.$$

It follows from Lemma 2.12 and Coleman's theorem that for each element $(\beta_n)_n$ of $\varprojlim(U_n(p)[p])$, there exists a Laurent series $f \in \widehat{\mathcal{O}}_F((X))$ and $\tilde{r} \in \mathbb{Z}_p$ such that $f \otimes \tilde{r} \in \widehat{\mathcal{O}}_F((X)) \otimes \mathbb{Z}_p$ and $(f \otimes \tilde{r})(\zeta_{p^{n+1}} - 1) = \beta_n^{\sigma_p^n}$ for all n. In fact, by the commutativity of the comp-functor and p-profinite completions,

$$\pi(\varprojlim(U_n(p)[p])) = \pi(U_\infty(p)[p]) = \pi(U_\infty(p))[p],$$

we can find $f \in \widehat{\mathcal{O}}_F((X))$ which is independent of n such that

$$f(\zeta_{p^{n+1}} - 1) \otimes \tilde{r}_n = \beta_n^{\sigma_p^n}$$

where $\tilde{r}_n \in \mathbb{Z}_p$. By the norm compatible property of β_n and $f(\zeta_{p^{n+1}} - 1)$, we can take $\tilde{r}_n$ as a fixed element $\tilde{r} \in \mathbb{Z}_p$ which is also independent of n. We finish the proof by letting $\tilde{f} = f \otimes \tilde{r} \in \widehat{\mathcal{O}}_F((X)) \otimes \mathbb{Z}_p$. ∎

3. Cohomology groups of the group of norm compatible elements. Kuz'min already showed that local norm compatible groups are cohomologically trivial. More precisely, for a prime divisor v of k lying over p, let $k_{v,\infty} = \bigcup_n k_{v,n}$ denote the cyclotomic $\mathbb{Z}_p$-extension of k_v. Let $\widehat{k}_{v,n}$ denote the p-adic completion of $k_{v,n}^\times$ and let $k_{v,n}^{\mathrm{comp}}$ denote the subgroup of $\widehat{k}_{v,n}$ which corresponds to $k_{v,\infty}$ via class field theory, i.e.,

$$\frac{\widehat{k}_{v,n}}{k_{v,n}^{\mathrm{comp}}} \cong G(k_{v,\infty}/k_{v,n}).$$

Then both $k_{v,n}^{\mathrm{comp}}$ and the torsion-free part $k_{v,n}^{\mathrm{comp}}/\mathrm{tor}_{\mathbb{Z}}(k_{v,n}^{\mathrm{comp}})$ are cohomologically trivial (cf. pp. 293–294 of [7]).

Over finitely generated $\mathbb{Z}_p$-modules, the "comp"-functor behaves in the following way. For $n \geq 1$ let $1 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 1$ be exact sequences of $\mathbb{Z}_p[G(k_{v,n}/k_v)]$-modules which are finitely generated as $\mathbb{Z}_p$-modules such that the following diagram commutes for all $m \geq n$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A_m & \xrightarrow{f_m} & B_m & \xrightarrow{g_m} & C_m & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N_{m,n}} & & \downarrow{\scriptstyle N_{m,n}} & & \downarrow{\scriptstyle N_{m,n}} & & \\ 1 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 1 \end{array}$$

Taking inverse limits with respect to the norm maps and using a compactness argument yields the exact sequence

$$1 \to \varprojlim_n A_n \xrightarrow{f_\infty} \varprojlim_n B_n \xrightarrow{g_\infty} \varprojlim_n C_n \to 1.$$

From the commutativity of the diagrams and Lemma 3.7 of [9], it follows that for each $n \geq 0$, there exist well defined restriction maps of f_n, g_n to the groups $A_n^{\text{comp}}, B_n^{\text{comp}}$ of norm compatible elements which induce the semi-exact sequence

$$1 \to A_n^{\text{comp}} \xrightarrow{f_n} B_n^{\text{comp}} \xrightarrow{g_n} C_n^{\text{comp}} \to 1.$$

More precisely, it can be shown easily that f_n is injective, $g_n(f_n(A_n^{\text{comp}})) = 1$ and

$$g_n(B_n^{\text{comp}}) = C_n^{\text{comp}}.$$

For the norm compatible elements of global fields, the "comp"-functor behaves less nicely than in the case of local fields since they are not compact and the cohomology groups are different from those of local fields. It seems difficult to find explicit computations for these cohomology groups. For the topological generator $\tau_n = \tau^{p^n}$ of $G(k_\infty/k_n)$, we denote its restriction to k_{n+i} by $\sigma_{n+i} = \tau_n|_{k_{n+i}}$, which is a generator of $G(k_{n+i}/k_n)$. Suppose that $\alpha_{n+i} \in k_{n+i}^{\text{comp}}$ and $N_{n+i,n}(\alpha_{n+i}) = 1$ where for $s \geq t$, $N_{s,t}$ denotes the norm map from k_s to k_t as defined in §2. By Hilbert's Theorem 90, there is $\beta_{n+i} \in k_{n+i}$ such that $\alpha_{n+i} = \beta_{n+i}^{\sigma_{n+i}-1}$. Since $\alpha_{n+i} \in k_{n+i}^{\text{comp}}$, for $j \geq i$ there exist $\alpha_{n+j}, \beta_{n+j} \in k_{n+j}$ such that $N_{n+j,n+i}(\alpha_{n+j}) = \alpha_{n+i}$ and $\alpha_{n+j} = \beta_{n+j}^{\sigma_{n+j}-1}$. Hence it follows from $(N_{n+j,n+i}\beta_{n+j})^{\sigma_{n+i}-1} = \beta_{n+i}^{\sigma_{n+i}-1}$ that

$$N_{n+j,n+i}\beta_{n+j} = \beta_{n+i} \bmod k_n^\times.$$

Let $\tilde{k}_{n+i}^{\text{comp}} = \pi(\varprojlim_{j\geq i}(k_{n+j}^\times/k_n^\times))$ denote the image of the natural projection

$$\pi : \varprojlim_{j\geq i}(k_{n+j}^\times/k_n^\times) \to k_{n+i}^\times/k_n^\times.$$

From the above argument, we have

$$(k_{n+i})^{\sigma_{n+i}-1} \cap k_{n+i}^{\text{comp}} = (\tilde{k}_{n+i}^{\text{comp}})^{\sigma_{n+i}-1}.$$

Hence, the one-dimensional cohomology group of the group of norm compatible elements is given by

$$H^1(G(k_{n+i}/k_n), k_{n+i}^{\mathrm{comp}}) = \frac{(\tilde{k}_{n+i}^{\mathrm{comp}})^{\sigma_{n+i}-1}}{(k_{n+i}^{\mathrm{comp}})^{\sigma_{n+i}-1}}.$$

Set

$$\Delta = \Big\{(x_{n+i}x_{n+i+1}^{-p}, x_{n+i+1}x_{n+i+2}^{-p}, \ldots) \in \prod_{j\geq i} k_n^{\times} \;\Big|\; x_{n+j} \in k_n^{\times} \text{ for all } j \geq i\Big\}.$$

For each $(\overline{\alpha}_{n+j})_{j\geq i} \in \varprojlim_{j\geq i}(k_{n+j}^{\times}/k_n^{\times})$ with $\overline{\alpha}_{n+j} = \alpha_{n+j} \bmod k_n^{\times}$, we have a well defined map $\delta((\overline{\alpha}_{n+j})_{j\geq i}) = (a_{n+j})_{j\geq i} \bmod \Delta$ where for each representative α_t of $\overline{\alpha}_t$, a_t is defined as $(N_{t+1,t}\alpha_{t+1})\alpha_t^{-1} = a_t \in k_n^{\times}$, for $t \geq n_i$. This leads to the exact sequence

$$1 \to \varprojlim_{j\geq i} k_{n+j} \to \varprojlim_{j\geq i}(k_{n+j}^{\times}/k_n^{\times}) \xrightarrow{\delta} \frac{\prod_{j\geq i} k_n^{\times}}{\Delta}$$

where the first map is the natural projection. Moreover, for each $\overline{\alpha}_{n+i} \in \tilde{k}_{n+i}^{\mathrm{comp}}$, we note that $(\overline{\alpha}_{n+j}^{\tau_{n+i}-1})_{j\geq i}$ forms a norm compatible sequence in $\prod_{j\geq i} k_{n+j}$. Thus, it can be shown easily that

$$(\tilde{k}_{n+i}^{\mathrm{comp}})^{\tau_{n+i}-1} = (k_{n+i}^{\tau_{n+i}-1})^{\mathrm{comp}}.$$

We have proved the following lemma.

Lemma 3.1. *Let $k_\infty = \bigcup k_n$ be a $\mathbb{Z}_p$-extension of k. Then*

$$H^1(G(k_{n+i}/k_n), k_{n+i}^{\mathrm{comp}}) = \frac{(k_{n+i}^{\sigma_{n+i}-1})^{\mathrm{comp}}}{(k_{n+i}^{\mathrm{comp}})^{\sigma_{n+i}-1}} = \frac{(U_{n+i}(p)^{\sigma_{n+i}-1})^{\mathrm{comp}}}{(U_{n+i}(p)^{\mathrm{comp}})^{\sigma_{n+i}-1}},$$

$$H^0(G(k_{n+i}/k_n), k_{n+i}^{\mathrm{comp}}) = \frac{(k_{n+i}^{\mathrm{comp}})^{G(k_{n+i}/k_n)}}{k_n^{\mathrm{comp}}} = \frac{(U_{n+i}(p)^{\mathrm{comp}})^{G(k_{n+i}/k_n)}}{U_n(p)^{\mathrm{comp}}}.$$

We now proceed to an explicit computation under a certain condition. Let $k_\infty = \bigcup k_n$ be the cyclotomic $\mathbb{Z}_p$-extension of k. When $k/\mathbb{Q}$ is Galois and there is only one prime of k lying over p, it was remarked on page 529 of [1] that $k^{\mathrm{univ}} = U_k(p)$. Moreover, it was also mentioned there that if the class number of k is prime to p, then the norm map from k_n^{univ} to k_{n-1}^{univ} is surjective for all $n > 0$. The last statement amounts to the equality

$$k_n^{\mathrm{univ}} = k_n^{\mathrm{comp}} = U_n(p).$$

When k is abelian, we can remove the condition that the class number of k is prime to p.

Lemma 3.2. *Let k be an arbitrary abelian field containing a primitive pth root of unity and there is only one prime of k lying over p. For the* *cyclotomic $\mathbb{Z}_p$-extension $k_\infty = \bigcup k_n$ we have, for all $n \geq 0$,*

$$k_n^{\mathrm{univ}} = k_n^{\mathrm{comp}} = U_n(p).$$

Proof. Let $\mathbb{Q}_n$ denote the intermediate field of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ such that $[\mathbb{Q}_n : \mathbb{Q}] = p^n$. Let $p^r s$ denote the conductor of k. Let $F = \mathbb{Q}(\mu_{p^r}) \cap k$. We fix a prime $\mathfrak{p}$ of k_n. For a subfield L of k_n, we will write $L_\mathfrak{p}$ for the local field which is completed at $\mathfrak{p}_L = \mathfrak{p} \cap \mathcal{O}_L$, the corresponding prime ideal inside the ring $\mathcal{O}_L$ of integers of L. In this setting, we have

$$[k : F] = [k_\mathfrak{p} : F_\mathfrak{p}]$$

since there is only one prime ideal of k lying above p. Since p is totally ramified over $F_n/\mathbb{Q}$, we have

$$[F_n : F] = [F_{n\mathfrak{p}} : F_\mathfrak{p}] \quad \text{and} \quad [F : \mathbb{Q}] = [F_\mathfrak{p} : \mathbb{Q}_p].$$

Then $\mathbb{Q}(\mu_{p^r})$ and F_n are linearly disjoint over F since $\mathbb{Q}(\mu_{p^r})$ is totally ramified over F of order prime to p, and F_n is totally ramified over F of p-power order. Moreover, $\mathbb{Q}(\mu_{p^r s})$ and $F_n\mathbb{Q}(\mu_{p^r})$ are linearly disjoint over $\mathbb{Q}(\mu_{p^r})$ since $\mathbb{Q}(\mu_{p^r s})$ is unramified over $\mathbb{Q}(\mu_{p^r})$ and $F_n\mathbb{Q}(\mu_{p^r})$ is totally ramified over $\mathbb{Q}(\mu_{p^r})$. Thus, $\mathbb{Q}(\mu_{p^r s})$ and F_n are linearly disjoint over F and hence its intermediate field k and F_n are also linearly disjoint over F.

$$\begin{tikzcd}
 & F_n(\mu_{p^r s}) \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathbb{Q}(\mu_{p^r s}) \arrow[dr, dash] & & k_n \arrow[dl, dash] \arrow[dr, dash] & \\
 & k \arrow[dr, dash] & & F_n \arrow[dl, dash] \\
 & & F &
\end{tikzcd}$$

The linear disjointness of the global fields described above also remains true over the corresponding local fields completed at the prime $\mathfrak{p}$. We have

$$[k_n : F] = [k : F][F_n : F] = [k_\mathfrak{p} : F_\mathfrak{p}][F_{n\mathfrak{p}} : F_\mathfrak{p}] = [k_{n\mathfrak{p}} : F_\mathfrak{p}],$$

which leads to $[k_n : \mathbb{Q}] = [k_{n\mathfrak{p}} : \mathbb{Q}_p]$ for all $n \geq 0$. Hence, there is only one prime lying over p in k_n for all $n \geq 0$. We apply again the remark on page 529 of [1] to obtain the equalities $k_n^{\mathrm{univ}} = k_n^{\mathrm{comp}} = U_n(p)$ for all $n \geq 0$ as desired. ∎

Under the same hypotheses of Lemma 3.2, it follows from Lemmas 3.1 and 3.2 that

$$H^0(G(k_m/k_n), k_m^{\mathrm{comp}}) = \frac{(U_m(p)^{\mathrm{comp}})^{G(k_m/k_n)}}{U_n(p)^{\mathrm{comp}}} = 1.$$

The one-dimensional cohomology is also trivial either by a direct computation of Lemma 3.1 or by computing the Herbrand quotient $Q(k_n^{\mathrm{comp}})$ of

k_n^{comp} which is given by

$$Q(k_n^{\mathrm{comp}}) = \frac{1}{(k_n : k)} \prod_{v \in S} (k_{n,v} : k_v) = 1$$

from the Herbrand quotient $Q(U_n(p))$ (cf. Corollary 2 of Ch. IX, §4 of [8]) of $U_n(p)$. Since the Galois group $G(k_m/k_n)$ is cyclic for all $m \geq n \geq 0$, under the hypotheses of Lemma 3.2 we have, for all $s \geq 0$,

$$H^s(G(k_m/k_n), k_m^{\mathrm{comp}}) = 1.$$

Acknowledgements. We would like to thank the referee for his helpful comments and suggestions.

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00007).

References

[1] F. Bertrandias et J.-J. Payan, *Γ-extensions et invariants cyclotomiques*, Ann. Sci. École Norm. Sup. 5 (1972), 517–543.
[2] R. Coleman, *Division values in local fields*, Invent. Math. 53 (1979), 96–116.
[3] R. Greenberg, *On a certain l-adic representation*, ibid. 21 (1973), 117–124.
[4] B. H. Gross, *p-adic L-series at $s = 0$*, J. Fac. Sci. Univ. Tokyo IA 28 (1981), 919–994.
[5] J.-F. Jaulent, *L'arithmetique des l-extensions*, Thèse, Univ. de Franche Comté, 1986.
[6] M. Kolster, *An idelic approach to the wild kernel*, Invent. Math. 103 (1991), 9–24.
[7] L. V. Kuz'min, *The Tate module of algebraic number fields*, Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972), 267–327 (in Russian).
[8] S. Lang, *Algebraic Number Theory*, Springer, 1994.
[9] S. Seo, *On first layers of Z_p-extensions*, http://asarc.kaist.ac.kr/ (recent papers).
[10] —, *On the Tate module of a number field*, http://asarc.kaist.ac.kr/ (recent papers).
[11] T. Tsuji, *Semi-local units modulo cyclotomic units*, J. Number Theory 78 (1999), 1–26.

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Received on 1.2.2010
and in revised form on 31.5.2011 (6293)