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Inhomogeneous Diophantine approximation on integer polynomials with non-monotonic error function

by

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1. Introduction and statements. In this paper we consider the problem of approximating real numbers by polynomials with a non-monotonic error function. First some notation is needed. Throughout, $P \in \mathbb{Z}[x]$ given by

$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$

is an integer polynomial with degree $\deg P = n$ and height

$$H(P) = \max_{0 \le j \le n} |a_j|.$$

Further, let $\mathcal{P}_n = \{P \in \mathbb{Z}[x] : \deg P \le n\}$ and

$$\mathcal{P}_n(H) = \{P \in \mathcal{P}_n : H(P) = H\}.$$

The Lebesgue measure of a measurable set $A \subset \mathbb{R}$ is denoted by $\mu(A)$. By $\ll$ and $\gg$ we will mean the Vinogradov symbols with implicit constants depending only on n.

In what follows, d is a fixed real number. Define a real-valued function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ and denote by $\mathcal{L}_{n,d}(\Psi)$ the set of $x \in \mathbb{R}$ such that the inequality

$$|P(x) + d| < \Psi(H(P)) \tag{1.1}$$

has infinitely many solutions $P \in \mathcal{P}_n$. The set $L_{n,d}(\Psi)$ consists of points satisfying an inhomogeneous Diophantine inequality. The homogeneous case is when $d \in \mathbb{Q}$ and the corresponding set is denoted by $\mathcal{L}_n(\Psi)$.

The main result of this paper is the following statement.

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THEOREM 1.1. *For $n \geq 2$,*

$$\mu(\mathcal{L}_{n,d}(\Psi)) = 0$$

if the sum $\sum_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges.

There are many results regarding this problem when Ψ is monotonic and $d \in \mathbb{Q}$. For $\Psi(H) = H^{-w}$, $w > n$, and $d \in \mathbb{Q}$ the theorem was proved by Sprindžuk [14]. For a general monotonic function Ψ such that $\sum_{h=1}^{\infty} \Psi^{1/n}(h) < \infty$ and $d \in \mathbb{Q}$ it was proved by Baker [2] who further conjectured that $\mu(L_n(\Psi)) = 0$ if the sum $\sum_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges. This was proved in 1989 by Bernik [8], and later Beresnevich [3] proved the corresponding divergence result. The first time that inequality (1.1) for any $d \in \mathbb{R}$ was considered was in [9] and a similar question in the p-adic case was answered in [10].

The above problems can be considered as questions concerning Diophantine approximation on the Veronese curve $\mathcal{V}_n = \{(x, x^2, \ldots, x^n) : x \in \mathbb{R}\}$. Regarding more general curves and surfaces, in 1998 Kleinbock and Margulis [13] established the Baker–Sprindžuk conjecture concerning homogeneous Diophantine approximation on manifolds. An inhomogeneous version was then proved by Beresnevich and Velani [7]. The significantly stronger Groshev type theory for dual Diophantine approximation on manifolds was established in [4], [6], and [11] for the homogeneous case and in [1] for the inhomogeneous case. In all of these results the function Ψ was assumed to be monotonic. In 2005 Beresnevich [5] proved Theorem 1.1 above without the condition that Ψ is monotonic for $d \in \mathbb{Q}$; he conjectured that the result should also hold for any non-degenerate curve in Euclidean space. This was proved in [12]. Here we extend this last result to the inhomogeneous setting for the Veronese curve $\mathcal{V}_n$. Note that using results from [12] (by taking $\mathbf{f} = (1, x, x^2, \ldots, x^n, d)$ and $\mathbf{a} = (a_0, a_1, \ldots, a_n, 1)$) we obtain

$$\mu(\mathcal{L}_{n,d}(\Psi)) = 0$$

if $\sum_{h=1}^{\infty} h^n\Psi(h) < \infty$. In Theorem 1.1 it is shown that this convergence condition can be weakened to $\sum_{h=1}^{\infty} h^{n-1}\Psi(h) < \infty$.

2. Proof of Theorem 1.1. First note that since $\sum_{h=1}^{\infty} h^{n-1}\Psi(h)$ converges, $h^{n-1}\Psi(h)$ tends to 0 as $h \to \infty$. Therefore,

$$\Psi(h) = o(h^{-n+1}). \tag{2.1}$$

Fix an arbitrary constant $0 < \theta < 1$. As the set of points x satisfying $|x| < \theta$ is arbitrarily small, without loss of generality it will be assumed from now on that

$$|x| \geq \theta. \tag{2.2}$$

Also note that $\mu(\mathcal{L}_{n,d}(\Psi)) = 0$ if $\mu(\mathcal{L}_{n,d}(\Psi) \cap I) = 0$ for each open interval I. Again, without loss of generality (only the constants change), fix the interval $I = (\theta, 1)$.

The next lemma will be used repeatedly.

LEMMA 2.1 (Borel–Cantelli). *Let (X, μ) be a measure space. Let A_i for $i = 1, 2, \ldots$ be a sequence of sets such that $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Then the set of points lying in infinitely many A_i has measure zero.*

The proof is now split into two parts and the following two sets are considered. Fix a real number v satisfying

$$0 < v < 1/3. \tag{2.3}$$

Define

$$\mathcal{L}_1(n, d) = \{x \in I : |P(x)+d| < H(P)^{-n+1},\ |P'(x)| < H(P)^{-v} \text{ i.m. } P \in \mathcal{P}_n\}$$

and

$$\mathcal{L}_2(n, d, \Psi) = \{x \in I : \\ |P(x) + d| < \Psi(H(P)),\ |P'(x)| \geq H(P)^{-v} \text{ i.m. } P \in \mathcal{P}_n\}$$

where i.m. should be read *for infinitely many*. Clearly, from (2.1),

$$\mathcal{L}_{n,d}(\Psi) \subset \mathcal{L}_1(n, d) \cup \mathcal{L}_2(n, d, \Psi).$$

It will be shown that each of the sets $\mathcal{L}_1(n, d)$ and $\mathcal{L}_2(n, d, \Psi)$ has Lebesgue measure zero.

2.1. The case of small derivative

PROPOSITION 2.2. *Let $n \geq 2$. Then $\mu(\mathcal{L}_1(n, d)) = 0$.*

First $\mathcal{L}_1(n, d)$ is written as a limsup set. For $P \in \mathcal{P}_n$ define

$$B(P) = \{x \in \mathbb{R} : |P(x) + d| < H(P)^{-n+1},\ |P'(x)| < H(P)^{-v}\}.$$

Then

$$\mathcal{L}_1(n, d) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \bigcup_{P \in \mathcal{P}_n^t} B(P),$$

where

$$\mathcal{P}_n^t := \{P \in \mathcal{P}_n : 2^t \leq H(P) < 2^{t+1}\}.$$

To prove the proposition it will be shown that a larger set (containing $\mathcal{L}_1(n, d)$) has measure zero and then the Inhomogeneous Transference Principle proved in [7] will be used. The Inhomogeneous Transference Principle allows the transfer of zero measure statements for homogeneous lim sup sets to inhomogeneous lim sup sets and is described below.

2.1.1. *Inhomogeneous Transference Principle.* Most of this section is adapted from [7]. For our purposes the two countable indexing sets $\mathbf{T}$ and $\mathcal{A}$ from [7] are the sets $\mathbf{T} = \mathbb{N} \cup \{0\}$ and $\mathcal{A} = \mathcal{P}_n$. Throughout, J denotes a finite open interval in $\mathbb{R}$ with closure denoted by $\bar{J}$. Let $\mathcal{H}$ and $\mathcal{I}$ be two maps from $(\mathbb{N} \cup \{0\}) \times \mathcal{P}_n \times \mathbb{R}$ into the set of open subsets of $\mathbb{R}$ such that

$$\mathcal{H}(t, P, \varepsilon) = \mathcal{I}_0^t(P, \varepsilon) \quad \text{and} \quad \mathcal{I}(t, P, \varepsilon) = \mathcal{I}_d^t(P, \varepsilon).$$

For the specific case considered in this article the sets $\mathcal{I}_0^t(P, \varepsilon)$ and $\mathcal{I}_d^t(P, \varepsilon)$ are defined as follows:

$$\mathcal{I}_d^t(P, \varepsilon) = \begin{cases} \{x \in I : |P(x) + d| < 2^{t(-n+1)}\varepsilon,\ |P'(x)| < 2^{-tv}\varepsilon\} & \text{if } P \in \mathcal{P}_n^t, \\ \emptyset & \text{else,} \end{cases}$$

and

$$(2.4) \qquad \mathcal{I}_0^t(P, \varepsilon) = \begin{cases} \{x \in I : |P(x)| < 2^{t(-n+1)}\varepsilon,\ |P'(x)| < 2^{-tv}\varepsilon\} & \text{if } P \in \bigcup_{s=0}^{t+1} \mathcal{P}_n^s, \\ \emptyset & \text{else.} \end{cases}$$

Let $\delta > 0$ and define the function $\phi_\delta(t) = 2^{\delta t}$. Also, define $\varPhi = \{\phi_\delta : 0 \le \delta < v/2\}$. For any $\phi \in \varPhi$ define

$$\mathcal{I}_d^t(\phi) = \bigcup_{P \in \mathcal{P}_n} \mathcal{I}_d^t(P, \phi(t)) = \bigcup_{P \in \mathcal{P}_n^t} \mathcal{I}_d^t(P, \phi(t))$$

and denote by $\varLambda_{\mathcal{I}}(\phi)$ the lim sup set

$$\varLambda_{\mathcal{I}}(\phi) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \mathcal{I}_d^t(\phi).$$

In order to use the Inhomogeneous Transference Principle from [7] we also define the homogeneous lim sup set

$$\varLambda_{\mathcal{H}}(\phi) = \bigcap_{N=1}^{\infty} \bigcup_{t=N}^{\infty} \mathcal{I}_0^t(\phi),$$

where

$$\mathcal{I}_0^t(\phi) = \bigcup_{P \in \mathcal{P}_n} \mathcal{I}_0^t(P, \phi(t)) = \bigcup_{s=0}^{t+1} \bigcup_{P \in \mathcal{P}_n^s} \mathcal{I}_0^t(P, \phi(t)).$$

Clearly, for any $0 \le \delta < v/2$,

$$\mathcal{L}_1(n, d) \subset \varLambda_{\mathcal{I}}(\phi_\delta)$$

holds. The use of the Transference Principle depends on the following two properties being satisfied.

Intersection property. Let $\varPhi$ denote a set of functions $\phi : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$. The triple $(\mathcal{H}, \mathcal{I}, \varPhi)$ is said to have the *intersection property* if for any $\phi \in \Phi$ there exists $\phi^* \in \Phi$ such that for all but finitely many $t \in \mathbb{N} \cup \{0\}$ and all distinct $P, \tilde{P} \in \mathcal{P}_n$,

$$\mathcal{I}_d^t(P, \phi(t)) \cap \mathcal{I}_d^t(\tilde{P}, \phi(t)) \subset \mathcal{I}_0^t(\phi^*). \tag{2.5}$$

CONTRACTING PROPERTY. Let $\{k_t\}_{t\in\mathbb{N}}$ be a sequence of positive numbers such that

$$\sum_{t\in\mathbb{N}\cup\{0\}} k_t < \infty. \tag{2.6}$$

The measure μ is said to be *contracting with respect to* $(\mathcal{I}, \Phi)$ if for any $\phi \in \Phi$ there exists $\phi^+ \in \Phi$ such that for all but finitely many t and all $P \in \mathcal{P}_n$ there exists a collection $C_{t,P}$ of balls B centred in $\bar{J}$ satisfying the following three conditions:

$$\bar{J} \cap \mathcal{I}_d^t(P, \phi(t)) \subset \bigcup_{B\in C_{t,P}} B, \tag{2.7}$$

$$\bar{J} \cap \bigcup_{B\in C_{t,P}} B \subset \mathcal{I}_d^t(P, \phi^+(t)), \tag{2.8}$$

$$\mu\big(5B \cap \mathcal{I}_d^t(P, \phi(t))\big) \le k_t \mu(5B). \tag{2.9}$$

We now state the theorem from [7].

THEOREM 2.3 (Inhomogeneous Transference Principle). *Suppose that* $(\mathcal{H}, \mathcal{I}, \Phi)$ *has the intersection property and that* μ *is contracting with respect to* $(\mathcal{I}, \Phi)$. *If* $\mu(\Lambda_{\mathcal{H}}(\phi)) = 0$ *for all* $\phi \in \Phi$, *then* $\mu(\Lambda_{\mathcal{I}}(\phi)) = 0$ *for all* $\phi \in \Phi$.

First the contracting and intersection properties are verified and then it will be shown that $\mu(\Lambda_{\mathcal{H}}(\phi_\delta)) = 0$. This will imply, using the Transference Principle, that $\Lambda_{\mathcal{I}}(\phi_\delta)$ has measure zero and further that $\mu(\mathcal{L}_1(n, d)) = 0$ as required.

2.1.2. *Verifying the intersection property.* Let $t \in \mathbb{N} \cup \{0\}$ and $P, \tilde{P} \in \mathcal{P}_n$ with $P \ne \tilde{P}$. Suppose that

$$x \in \mathcal{I}_d^t(P, \phi_\delta(t)) \cap \mathcal{I}_d^t(\tilde{P}, \phi_\delta(t)).$$

Then the following inequalities hold:

$$|P(x) + d| < \phi_\delta(t)2^{t(-n+1)} \quad \text{and} \quad |\tilde{P}(x) + d| < \phi_\delta(t)2^{t(-n+1)},$$
$$|P'(x)| < \phi_\delta(t)2^{-vt} \quad \text{and} \quad |\tilde{P}'(x)| < \phi_\delta(t)2^{-vt}.$$

Let $R(x) = (P(x) + d) - (\tilde{P}(x) - d)$. Then

$$|R(x)| < 2\phi_\delta(t)2^{t(-n+1)} < \phi_{\delta'}(t)2^{t(-n+1)},$$
$$|R'(x)| < 2^{1-vt}\phi_\delta(t) < 2^{-vt}\phi_{\delta'}(t)$$

for all $t > 1/(v/2-\delta)$ and where $\phi_{\delta'} \in \Phi$. Clearly R cannot be constant for $n \geq 2$ and $t \geq 2$, so $R \in \bigcup_{s=0}^{t+1} \mathcal{P}_n^s$. Thus, $x \in \mathcal{I}_0^t(R, \phi_{\delta'}(t))$ and (2.5) is satisfied with $\phi^* = \phi_{\delta'}$.

2.1.3. *Verifying the contracting property.* The following lemma from [13, Lemma 3.1 and Proposition 3.2] will be used.

LEMMA 2.4. *Let $I \subset \mathbb{R}$, $T \in \mathbb{R}[x]$ be a polynomial of degree at most n and $K = \sup_{x\in I} |T(x)|$. Then*

$$\mu(\{x \in I : |T(x)| < \varepsilon\}) \leq 2n(n+1)^{1/n} K^{-1/n} \varepsilon^{1/n} \mu(I).$$

It clearly implies that there exists a constant $C > 0$ such that

$$\mu(\{x \in I : |\mathbf{F}_{t,P}(x)| < \varepsilon\}) \leq C\varepsilon^{1/n}\mu(I)$$

where

$$\mathbf{F}_{t,P}(x) := \max\{2^{t(n-1)}2^{-vt}|P(x)+d|, |P'(x)|\}.$$

By definition, for $P \in \mathcal{P}_n$,

$$\mathcal{I}_d^t(P, \phi_\delta(t)) = \begin{cases} \{x \in I : \mathbf{F}_{t,P}(x) < \phi_\delta(t)2^{-vt}\} & \text{if } P \in \mathcal{P}_n^t, \\ \emptyset & \text{else.} \end{cases} \tag{2.10}$$

Next, given $\phi_\delta \in \Phi$ let

$$\phi_\delta^+ := \phi_{(\delta+v/2)/2}.$$

Clearly, $\phi_\delta^+ \in \Phi$ and $\phi_\delta(t) \leq \phi_\delta^+(t)$ for all $t \in \mathbb{N}\cup\{0\}$; therefore,

$$\mathcal{I}_d^t(P, \phi_\delta(t)) \subset \mathcal{I}_d^t(P, \phi_\delta^+(t)). \tag{2.11}$$

Let J be a sufficiently small open interval such that $5J \subset I$. The collection $C_{t,P}$ will consist of intervals $B(x)$, each centred at a point $x \in J$, which satisfy conditions (2.6)–(2.9) for an appropriate sequence k_t; they are constructed in the following way. Let $P \in \mathcal{P}_n$. If $\mathcal{I}_d^t(P, \phi_\delta(t)) = \emptyset$ then $C_{t,P} = \emptyset$. Now assume that $\mathcal{I}_d^t(P, \phi_\delta(t)) \neq \emptyset$. By the definition of Φ and (2.3), it follows that

$$\mathcal{I}_d^t(P, \phi_\delta^+(t)) \subset \{x \in I : |P(x)+d| < 2^{-t(n-7/6)}\}.$$

By Lemma 2.4 and $\sup_{x\in 5J} |P(x)+d| > 0$,

$$\begin{aligned} \mu\big(\mathcal{I}_d^t(P, \phi_\delta^+(t)) \cap J\big) &\leq \mu(\{x \in J : |P(x)+d| < 2^{-t(n-7/6)}\}) \\ &\ll 2^{-t(1-7/6n)}\mu(J) \end{aligned}$$

for sufficiently large t. Hence,

$$J \not\subset \mathcal{I}_d^t(P, \phi_\delta^+(t)) \tag{2.12}$$

for sufficiently large t and $n \geq 2$.

By (2.11) and the fact that $\mathcal{I}_d^t(P,\phi_\delta^+(t))$ is open, for every $x \in \bar{J} \cap \mathcal{I}_d^t(P,\phi_\delta(t))$ there is an open interval $B'(x)$ containing x such that

$$B'(x) \subset \mathcal{I}_d^t(P,\phi_\delta^+(t)).$$

Hence, by (2.12), and the fact that J is bounded, there exists a scaling factor $\tau \geq 1$ such that the open interval $B(x) := \tau B'(x)$ satisfies

$$\begin{aligned} \bar{J} \cap B(x) &\subset \mathcal{I}_d^t(P,\phi_\delta^+(t)), \\ \bar{J} \cap 5B(x) &\not\subset \mathcal{I}_d^t(P,\phi_\delta^+(t)), \\ 5B(x) &\subset 5J. \end{aligned} \tag{2.13}$$

Let

$$C_{t,P} := \{B(x) : x \in \bar{J} \cap \mathcal{I}_d^t(P,\phi_\delta(t))\}.$$

By (2.13) and the construction, (2.7) and (2.8) are automatically satisfied. Consider any interval $B \in C_{t,P}$. By (2.10) and (2.13),

$$\sup_{x\in 5B} \mathbf{F}_{t,P}(x) \geq \sup_{x\in \bar{J}\cap 5B} \mathbf{F}_{t,P}(x) \geq \phi_\delta^+(t)2^{-vt}. \tag{2.14}$$

On the other hand, by (2.10),

$$\sup_{x\in \mathcal{I}_d^t(P,\phi_\delta(t))\cap 5B} \mathbf{F}_{t,P}(x) \leq \phi_\delta(t)2^{-vt}. \tag{2.15}$$

Let $\delta^* = \frac{1}{4}(v-2\delta) > 0$. Then, using (2.14), (2.15) and the definitions of ϕ_δ and ϕ_δ^+, we obtain

$$\sup_{x\in \mathcal{I}_d^t(P,\phi_\delta(t))\cap 5B} \mathbf{F}_{t,P}(x) \leq 2^{-\delta^* t} \sup_{x\in 5B} \mathbf{F}_{t,P}(x).$$

Again, from Lemma 2.4 it follows by (2.13) and (2.15) that

$$\begin{aligned} \mu(\mathcal{I}_d^t(P,\phi_\delta(t)) \cap 5B) &\leq \mu\Big(\Big\{x \in 5B : \mathbf{F}_{t,P}(x) \leq 2^{-\delta^* t} \sup_{x\in 5B} \mathbf{F}_{t,P}(x)\Big\}\Big) \\ &\leq C2^{-\delta^* t/n}\mu(5B) \end{aligned}$$

for sufficiently large t. This verifies (2.9) with $k_t := C2^{-\delta^* t/n}$ and it is easily seen that the convergence condition (2.6) is satisfied.

2.1.4. *Establishing* $\mu(\Lambda_{\mathcal{H}}(\phi_\delta)) = 0$. For this, Theorem 1.4 of [11] is used. In the notation of that paper take $\mathbf{f} = (x, x^2, \ldots, x^n)$, $d = 1$, $U = \mathbb{R}$ and $T_1 = \cdots = T_n = T$, to obtain the next result.

THEOREM 2.5 ([11]). *Let $x_0 \in I$. There exists an interval $J \subset I$ containing x_0 such that for any interval $B \subset J$ there exists a constant $E > 0$ such that for any choice of real numbers ω, K, T satisfying the inequalities*

$$0 < \omega \leq 1, \quad T \geq 1, \quad K > 0, \quad \omega K T^{n-1} \leq 1$$

the set

$$S(\omega,K,T) := \left\{ x\in B : \text{there exists } P\in\mathcal{P}_n \text{ such that } \begin{array}{l} |P(x)|<\omega, \\ |P'(x)|<K, \\ 0<H(P)<T \end{array} \right\}$$

has measure at most $E\epsilon^{1/(2n-1)}\mu(B)$, *where*

$$\epsilon := \max\bigl(\omega, (\omega K T^{n-1})^{1/(n+1)}\bigr).$$

Fix $\delta\in[0,v/2)$. It then follows from (2.4) that

$$\mathcal{I}_0^t(\phi_\delta) = \bigcup_{s=0}^{t+1}\bigcup_{P\in\mathcal{P}_n^s}\mathcal{I}_0^t(P,\phi_\delta(t)) = S(\omega,K,T)$$

with $\omega=\phi_\delta(t)2^{t(-n+1)}$, $K=\phi_\delta(t)2^{-vt}$ and $T=2^{t+2}$. By (2.3), we have $\epsilon\ll 2^{-4\delta^* t/(n+1)}$. Thus, Theorem 2.5 implies that

$$\mu(\mathcal{I}_0^t(\phi_\delta)) \ll 2^{-\beta t},$$

where $\beta := 4\delta^*/((n+1)(2n-1))$ is a positive constant. This finally gives

$$\sum_{t\in\mathbb{N}}\mu(\mathcal{I}_0^t(\phi_\delta)) \ll \sum_{t=0}^{\infty}2^{-\beta t} < \infty.$$

Therefore, by the Borel–Cantelli lemma $\mu(\varLambda_{\mathcal{H}}(\phi_\delta))=0$ for all $\delta\in[0,v/2)$. By the Inhomogeneous Transference Principle this further implies that $\mu(\varLambda_I(\phi_\delta))=0$ as required. The proposition has now been proved.

2.2. The case of large derivative. This subsection is devoted to proving the following proposition.

PROPOSITION 2.6. *Let* $n\ge 2$. *Then* $\mu(\mathcal{L}_2(n,d,\varPsi))=0$.

Let $D_n(H)$ be the set of points $x\in I$ which satisfy

$$|P(x)+d|<\varPsi(H) \quad\text{and}\quad |P'(x)|\ge H^{-v} \tag{2.16}$$

for some polynomial $P\in\mathcal{P}_n(H)$. Clearly,

$$\mathcal{L}_2(n,d,\varPsi) = \bigcap_{N=1}^{\infty}\bigcup_{H=N}^{\infty}D_n(H).$$

Define $\mathcal{P}_{n,j}(H)$ to be the set

$$\mathcal{P}_{n,j}(H) = \Bigl\{P\in\mathcal{P}_n(H) : j = \max_{|a_k|=H,\,0\le k\le n} k\Bigr\}$$

for $j=1,\dots,n$. Then $\mathcal{P}_n(H)=\bigcup_{j=0}^{n}\mathcal{P}_{n,j}(H)$. For each $P\in\mathcal{P}_{n,j}(H)$ define $\sigma_0(P,d)$ to be the set of points for which the inequalities in (2.16) hold, so that

$$D_n(H) = \bigcup_{j=0}^{n} \bigcup_{P \in \mathcal{P}_{n,j}(H)} \sigma_0(P, d).$$

For convenience we will occasionally use P_d to denote the polynomial $P(x) + d$. Clearly, for all $x \in \mathbb{R}$, $P^{(j)}(x) = P_d^{(j)}(x)$ for $j = 1, \ldots, n$.

2.2.1. *Case 1*: $n \geq 3$. The roots of any polynomial P will be denoted by $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. For each root of P define the sets

$$S_P(\alpha_j) = \Big\{x \in \mathbb{R} : |x - \alpha_j| = \min_{1 \leq i \leq n} |x - \alpha_i|\Big\}, \qquad 1 \leq j \leq n.$$

Clearly, for each P, $x \in S_P(\alpha_j)$ for at least one $j \in \{1, \ldots, n\}$. During the proof the points x will be restricted to a set $S_P(\alpha_j)$ for a fixed j and for simplicity we will take $j = 1$. The following easy lemma will be used in what follows.

Lemma 2.7. *Let P be a polynomial with root α_1 such that $P'(\alpha_1) \neq 0$. Then, if $x \in S_P(\alpha_1)$,*

$$|x - \alpha_1| < n|P(x)|\,|P'(x)|^{-1}.$$

Proof. As

$$P(x) = a_n(x - \alpha_1) \cdots (x - \alpha_n), \qquad P'(x) = a_n \sum_{j=1}^{n} \Big((x - \alpha_j)^{-1} \prod_{i=1}^{n} (x - \alpha_i)\Big)$$

we have

$$\frac{|P'(x)|}{|P(x)|} \leq \sum_{j=1}^{n} \frac{1}{|x - \alpha_j|} \leq \frac{n}{|x - \alpha_1|}. \ \blacksquare$$

For $x \in I \cap S_{P_d}(\alpha) \cap \sigma_0(P, d)$ such that $P'(x) \neq 0$ let $\sigma'(P_d, \alpha)$ denote the interval defined by the inequality

$$|x - \alpha| < n|P(x) + d|\,|P'(x)|^{-1} \leq n\Psi(H)H^v.$$

The last inequality follows from Lemma 2.7. Now, the Taylor series of $P' = P_d'$ is evaluated in the neighbourhood of α. Estimating each term, using (2.1) and the fact that $v < 1/3$, $n \geq 3$, gives

$$|P^{(j)}(\alpha)(x - \alpha)^{j-1}| \ll H(\Psi(H)H^v)^{j-1} \ll H^{1+(j-1)(-n+1+v)} < H^{-v-\varepsilon}$$

for $j = 2, \ldots, n$ and H sufficiently large. Further, since $|P'(x)| \geq H^{-v}$, we have

$$H^{-v}/2 \leq |P'(x)|/2 < |P'(\alpha)| < 2|P'(x)|.$$

Therefore, $\sigma'(P_d, \alpha)$ is contained in the interval $\sigma(P_d, \alpha)$ defined by the inequality

$$|x - \alpha| < 2n|P(x) + d|\,|P'(\alpha)|^{-1} \leq 2n\Psi(H)|P'(\alpha)|^{-1}. \tag{2.17}$$

For each polynomial $P \in \mathcal{P}_{n,j}(H)$ let A_{P_d} be the set

$$A_{P_d} = \{\alpha \in I : P_d(\alpha) = 0 \text{ and } |P'(\alpha)| > H(P)^{-v}/2\}.$$

Thus, $\sigma_0(P, d) \subseteq \sigma(P_d) = \bigcup_{\alpha \in A_{P_d}} \sigma(P_d, \alpha)$.

The proof is now subdivided into three parts depending on the size of $P'(\alpha)$ when $x \in S_{P_d}(\alpha)$. The three subcases to consider are

$$\begin{gathered} |P'(\alpha)| > c_0 H(P)^{1/2}, \\ 1 < |P'(\alpha)| \leq c_0 H(P)^{1/2}, \\ \tfrac{1}{2} H(P)^{-v} < |P'(\alpha)| \leq 1 \end{gathered} \tag{2.18}$$

for some constant $c_0 > 0$. These three inequalities partition the roots of P_d and are labelled $A_{P_d}^{(i)}$, $i = 1, 2, 3$, respectively.

Proposition 2.8. *Assume that $\sum_{h=1}^{\infty} h^{n-1} \Psi(h) < \infty$. The set of points $x \in I \cap S_{P_d}(\alpha)$ with $\alpha \in A_{P_d}^{(1)}$ which satisfy*

$$|P_d(x)| = |P(x) + d| < \Psi(H(P)), \qquad |P'(x)| \geq H(P)^{-v}$$

for infinitely many $P \in \mathcal{P}_n$ has measure zero.

Proof. Let $c_1 = c_1(n, d)$ be a constant to be chosen later. For each $P \in \mathcal{P}_{n,j}(H)$ and $\alpha \in A_{P_d}^{(1)}$ define the set $\sigma_1(P_d, \alpha)$ of points $x \in I$ which satisfy

$$|x - \alpha| < c_1 |P'(\alpha)|^{-1}.$$

From (2.17), for H sufficiently large, $\sigma(P_d, \alpha) \subset \sigma_1(P_d, \alpha)$ and

$$\mu(\sigma(P_d, \alpha)) < 2nc_1^{-1} \Psi(H) \mu(\sigma_1(P_d, \alpha)). \tag{2.19}$$

Now the Taylor series of P_d on $\sigma_1(P_d, \alpha)$ is evaluated. Each term is estimated to obtain

$$\begin{gathered} |P_d(\alpha)| = |P(\alpha) + d| = 0, \\ |P'(\alpha)(x - \alpha)| < c_1, \\ |P^{(j)}(\alpha)(x - \alpha)^j| < c_1^j n^{j+1} H (c_0 H^{1/2})^{-j} \leq n^3 c_1^2 c_0^{-2} \end{gathered}$$

for $2 \leq j \leq n$ and H sufficiently large. Choose $c_1 = c_1(\theta) < \theta/8$ (where θ is defined in (2.2)) such that $n^4 c_1 c_0^{-2} < 1$. Then $|P(x) + d| < 2c_1$ for H sufficiently large.

The set $\mathcal{P}_{n,j}(H)$ is now subdivided into sets with the same coefficients. Let $\mathbf{b}_1$ denote the $(n-1)$-tuple $(a_n, a_{n-1}, \ldots, a_{i+1}, a_{i-1}, \ldots, H, \ldots, a_0)$, where $|a_j| = H$, $i \neq j$, $i \neq 0$; let the subclass of polynomials $P \in \mathcal{P}_{n,j}(H)$ with the same $(n-1)$-tuple of coefficients $\mathbf{b}_1$ be denoted by $\mathcal{P}_{n,j}^{\mathbf{b}_1}(H)$. Then $\mathcal{P}_{n,j}(H) = \bigcup_{\mathbf{b}_1} \mathcal{P}_{n,j}^{\mathbf{b}_1}(H)$ and the number of subclasses is $\ll H^{n-1}$. Let $P, \tilde{P} \in P_{n,j}^{\mathbf{b}_1}(H)$, with $P \neq \tilde{P}$, and assume that $\sigma_1(P_d, \alpha) \cap \sigma_1(\tilde{P}_d, \tilde{\alpha}) \neq \emptyset$.

Let $x \in \sigma_1(P_d, \alpha) \cap \sigma_1(\tilde{P}_d, \tilde{\alpha})$ and let $R(x) = \tilde{P}_d(x) - P_d(x) = a_i' x^i$ for some $a_i' \in \mathbb{Z} \setminus 0$. Then, by (2.2),

$$\theta < |R(x)| \leq 4c_1 < \theta/2,$$

which is a contradiction. Hence, $\sigma_1(P_d, \alpha) \cap \sigma_1(\tilde{P}_d, \tilde{\alpha}) = \emptyset$ and

$$\sum_{P \in P_{n,j}^{\mathbf{b}_1}(H)} \sum_{\alpha \in A_{P_d}^{(1)}} \mu(\sigma_1(P_d, \alpha)) \leq \mu(I).$$

Together with (2.19) this gives

$$\sum_{P \in P_{n,j}^{\mathbf{b}_1}(H)} \sum_{\alpha \in A_{P_d}^{(1)}} \mu(\sigma(P_d, \alpha)) \ll \Psi(H)\mu(I),$$

which further implies that

$$\sum_{H=1}^{\infty} \sum_{j=0}^{n} \sum_{\mathbf{b}_1 \in \mathbb{Z}^{n-1},\, |\mathbf{b}_1| \leq H} \sum_{P \in P_{n,j}^{\mathbf{b}_1}(H)} \sum_{\alpha \in A_{P_d}^{(1)}} \mu(\sigma(P_d, \alpha)) \ll \sum_{H=1}^{\infty} H^{n-1}\Psi(H)\mu(I) < \infty.$$

The proof of the proposition can now be completed using the Borel–Cantelli lemma. ∎

Proposition 2.9. *Assume that* $\sum_{h=1}^{\infty} h^{n-1}\Psi(h) < \infty$. *The set of points* $x \in I \cap S_{P_d}(\alpha)$ *with* $\alpha \in A_{P_d}^{(2)}$ *which satisfy*

$$|P_d(x)| = |P(x) + d| < \Psi(H(P)), \qquad |P'(x)| \geq H(P)^{-v}$$

for infinitely many $P \in \mathcal{P}_n$ *has measure zero.*

Proof. Let $P \in \mathcal{P}_{n,j}(H)$ and $\alpha \in A_{P,d}^{(2)}$. Define $\sigma_2(P_d, \alpha) \supset \sigma(P_d, \alpha)$ to be the set of points $x \in I$ which satisfy the inequality

$$|x - \alpha| < H^{-1}|P'(\alpha)|^{-1}.$$

Clearly,

$$\mu(\sigma(P_d, \alpha)) < 2nH\Psi(H)\mu(\sigma_2(P_d, \alpha)). \tag{2.20}$$

Again, $\mathcal{P}_{n,j}(H)$ is subdivided into sets which have the same coefficients. Let $\mathbf{b}_2$ be the $(n-2)$-tuple $(a_n, a_{n-1}, \dots, a_{l+1}, a_{l-1}, \dots, H, \dots, a_{k+1}, a_{k-1}, \dots, a_0)$, where $|a_j| = H$, $l, k \neq j$, $l, k \neq 0$, and $l > k$. Denote the subclass of polynomials with the same $(n-2)$-tuple $\mathbf{b}_2$ of coefficients by $\mathcal{P}_{n,j}^{\mathbf{b}_2}(H)$. Then $\mathcal{P}_{n,j}(H) = \bigcup_{\mathbf{b}_2} \mathcal{P}_{n,j}^{\mathbf{b}_2}(H)$. The number of classes is $\ll H^{n-2}$. We now use Sprindžuk's method of essential and inessential intervals; see [14] for more details. The interval $\sigma_2(P_d, \alpha)$ is called *essential* if

$$\mu(\sigma_2(P_d, \alpha) \cap \sigma_2(\tilde{P}_d, \tilde{\alpha})) \leq \frac{\mu(\sigma_2(P_d, \alpha))}{2}$$

for all $\tilde{P}_d \in P^{\mathbf{b}_2}_{n,j}(H)$ and all roots $\tilde{\alpha} \in A^{(2)}_{\tilde{P}_d}$ of $\tilde{P}$, $P \neq \tilde{P}$. Otherwise it is called *inessential*.

First, the essential polynomials are investigated. By definition

$$\sum_{P\in P^{\mathbf{b}_2}_{n,j}(H)} \sum_{\substack{\alpha\in A^{(2)}_{P_d}\\ \sigma_2(P_d,\alpha)\text{ essential}}} \mu(\sigma_2(P_d,\alpha)) \ll \mu(I).$$

From this and (2.20),

$$\sum_{\substack{\mathbf{b}_2\in\mathbb{Z}^{n-2}\\ |\mathbf{b}_2|\le H}} \sum_{P\in P^{\mathbf{b}_2}_{n,j}(H)} \sum_{\substack{\alpha\in A^{(2)}_{P_d}\\ \sigma_2(P_d,\alpha)\text{ essential}}} \mu(\sigma(P_d,\alpha)) \ll H^{n-1}\Psi(H)\mu(I).$$

Hence,

$$\sum_{H=1}^{\infty}\sum_{j=0}^{n} \sum_{\substack{\mathbf{b}_2\in\mathbb{Z}^{n-2}\\ |\mathbf{b}_2|\le H}} \sum_{P\in P^{\mathbf{b}_2}_{n,j}(H)} \sum_{\substack{\alpha\in A^{(2)}_{P_d}\\ \sigma_2(P_d,\alpha)\text{ essential}}} \mu(\sigma(P_d,\alpha)) < \infty.$$

Therefore, by the Borel–Cantelli lemma, the set of points x which satisfy (2.16) for infinitely many essential intervals is of measure zero.

Now we consider an inessential interval $\sigma_2(P_d,\alpha)$. By definition, there is a polynomial $\tilde{P} \in \mathcal{P}^{\mathbf{b}_2}_{n,j}(H)$ such that $\mu(\sigma_2(P_d,\alpha)\cap\sigma_2(\tilde{P}_d,\tilde{\alpha})) > \frac{1}{2}(\sigma_2(P_d,\alpha))$. Let $x \in \sigma_2(P_d,\alpha)\cap\sigma_2(\tilde{P}_d,\tilde{\alpha})$. The polynomial P_d is now developed as a Taylor series on the interval $\sigma_2(P_d,\alpha)$ and each term is estimated from above to obtain

$$\begin{gathered} |P'(\alpha)(x-\alpha)| \ll H^{-1},\\ |P^{(j)}(\alpha)(x-\alpha)^j| \ll H^{1-j}|P'(\alpha)|^{-j} \ll H^{1-j}, \quad 2\le j\le n. \end{gathered}$$

The last inequality follows from (2.18). Hence,

$$|P(x)+d| \ll H^{-1}. \tag{2.21}$$

The derivative P' is also developed as a Taylor series on $\sigma_2(P_d,\alpha)$ to obtain

$$\begin{aligned} |P'(x)| &\le |P'(\alpha)| + \sum_{j=2}^{n}((j-1)!)^{-1}|P^{(j)}(\alpha)(x-\alpha)^{j-1}| \\ &\ll H^{1/2} + \sum_{j=2}^{n} H^{2-j}|P'(\alpha)|^{-(j-1)} \ll H^{1/2}. \end{aligned} \tag{2.22}$$

Consider the new polynomial $R(x) = \tilde{P}_d(x) - P_d(x) = a'_k x^k + a'_l x^l$ with $a'_k, a'_l \in \mathbb{Z}$ not both zero, where both P_d and $\tilde{P}_d$ belong to $P^{\mathbf{b}_2}_{n,j}(H)$. By (2.21) and (2.22), the inequalities

$$|R(x)| \ll H^{-1}, \quad |R'(x)| \ll H^{1/2}$$

hold on $\sigma_2(P_d,\alpha)\cap\sigma_2(\tilde{P}_d,\tilde{\alpha})$. It is relatively straightforward to show that $|a_i'|\ll H^{1/2}$ for $i=k,l$ so that $H(R)\ll H^{1/2}$. Therefore, $|a_k'x^k+a_l'x^l|\ll H(R)^{-2}$. Divide by x^k. Then, using (2.2), we have $|a_l'x^{l-k}+a_k'|\ll H(R)^{-2}$, which holds infinitely often only on a set of measure zero by Khinchin's theorem. Thus, the measure of the set of x which lie in infinitely many inessential intervals is zero. ∎

PROPOSITION 2.10. *Assume that* $\sum_{h=1}^{\infty}h^{n-1}\Psi(h)<\infty$. *The set of points* $x\in I\cap S_{P_d}(\alpha)$, $\alpha\in A_{P_d}^{(3)}$, *which satify*

$$|P_d(x)|=|P(x)+d|<\Psi(H(P)),\qquad |P'(x)|\ge H(P)^{-v}$$

for infinitely many $P\in\mathcal{P}_n$ *has measure zero.*

Proof. This is very similar to the previous case so some of the details will be omitted.

For each P_d with root $\alpha\in A_{P,d}^{(3)}$ and $P\in\mathcal{P}_{n,j}(H)$ define the set $\sigma_2(P_d,\alpha)$ and the $(n-2)$-tuple $\mathbf{b}_2$ as above. Again, we use essential and inessential intervals. Summing over the essential intervals gives

$$\sum_{H=1}^{\infty}\sum_{j=0}^{n}\sum_{\substack{\mathbf{b}_2\in\mathbb{Z}^{n-2}\\|\mathbf{b}_2|\le H}}\sum_{P\in P_{n,j}^{\mathbf{b}_2}(H)}\sum_{\substack{\alpha\in A_{P_d}^{(3)}\\ \sigma_2(P_d,\alpha)\text{ essential}}}\mu(\sigma(P_d,\alpha))\le\sum_{H=1}^{\infty}H^{n-1}\Psi(H)\mu(I).$$

Thus, using the Borel–Cantelli lemma, the set of x lying in infinitely many essential intervals has zero measure.

Now let $\sigma_2(P_d,\alpha)$ be an inessential interval. Using Taylor's formula for P_d on $\sigma_2(P_d,\alpha)$, we obtain

$$|P'(\alpha)(x-\alpha)|\ll H^{-1},$$
$$|P^{(j)}(\alpha)(x-\alpha)^j|\ll HH^{-j}|P'(\alpha)|^{-j}\ll H^{2v-1},\qquad 2\le j\le n.$$

For the last part the fact that $v<1/3$ was used. Thus,

$$|P_d(x)|=|P(x)+d|\ll H^{2v-1}.\tag{2.23}$$

Similarly develop P' as a Taylor series on $\sigma_2(P_d,\alpha)$ to obtain

$$\begin{aligned}|P'(x)|&\le|P'(\alpha)|+\sum_{j=2}^{n}((j-1)!)^{-1}|P^{(j)}(\alpha)(x-\alpha)^{j-1}|\\&\ll 1+\sum_{j=2}^{n}HH^{-j+1}|P'(\alpha)|^{-(j-1)}\ll H^{v}\end{aligned}\tag{2.24}$$

since $v<1/3$.

As before let $x \in \sigma_2(P_d, \alpha) \cap \sigma_2(\tilde{P}_d, \tilde{\alpha})$ and let $R(x) = \tilde{P}_d(x) - P_d(x)$ with $P_d, \tilde{P}_d \in P_{n,j}^{\mathbf{b}_2}(H)$. For R the inequalities $|R(x)| \ll H^{2v-1}$ and $|R'(x)| \ll H^v$ hold; these follow from (2.23) and (2.24). As in Proposition 2.9 it is possible to show that $|a_i'| \ll H^v$ $(i = k, l)$ so that $H(R) = \max\{|a_k'|, |a_l'|\} \ll H^v$. By (2.2) and (2.23),

$$|R(x)| = |a_l' x^{l-k} + a_k'| \ll H(R)^{(2v-1)/v} \ll H(R)^{-1}$$

for $v < 1/3$. By Khinchin's theorem the last inequality holds infinitely often only for a set of measure zero. Hence, the measure of the set of x lying in infinitely many inessential intervals is also zero. ∎

The three propositions complete the proof of Proposition 2.6 in the case $n \geq 3$.

2.2.2. *Case 2*: $n = 2$. The proof splits into two parts. If $|P'(x)| > c_2$ for some constant $c_2 \geq 1$ then we follow the proof of Proposition 2.6 until the start of Propostion 2.10, in each case replacing H^{-v} by c_2. The only other change is that instead of restricting to the sets $\mathcal{P}_2^{\mathbf{b}_2}(H)$ we restrict to the set $\mathcal{P}_2(H)$ in Proposition 2.9.

Next, the case $H^{-v} \leq |P'(x)| \leq c_2$ is considered. For a given polynomial $P \in \mathcal{P}_2(H)$ we redefine $\sigma_0(P, d)$ to be the set of solutions of $|P(x) + d| < \Psi(H)$ and $H^{-v} \leq |P'(x)| \leq c_2$. Let

$$\beta = \inf_{x \in \sigma_0(P,d)} |P'(x)|.$$

It is readily verified that $\sigma_0(P, d)$ consists of at most two intervals of length at most $4\Psi(H)\beta^{-1}$. For every P_d define a point $\gamma \in \sigma_0(P, d)$ such that $|P'(\gamma)| \leq 2\beta$. Then $\mu(\sigma_0(P, d)) \ll \Psi(H)|P'(\gamma)|^{-1}$. The choice of γ also implies that $H^{-v} \leq |P'(\gamma)| \leq c_2$. After this, the proof follows the same lines as in Proposition 2.10 except that instead of restricting to the sets $\mathcal{P}_2^{\mathbf{b}_2}(H)$ we restrict to the set $\mathcal{P}_2(H)$ and α is replaced by γ.

The two cases complete the proof of Proposition 2.6 for $n = 2$ and hence of Theorem 1.1. ∎

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