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Optimality of Chebyshev bounds for Beurling generalized numbers

by

HAROLD G. DIAMOND (Urbana, IL) and WEN-BIN ZHANG* (Chicago, IL)

1. Introduction. Let $N(x)$ and $\pi(x)$ denote the counting function of integers and the counting function of primes, respectively, in a Beurling generalized (henceforth, g-) number system $\mathcal{N}$. By analogy with classical prime number theory, the inequalities

$$x/\log x \ll \pi(x) \ll x/\log x$$

are called *Chebyshev bounds* for the system $\mathcal{N}$. Several conditions have been given for such bounds ([Di1], [Zh], [Vn1]). It was conjectured by the first author [Di3] that these bounds held if

$$\int_1^\infty x^{-2}|N(x) - Ax|\,dx < \infty, \tag{1.1}$$

but this was disproved by an example of J.-P. Kahane ([Ka1], [Ka2]). In [Vn1] it was shown that (1.1) together with the additional pointwise bound

$$(N(x) - Ax)x^{-1}\log x = o(1)$$

implies the Chebyshev upper bound $\pi(x) \ll x/\log x$. The second condition was weakened by the present authors [DZ] to

$$(N(x) - Ax)\,x^{-1}\log x = O(1) \tag{1.2}$$

and, still weaker, the average bound

$$\int_1^x |N(u) - Au|u^{-1}\log u\,du \ll x. \tag{1.3}$$

In this paper, we shall show that the conditions (1.1) and (1.2) (resp. (1.3)) are essentially best-possible for Chebyshev bounds.

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* Cantonese: Chung Man Ping.

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Added in proof. The Chebyshev upper estimate was also recently established under (1.1) and (1.2) by J. Vindas [Vn2].

Main Theorem 1.1. *Given any positive-valued function $f(x)$ on $[1,\infty)$ such that $f(x)$ is increasing and $f(x) \to \infty$ as $x \to \infty$, there exists a g-number system $\mathcal{N}_B$ such that:*

(1) *The associated zeta function $\zeta_B(s)$ is analytic on the open half-plane $\{s = \sigma + it : \sigma > 1\}$. Also, $(s-1)\zeta_B(s)$ has a continuous extension to the closed half-plane $\{\sigma \ge 1\}$ and $it\zeta_B(1+it) \ne 0$.*

(2) *The counting function $N_B(x)$ of the g-integers satisfies*

$$\int_1^\infty x^{-2}|N_B(x) - Ax|\,dx < \infty \tag{1.4}$$

and

$$N_B(x) - Ax = O\left(\frac{xf(x)}{\log x}\right) \tag{1.5}$$

with some constant $A > 0$.

(3) *The counting function $\pi_B(x)$ of the g-primes satisfies*

$$\limsup_{x\to\infty} \frac{\pi_B(x)}{x/\log x} = \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\pi_B(x)}{x/\log x} = 0.$$

In other words, if the right side of (1.2) is replaced by an unbounded function f, no matter how slowly it grows, then there exists a g-number system satisfying (1.1) for which the Chebyshev bounds fail.

2. The generalized primes. We construct our g-prime system following an idea from [Ka2]. The proof is divided into several lemmas. We begin by creating from f another function which grows at least as slowly and has several useful analytical properties.

Lemma 2.1. *Given $f(x)$ satisfying the conditions of Theorem* 1.1, *there exists a function $k(x)$ defined on $[1,\infty)$ such that:*

(1) *$k(x) \ge 1$ for $x \ge 1$ and $k(x) \ll f(x)$.*
(2) *$k(x)$ is increasing and $k(x) \to \infty$ as $x \to \infty$.*
(3) *$k(x)$ is differentiable and $(\log x)/k(x)$ is increasing on $(1,\infty)$.*

Proof. First, let

$$f_1(x) := \min\{f(x), \log\log(e^e x)\}, \qquad x \ge 1.$$

We have $0 < f_1(x) \le f(x)$ for $x \ge 1$. Moreover, $f_1(x)$ is increasing and $f_1(x) \to \infty$ as $x \to \infty$.

Next, let

$$f_2(x) := x^{-1}\int_1^x f_1(t)\,dt, \qquad x \ge 1.$$

We have

$$0 \le f_2(x) \le \frac{x-1}{x} f_1(x) \le f_1(x), \qquad x \ge 1.$$

Also, $f_2(x)$ is increasing, since for $\Delta x \ge 0$,

$$\begin{aligned} f_2(x+\Delta x) &\ge \frac{1}{x+\Delta x}\Big(\int_1^x f_1(t)\,dt + f_1(x)\Delta x\Big) \\ &\ge \frac{1}{x+\Delta x}\bigg(\int_1^x f_1(t)\,dt + \frac{\Delta x}{x}\int_1^x f_1(t)\,dt\bigg) = f_2(x). \end{aligned}$$

Also, $f_2(x) \to \infty$ as $x \to \infty$, for

$$f_2(x) > \frac{1}{x}\int_{x/2}^x f_1(t)\,dt \ge \frac{1}{2} f_1(x/2) \to \infty.$$

Moreover, $f_2(x)$ is continuous.

Then let

$$f_3(x) := 1 + x^{-1}\int_1^x f_2(t)\,dt.$$

As before, we have

$$1 \le f_3(x) \le 1 + f_2(x) \le 1 + f_1(x) \le 1 + f(x), \qquad x \ge 1.$$

Also, $f_3(x)$ is increasing and $f_3(x) \to \infty$ as $x \to \infty$. Moreover, $f_3(x)$ is differentiable at all points of $(1,\infty)$, since f_2 is continuous there.

Finally, we set

$$k(x) = f_3(\log\log(e^e x)), \qquad x \ge 1.$$

For $x \ge 1$ we have

$$1 \le k(x) \le 1 + f(\log\log(e^e x)),$$

and from the definition of $f_1(x)$, $k(x) \ll \log\log\log\log x$. Also, $k(x)$ is increasing and $k(x) \to \infty$. Moreover,

$$\left(\frac{\log x}{k(x)}\right)' = \frac{1}{xk(x)}\left(1 - \frac{f_3'(\log\log(e^e x))}{f_3(\log\log(e^e x))}\,\frac{\log x}{\log(e^e x)}\right).$$

Note that $f_3(y) > 1$ and that

$$0 \le f_3'(y) = \frac{f_2(y)}{y} - \frac{\int_1^y f_2(t)\,dt}{y^2} < \frac{f_2(y)}{y} < \frac{\log\log(e^e y)}{y} < 1$$

for $y > 1$. Therefore, for $x > 1$,

$$\left(\frac{\log x}{k(x)}\right)' \geq \frac{1}{xk(x)}\left(1 - \frac{\log\log(e^e \log\log(e^e x))}{\log\log(e^e x)}\right) > 0,$$

i.e., $(\log x)/k(x)$ is increasing for $x > 1$. ∎

Using $k(x)$, we next determine a sparse sequence for our construction. Since $k(x)$ increases monotonically to infinity, there exists a sequence $c_1, c_2, \ldots$ such that

$$\sum_{n \geq 1} 1/\sqrt{k(c_n)} < \infty.$$

Next, define another sequence (A_n) recursively by taking $A_1 = e$ and $A_{n+1} = \max\{e^{A_n}, c_{n+1}\}$. Note that the sequence $(\log A_n)$ grows faster than exponentially. We have

$$\sum_{n \geq 1} \frac{\log k(n)}{k(A_n)} < \infty \tag{2.1}$$

since $k(x)$ is increasing and

$$k(A_n)^{1/2} \geq \tfrac{1}{2} \log k(A_n) \geq \tfrac{1}{2} \log k(n).$$

Now we construct the g-prime set of the theorem. Let n_0 be a positive integer; it is to be taken large enough to satisfy each of several conditions below. From here onwards, p denotes a *rational* prime, $\mathcal{P}$ the set of all such, and $\pi(x)$ the counting function of the rational primes. We take

$$\mathcal{P}_B = \Big(\mathcal{P} \setminus \bigcup_{n \geq n_0} \{p \in [A_n, \sqrt{k(n)}\, A_n]\}\Big)$$
$$\cup \bigcup_{n \geq n_0} \{A_n \text{ with multiplicity } [A_n \log k(n)/(2 \log A_n)]\}.$$

In words, $\mathcal{P}_B$ consists of an initial string of rational primes, then a g-prime A_{n_0} having high multiplicity (a "pulse"), followed by a long interval having no g-primes, after which comes a longer interval of rational primes, then A_{n_0+1} appears, and the cycle repeats. We shall see that the multiplicity of A_n has been balanced with the length of the subsequent dead interval to achieve a positive density of g-integers. Also, note that the intervals $[A_n, \sqrt{k(n)}\, A_n]$ are pairwise non-overlapping for sufficiently large n_0, since $k(n) \leq 1 + \log\log(e^e n)$ and $A_{n+1} \geq \exp A_n$. To make formulas easier to read, *we shall generally write $A_n^\star$ in place of $\sqrt{k(n)}\, A_n$.*

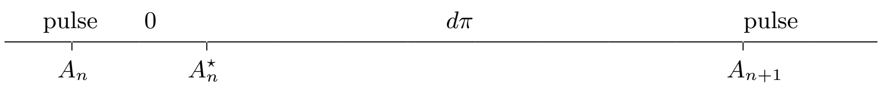

Fig 1. $d\pi_B$ on one interval

We shall show that the set of g-primes $\mathcal{P}_B$ and associated g-integers $\mathcal{N}_B$ satisfies the conditions of the theorem. We begin with the failure of the Chebyshev bounds.

3. Chebyshev bounds and the zeta function

LEMMA 3.1. *Property* (3) *of the theorem is satisfied.*

Proof. First, there exists a sequence on which $\pi(x)$ is too large. Indeed,

$$\frac{\pi_B(A_n)}{A_n/\log A_n} \ge \frac{[A_n \log k(n)/(2\log A_n)]}{A_n/\log A_n} \to \infty \quad \text{as } n \to \infty.$$

Next, we show that $\pi(x)$ is too small on the points $x = A_n^\star$, the end of the "dead zones". We begin with an inductive argument to show that

$$\pi_B(A_n-) \le \pi(A_n-). \tag{3.1}$$

This relation holds trivially (with equality) for $n = n_0$. Note that the number of rational primes inhabiting each dead zone is

$$\pi(A_n^\star) - \pi(A_n) \sim \frac{A_n\, k(n)^{1/2}}{\log A_n + (1/2)\log k(n)} > \frac{A_n \log k(n)}{2\log A_n}$$

for $n \ge n_0$. Hence, from the definition of $\mathcal{P}_B$,

$$\begin{aligned}\pi_B(A_{n+1}-) &= \{\pi(A_{n+1}-) - \pi(A_n^\star)\} + \{\pi_B(A_n) - \pi_B(A_n-)\} + \pi_B(A_n-)\\ &\le \{\pi(A_{n+1}-) - \pi(A_n^\star)\} + \frac{A_n \log k(n)}{2\log A_n} + \pi_B(A_n-) < \pi(A_{n+1}-).\end{aligned}$$

Thus (3.1) holds. It follows that, as $n \to \infty$,

$$\begin{aligned}\frac{\pi_B(A_n^\star)}{A_n^\star/\log A_n^\star} &= \frac{\pi_B(A_n)}{A_n^\star/\log A_n^\star}\\ &\le \frac{\pi(A_n) + A_n \log k(n)/(2\log A_n)}{A_n^\star/\log A_n^\star} \ll \frac{\log k(n)}{k(n)^{1/2}} \to 0. \ \blacksquare\end{aligned}$$

Our further analysis uses an auxiliary system appearing in [Di2]. Let

$$d\pi_0 := d(\pi_B - \pi)_v,$$

the variation of $d(\pi_B - \pi)$;

$$d\Pi_0(x) := \sum_{\ell \ge 1} \frac{1}{\ell} d\pi_0(x^{1/\ell});$$

and

$$N_0(x) := 1 + \sum_{n \ge 1} \frac{1}{n!} \int_1^x d\Pi_0^{*n},$$

where the last expression denotes the n-fold multiplicative convolution of $d\Pi_0$ with itself. Note that $d\pi_0(u) = d\Pi_0(u) = 0$ on $\{u : u < A_{n_0}\}$ and $d\pi_0(u) = 0$ on each interval $(A_n^\star, A_{n+1})$ with $n \ge n_0$.

Also, we need a preliminary estimate.

LEMMA 3.2.

$$\sum_{A_m<p\le A_m^\star} p^{-1} = \frac{\log k(m)}{2\log A_m} - \frac{1}{8}\left(\frac{\log k(m)}{\log A_m}\right)^2 + O\left(\frac{\log k(m)}{\log^2 A_m}\right). \tag{3.2}$$

Proof. In Stieltjes integral form, the left-hand side of (3.2) is

$$\int_{A_m}^{A_m^\star} \frac{dt}{t\log t} + \int_{A_m}^{A_m^\star} \frac{1}{t}\left\{d\pi(t) - \frac{dt}{\log t}\right\} =: I_1 + I_2,$$

say. We have

$$\begin{aligned} I_1 &= \log\left\{\frac{\log(A_m\, k(m)^{1/2})}{\log A_m}\right\} = \log\left\{1 + \frac{\log k(m)}{2\log A_m}\right\} \\ &= \frac{\log k(m)}{2\log A_m} - \frac{1}{8}\left(\frac{\log k(m)}{\log A_m}\right)^2 + O\left(\frac{\log^3 k(m)}{\log^3 A_m}\right). \end{aligned}$$

For I_2, use integration by parts and the classical prime number theorem error bound

$$R(x) := \int_2^x \left\{d\pi(t) - \frac{dt}{\log t}\right\} \ll \frac{x}{\log^2 x}. \tag{3.3}$$

We find

$$\begin{aligned} I_2 &= \frac{R(A_m^\star)}{A_m^\star} - \frac{R(A_m)}{A_m} + \int_{A_m}^{A_m^\star} \frac{R(t)}{t^2}\, dt \\ &\ll \frac{1}{\log^2 A_m} + \int_{A_m}^{A_m\sqrt{k(m)}} \frac{O(1)\, dt}{t\log^2 t} \ll \frac{\log k(m)}{\log^2 A_m}. \ \blacksquare \end{aligned}$$

LEMMA 3.3.

$$\int_1^\infty x^{-1}\frac{\log x}{k(x)}\, d\Pi_0(x) < \infty. \tag{3.4}$$

Proof. We first note that

$$\begin{aligned} &\int_1^\infty x^{-1}\frac{\log x}{k(x)}\, d\pi_0(x) \\ &\qquad = \sum_{n\ge n_0}\left(A_n^{-1}\frac{\log A_n}{k(A_n)}\left[\frac{A_n\log k(n)}{2\log A_n}\right] + \sum_{A_n<p\le A_n^\star} p^{-1}\frac{\log p}{k(p)}\right). \end{aligned}$$

Then, by the monotonicity of $\log x$ and of $k(x)$ and the last lemma,

$$\sum_{A_n<p\le A_n^\star} p^{-1}\frac{\log p}{k(p)} \le \frac{\log A_n^\star}{k(A_n)}\sum_{A_n<p\le A_n^\star} p^{-1} \ll \frac{\log A_n^\star}{k(A_n)}\,\frac{\log k(n)}{\log A_n}.$$

Since $\log A_n^\star \ll \log A_n$, we have

$$\int_1^\infty x^{-1}\frac{\log x}{k(x)}\,d\pi_0(x) \ll \sum_{n\ge n_0}\frac{\log k(n)}{k(A_n)} < \infty$$

by (2.1). Finally, the left-hand side of (3.4) equals

$$\begin{aligned}\sum_{\ell\ge 1}\frac{1}{\ell}\int_1^\infty x^{-1}\frac{\log x}{k(x)}\,d\pi_0(x^{1/\ell}) &= \sum_{\ell\ge 1}\frac{1}{\ell}\int_1^\infty u^{-\ell}\,\frac{\ell\log u}{k(u^\ell)}\,d\pi_0(u)\\ &\le \frac{1}{1-A_{n_0}^{-1}}\int_1^\infty u^{-1}\frac{\log u}{k(u)}\,d\pi_0(u) < \infty. \ \blacksquare\end{aligned}$$

The zeta function for $\mathcal{N}_B$ is defined, analogously to the Riemann zeta function, by the Mellin integral

$$\zeta_B(s) := \int_{1-}^\infty u^{-s}\,dN_B(u).$$

We now show that $\zeta_B(s)$ does have the expected properties.

LEMMA 3.4. *$\zeta_B(s)$ is analytic for $\sigma > 1$, and $(s-1)\zeta_B(s)$ has a continuous extension to the closed half-plane $\sigma \ge 1$. Moreover, $it\,\zeta_B(1+it) \ne 0$.*

Proof. We write

$$\zeta_B(s) = \exp\Big\{\int_1^\infty x^{-s}\,d\Pi_B(x)\Big\} = \zeta(s)\exp\Big\{\int_1^\infty x^{-s}\,d(\Pi_B-\Pi)(x)\Big\},$$

where $\zeta(s)$ is the Riemann zeta function and $\Pi(x) = \sum_{\ell\ge 1}\ell^{-1}\,\pi(x^{1/\ell})$. Note that $d(\Pi_B-\Pi)_v \le d\Pi_0$ by the triangle inequality. Since $(\log x)/k(x) \gg 1$ for $x \ge A_{n_0}$, Lemma 3.3 implies that the last integral converges absolutely for $\sigma \ge 1$. Hence $\zeta_B(s)$ is analytic on $\{s : \sigma > 1\}$ and, by familiar properties of the Riemann zeta function,

$$(s-1)\zeta_B(s) = (s-1)\zeta(s)\exp\Big\{\int_1^\infty x^{-s}\,d(\Pi_B-\Pi)(x)\Big\}$$

has a continuous extension to $\sigma \ge 1$ and furthermore

$$it\zeta_B(1+it) = it\zeta(1+it)\exp\Big\{\int_1^\infty x^{-(1+it)}\,d(\Pi_B-\Pi)(x)\Big\} \ne 0.$$

Thus, property (1) of the theorem is proved. $\blacksquare$

4. The counting function $N_B(x)$. Our remaining job is to give estimates for $N_B(x)$, to establish property (2) of the theorem. We first have

LEMMA 4.1.
$$\int_1^\infty x^{-1}\frac{\log x}{k(x)}\,dN_0(x)<\infty.$$

Proof. Recall that $k(x)$ is increasing. Hence
$$1+\frac{\log(x_1\cdots x_n)}{k(x_1\cdots x_n)}\le\left(1+\frac{\log x_1}{k(x_1)}\right)\cdots\left(1+\frac{\log x_n}{k(x_n)}\right)$$
for $x_i\ge A_{n_0}$, $i=1,\ldots,n$. Then we have
$$\begin{aligned}\int_1^\infty x^{-1}\left(1+\frac{\log x}{k(x)}\right)d\Pi_0^{*n}(x)&\le\int_1^\infty x^{-1}\left\{\left(1+\frac{\log x}{k(x)}\right)d\Pi_0(x)\right\}^{*n}\\&=\int_1^\infty\left\{x^{-1}\left(1+\frac{\log x}{k(x)}\right)d\Pi_0(x)\right\}^{*n}\\&=\left\{\int_1^\infty x^{-1}\left(1+\frac{\log x}{k(x)}\right)d\Pi_0(x)\right\}^{n}.\end{aligned}$$
Therefore, by Lemma 3.3,
$$\int_1^\infty x^{-1}\left(1+\frac{\log x}{k(x)}\right)dN_0(x)\le\exp\left\{\int_1^\infty x^{-1}\left(1+\frac{\log x}{k(x)}\right)d\Pi_0(x)\right\}<\infty.\ \blacksquare$$

By the fundamental relation between dN and $d\Pi$ (resp. dN_B and $d\Pi_B$) and the homomorphic property of exponentials we have
$$dN_B=\exp\{d\Pi_B\}=\exp\{d\Pi+d(\Pi_B-\Pi)\}=dN*\exp\{d(\Pi_B-\Pi)\}.$$
Thus the counting function of g-integers satisfies
$$\begin{aligned}N_B(x)&=\int_{1-}^x N\left(\frac{x}{t}\right)\exp\{d(\Pi_B-\Pi)\}(t)\\&=N(x)+\int_1^x N\left(\frac{x}{t}\right)\sum_{n\ge1}\frac{1}{n!}\,d(\Pi_B-\Pi)^{*n}(t)\\&=x+\theta(x)+x\sum_{n\ge1}\frac{1}{n!}\int_1^x t^{-1}\,d(\Pi_B-\Pi)^{*n}(t)\\&\quad+\sum_{n\ge1}\frac{1}{n!}\int_1^x\theta\left(\frac{x}{t}\right)d(\Pi_B-\Pi)^{*n}(t)\end{aligned}\tag{4.1}$$
with $N(x)$ the counting function of rational integers and $\theta(x)=N(x)-x$.

Let

$$c_1 := \int_1^\infty t^{-1}\, d(\Pi_B - \Pi)(t),$$

an absolutely convergent integral by Lemma 3.3. As we saw in the proof of Lemma 4.1,

$$\int_1^\infty t^{-1}\, d(\Pi_B - \Pi)^{*n}(t)$$

is absolutely convergent; it equals

$$\Big(\int_1^\infty t^{-1}\, d(\Pi_B - \Pi)(t)\Big)^n = c_1^n.$$

Add and subtract terms $\int_x^\infty t^{-1}\, d(\Pi_B - \Pi)^{*n}(t)$ and rewrite (4.1) as

$$N_B(x) = Ax + xE(x), \tag{4.2}$$

where

$$A = 1 + \sum_{n\ge 1} \frac{1}{n!} \int_1^\infty t^{-1}\, d(\Pi_B - \Pi)^{*n}(t) = e^{c_1}$$

and

$$E(x) := x^{-1}\theta(x) - E_1(x) + E_2(x) \tag{4.3}$$

with

$$E_1(x) := \sum_{n\ge 1} \frac{1}{n!} \int_x^\infty t^{-1} d(\Pi_B - \Pi)^{*n}(t) \tag{4.4}$$

and

$$E_2(x) := x^{-1} \sum_{n\ge 1} \frac{1}{n!} \int_1^x \theta\Big(\frac{x}{t}\Big)\, d(\Pi_B - \Pi)^{*n}(t).$$

Also, Lemmas 4.1 and 2.1(3) together imply that

$$\zeta_0(s) := \int_{1-}^\infty x^{-s}\, dN_0(x)$$

converges absolutely for $\sigma \ge 1$. Hence, $\zeta_0(s)$ is analytic on $\sigma > 1$ and continuous on $\sigma \ge 1$.

Lemma 4.2. *We have*

$$\frac{N_0(x)}{x} \ll \frac{k(x)}{\log x} \tag{4.5}$$

and hence

$$|E_2(x)| \le \frac{N_0(x)}{x} \ll \frac{k(x)}{\log x}. \tag{4.6}$$

Also,

$$\int_1^\infty x^{-1}|E_2(x)|\,dx < \infty. \tag{4.7}$$

Proof. By Lemma 4.1,

$$\int_{A_{n_0}}^x y^{-1}\frac{\log y}{k(y)}\,dN_0(y) < \int_1^\infty y^{-1}\frac{\log y}{k(y)}\,dN_0(y) < \infty.$$

The left-hand side equals, by integration by parts,

$$x^{-1}\frac{\log x}{k(x)}\,N_0(x) - A_{n_0}^{-1}\frac{\log A_{n_0}}{k(n_0)}\,N_0(A_{n_0}) \\ + \int_{A_{n_0}}^x N_0(y)y^{-2}\left(\frac{\log y - 1}{k(y)} + \frac{yk'(y)\log y}{k^2(y)}\right)dy.$$

Recalling that $k'(x) \ge 0$ and noting that $\log A_{n_0} \ge 1$, we have

$$\int_{A_{n_0}}^x y^{-1}\frac{\log y}{k(y)}\,dN_0(y) \ge x^{-1}\frac{\log x}{k(x)}\,N_0(x) - A_{n_0}^{-1}\frac{\log A_{n_0}}{k(A_{n_0})}\,N_0(A_{n_0}).$$

Thus, (4.5) follows. Next,

$$|E_2(x)| \le \frac{1}{x}\sum_{n\ge 1}\frac{1}{n!}\int_1^x d\Pi_0^{*n}(t) < \frac{N_0(x)}{x}$$

and (4.6) follows. Moreover, by Lemma 4.1 again,

$$\int_1^\infty x^{-s}\frac{N_0(x)}{x}\,dx = \frac{\zeta_0(s)}{s}$$

for $\sigma \ge 1$. Hence

$$\int_1^\infty x^{-1}|E_2(x)|\,dx \le \int_1^\infty x^{-2}N_0(x)\,dx = \zeta_0(1) < \infty. \ \blacksquare$$

The analysis of $E_1(x)$ requires a more delicate argument.

5. Fundamental estimates

LEMMA 5.1. *For $n \ge n_0$, a sufficiently large number, we have*

$$\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t)\Big| \le \begin{cases} \frac{1}{4}(\log k(n_0)/\log A_{n_0})^2 & \text{if } 1 \le x \le A_{n_0}, \\ \log k(n)/\log A_n & \text{if } A_n < x \le A_n^\star, \\ \frac{1}{4}(\log k(n+1)/\log A_{n+1})^2 & \text{if } A_n^\star < x \le A_{n+1}. \end{cases} \tag{5.1}$$

Also, for $\ell \ge 2$,

$$\Big|\int_x^\infty t^{-\ell}\,d(\pi_B-\pi)(t)\Big| \le \begin{cases} A_{n_0}^{-\ell+1}\log k(n_0)/\log A_{n_0} & \text{if } 1 \le x \le A_{n_0}, \\ 2A_n^{-\ell+1}/\log A_n & \text{if } A_n < x \le A_n^\star, \\ A_{n+1}^{-\ell+1}\log k(n+1)/\log A_{n+1} & \text{if } A_n^\star < x \le A_{n+1}. \end{cases} \tag{5.2}$$

Proof. For $A_n^\star < x \le A_{n+1}$, $n \ge n_0$, or $1 \le x \le A_{n_0}$ (i.e., $n+1 = n_0$),

$$\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t) = \sum_{m\ge n+1}\bigg(A_m^{-1}\bigg[\frac{A_m\log k(m)}{2\log A_m}\bigg] - \sum_{A_m<p\le A_m^\star} p^{-1}\bigg).$$

By Lemma 3.2,

$$\begin{aligned} &A_m^{-1}\bigg[\frac{A_m\log k(m)}{2\log A_m}\bigg] - \sum_{A_m<p\le A_m^\star} p^{-1} \\ &= \frac{\log k(m)}{2\log A_m} + O(A_m^{-1}) - \bigg\{\frac{\log k(m)}{2\log A_m} - \frac{1}{8}\bigg(\frac{\log k(m)}{\log A_m}\bigg)^2 + O\bigg(\frac{\log k(m)}{\log^2 A_m}\bigg)\bigg\} \\ &= \frac{1}{8}\bigg(\frac{\log k(m)}{\log A_m}\bigg)^2 + O\bigg(\frac{\log k(m)}{\log^2 A_m}\bigg). \end{aligned}$$

Therefore we have

$$\begin{aligned} \Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t)\Big| &= \sum_{m\ge n+1}\bigg\{\frac{1}{8}\bigg(\frac{\log k(m)}{\log A_m}\bigg)^2 + O\bigg(\frac{\log k(m)}{\log^2 A_m}\bigg)\bigg\} \\ &\le \frac{1}{4}\bigg(\frac{\log k(n+1)}{\log A_{n+1}}\bigg)^2 \end{aligned}$$

for n_0 large enough. This proves the first and the third inequalities of (5.1).

For $A_n < x \le A_n^\star$, $n \ge n_0$, by the definition of $\mathcal{P}_B$,

$$\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t) = -\sum_{x<p\le A_n^\star} p^{-1} + \int_{A_{n+1}}^\infty t^{-1}\,d(\pi_B-\pi)(t).$$

From the third inequality of (5.1), just proved,

$$\Big|\int_{A_{n+1}}^{\infty} t^{-1}\, d(\pi_B-\pi)(t)\Big| \le \frac{1}{4}\left(\frac{\log k(n+1)}{\log A_{n+1}}\right)^2.$$

Also, by (3.2),

$$\sum_{x<p\le A_n^\star} p^{-1} \le \sum_{A_n<p\le A_n^\star} p^{-1} \le \frac{\log k(n)}{2\log A_n} + O\left(\left\{\frac{\log k(n)}{\log A_n}\right\}^2\right).$$

Hence

$$\Big|\int_{x}^{\infty} t^{-1}\, d(\pi_B-\pi)(t)\Big| \le \frac{\log k(n)}{\log A_n}.$$

This proves the second inequality of (5.1).

Now suppose that $\ell \ge 2$. For $A_n^\star < x \le A_{n+1},\ n \ge n_0$, or $1 \le x \le A_{n_0}$ (i.e., $n+1 = n_0$), we have in a similar way

$$\int_{x}^{\infty} t^{-\ell}\, d(\pi_B-\pi)(t) = \sum_{m\ge n+1}\left(A_m^{-\ell}\left[\frac{A_m \log k(m)}{2\log A_m}\right] - \sum_{A_m<p\le A_m^\star} p^{-\ell}\right).$$

Applying the method used in proving Lemma 3.2, write

$$\sum_{A_m<p\le A_m^\star} p^{-\ell} = \int_{A_m}^{A_m^\star} \frac{dt}{t^\ell \log t} + \int_{A_m}^{A_m^\star} t^{-\ell}\left\{d\pi(t) - \frac{dt}{\log t}\right\} =: I_1' + I_2',$$

say. We have, by integration by parts,

$$I_1' = \frac{A_m^{1-\ell}}{(\ell-1)\log A_m} - \frac{(A_m^\star)^{1-\ell}}{(\ell-1)\log A_m^\star} + O\left(\frac{A_m^{1-\ell}}{\log^2 A_m}\right).$$

For I_2', apply integration by parts and the prime number estimate (3.3). We find

$$I_2' = R(t)t^{-\ell}\Big|_{A_m}^{A_m^\star} + \ell\int_{A_m}^{A_m^\star} R(t)t^{-\ell-1}\, dt \ll \frac{A_m^{1-\ell}}{\log^2 A_m}.$$

Together, these estimates imply that

$$\sum_{A_m<p\le A_m^\star} p^{-\ell} = \frac{(1+o(1))A_m^{1-\ell}}{(\ell-1)\log A_m}, \tag{5.3}$$

provided that m is sufficiently large. Thus

$$\left|A_m^{-\ell}\left[\frac{A_m \log k(m)}{2\log A_m}\right] - \sum_{A_m<p\le A_m^\star} p^{-\ell}\right| \le \frac{A_m^{-\ell+1}\log k(m)}{2\log A_m},$$

and so we get

$$\Big|\int_x^\infty t^{-\ell}\,d(\pi_B-\pi)(t)\Big| \le \sum_{m\ge n+1}\frac{A_m^{-\ell+1}\log k(m)}{2\log A_m} \le \frac{A_{n+1}^{-\ell+1}\log k(n+1)}{\log A_{n+1}}.$$

Now suppose $A_n < x \le A_n^\star,\ n \ge n_0$. We have

$$\int_x^\infty t^{-\ell}\,d(\pi_B-\pi)(t) = -\sum_{x<p\le A_n^\star} p^{-\ell} + \int_{A_{n+1}}^\infty t^{-\ell}\,d(\pi_B-\pi)(t).$$

The sum is clearly bounded above by $\sum_{A_n<p\le A_n^\star} p^{-\ell}$, and the last sum equals

$$\frac{(1+o(1))A_n^{1-\ell}}{(\ell-1)\log A_n}$$

by the first relation in (5.3). If we combine this estimate with the inequality derived when $A_n^\star < x \le A_{n+1}$, $n \ge n_0$, we find

$$\Big|\int_x^\infty t^{-\ell}\,d(\pi_B-\pi)(t)\Big| \le \frac{(1+o(1))A_n^{-\ell+1}}{(\ell-1)\log A_n} + \frac{A_{n+1}^{-\ell+1}\log k(n+1)}{\log A_{n+1}} \le \frac{2A_n^{-\ell+1}}{\log A_n}.$$

This completes the proof of (5.2). ∎

Lemma 5.2. *For n_0 sufficiently large,*

$$c_2 := \int_1^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)(t)\Big|\,dx \le 2\frac{\log^2 k(n_0)}{\log A_{n_0}}.$$

Proof. By (5.1),

$$\int_1^{A_{n_0}} x^{-1}\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t)\Big|\,dx \le \frac{\log^2 k(n_0)}{4\log A_{n_0}},$$

$$\int_{A_n}^{A_n^\star} x^{-1}\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t)\Big|\,dx \le \frac{\log^2 k(n)}{2\log A_n},$$

$$\int_{A_n^\star}^{A_{n+1}} x^{-1}\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t)\Big|\,dx \le \frac{\log^2 k(n+1)}{4\log A_{n+1}}.$$

Hence

$$\int_1^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t)\Big|\,dx$$

$$\le \frac{\log^2 k(n_0)}{4\log A_{n_0}} + \sum_{n\ge n_0}\Big(\frac{\log^2 k(n)}{2\log A_n} + \frac{\log^2 k(n+1)}{4\log A_{n+1}}\Big) \le \frac{\log^2 k(n_0)}{\log A_{n_0}}$$

for n_0 sufficiently large. Also, for $\ell \geq 2$,

$$\int_1^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t^{1/\ell})\Big|\,dx = \int_1^\infty x^{-1}\Big|\int_{x^{1/\ell}}^\infty u^{-\ell}\,d(\pi_B-\pi)(u)\Big|\,dx$$
$$= \ell\int_1^\infty y^{-1}\Big|\int_y^\infty u^{-\ell}\,d(\pi_B-\pi)(u)\Big|\,dy.$$

By (5.2), in a similar way, the right side of the last equation is at most

$$\ell\bigg(A_{n_0}^{-\ell+1}\log k(n_0) + \sum_{n\geq n_0}\bigg\{\frac{A_n^{-\ell+1}\log k(n)}{\log A_n} + A_{n+1}^{-\ell+1}\log k(n+1)\bigg\}\bigg)$$
$$< 2\ell A_{n_0}^{-\ell+1}\log k(n_0).$$

Hence

$$\int_1^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\mathit{\Pi}_B-\mathit{\Pi})(t)\Big|\,dx$$
$$= \int_1^\infty x^{-1}\Big|\int_x^\infty \sum_{\ell\geq 1}\frac{1}{\ell}t^{-1}\,d(\pi_B-\pi)(t^{1/\ell})\Big|\,dx$$
$$\leq \sum_{\ell\geq 1}\frac{1}{\ell}\int_1^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\pi_B-\pi)(t^{1/\ell})\Big|\,dx$$
$$\leq \frac{\log^2 k(n_0)}{\log A_{n_0}} + 2\sum_{\ell\geq 2}A_{n_0}^{-\ell+1}\log k(n_0) \leq \frac{2\log^2 k(n_0)}{\log A_{n_0}}. \ \blacksquare$$

6. Proof of the theorem. It remains to study $E_1(x)$ (defined in (4.4)).

LEMMA 6.1. *For $x > 1$,*

$$E_1(x) \ll \frac{k(x)}{\log x}. \tag{6.1}$$

Proof. We have

$$\Big|\int_x^\infty t^{-1}\,d(\mathit{\Pi}_B-\mathit{\Pi})^{*n}(t)\Big| \leq \int_x^\infty t^{-1}\,d\mathit{\Pi}_0^{*n}(t) \leq \frac{k(x)}{\log x}\int_x^\infty t^{-1}\frac{\log t}{k(t)}\,d\mathit{\Pi}_0^{*n}(t),$$

since $(\log t)/k(t)$ is increasing. It follows that

$$|E_1(x)| \leq c_3\frac{k(x)}{\log x},$$

where

$$c_3 := \int_{1+}^\infty t^{-1}\frac{\log t}{k(t)}\,dN_0(t) < \infty$$

by Lemma 4.1. $\blacksquare$

LEMMA 6.2.

$$\int_1^\infty x^{-1}|E_1(x)|\,dx < \infty. \tag{6.2}$$

Proof. We have

$$\int_1^\infty x^{-1}|E_1(x)|\,dx = \int_1^\infty x^{-1}\Big|\sum_{\ell\ge 1}\frac{1}{\ell!}\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t)\Big|\,dx \le I_1+I_2,$$

say, where

$$I_1 := \int_1^\infty x^{-1}\Big|\sum_{1\le\ell\le\log x/\log A_{n_0}}\frac{1}{\ell!}\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t)\Big|\,dx,$$

$$I_2 := \int_1^\infty x^{-1}\Big|\sum_{\ell>\log x/\log A_{n_0}}\frac{1}{\ell!}\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t)\Big|\,dx.$$

Recall that $(\varPi_B-\varPi)(x)=0$ for $x<A_{n_0}$, so there is no contribution to the integrals unless $\log x/\log A_{n_0}\ge 1$. For $\ell>\log x/\log A_{n_0}$, i.e., $A_{n_0}^\ell > x$, we have

$$\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t) = \int_1^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t) = \Big(\int_1^\infty t^{-1}\,d(\varPi_B-\varPi)(t)\Big)^\ell = c_1^\ell.$$

Hence

$$\sum_{\ell>\log x/\log A_{n_0}}\frac{1}{\ell!}\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t) = \sum_{\ell>\log x/\log A_{n_0}}\frac{1}{\ell!}c_1^\ell,$$

and therefore

$$\begin{aligned} I_2 &\le \int_1^\infty x^{-1}\Big(\sum_{\ell>\log x/\log A_{n_0}}\frac{|c_1|^\ell}{\ell!}\Big)\,dx \\ &= \sum_{\ell\ge 1}\frac{|c_1|^\ell}{\ell!}\int_1^{A_{n_0}^\ell} x^{-1}\,dx = |c_1|e^{|c_1|}\log A_{n_0}. \end{aligned} \tag{6.3}$$

Next, we have

$$\begin{aligned} I_1 \le{}& \int_{A_{n_0}+}^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)(t)\Big|\,dx \\ &+\sum_{\ell\ge 2}\frac{1}{\ell!}\int_{A_{n_0}^\ell+}^\infty x^{-1}\Big|\int_x^\infty t^{-1}\,d(\varPi_B-\varPi)^{*\ell}(t)\Big|\,dx. \end{aligned}$$

For $\ell \geq 2$,

$$\int_{A_{n_0}^{\ell}+}^{\infty} x^{-1}\Big|\int_{x}^{\infty} t^{-1}\,d(\Pi_B-\Pi)^{*\ell}(t)\Big|\,dx$$

$$\leq \int_{A_{n_0}^{\ell}}^{\infty} x^{-1}\Big(\int_{A_{n_0}^{\ell-1}}^{\infty}\Big|\int_{x/v}^{\infty} u^{-1}\,d(\Pi_B-\Pi)(u)\Big|v^{-1}\,d\Pi_0^{*\ell-1}(v)\Big)\,dx$$

$$= \int_{A_{n_0}^{\ell-1}}^{\infty}\Big(\int_{A_{n_0}^{\ell}}^{\infty} x^{-1}\Big|\int_{x/v}^{\infty} u^{-1}\,d(\Pi_B-\Pi)(u)\Big|\,dx\Big)v^{-1}\,d\Pi_0^{*\ell-1}(v).$$

Letting $x/v = y$, the inner integral on the right-hand side becomes

$$\int_{A_{n_0}^{\ell}/v}^{\infty} \frac{1}{vy}\Big|\int_{y}^{\infty} u^{-1}\,d(\Pi_B-\Pi)(u)\Big|v\,dy$$

$$\leq \int_{1}^{\infty} y^{-1}\Big|\int_{y}^{\infty} u^{-1}\,d(\Pi_B-\Pi)(u)\Big|\,dy = c_2 < \infty$$

by Lemma 5.2. Therefore,

$$\int_{A_{n_0}^{\ell}+}^{\infty} x^{-1}\Big|\int_{x}^{\infty} t^{-1}\,d(\Pi_B-\Pi)^{*\ell}(t)\Big|\,dx \leq c_2\int_{A_{n_0}^{\ell-1}}^{\infty} v^{-1}\,d\Pi_0^{*\ell-1}(v)$$

$$\leq c_2\Big(\int_{1}^{\infty} v^{-1}\,d\Pi_0(v)\Big)^{\ell-1} = c_2c_4^{\ell-1},$$

where

$$c_4 := \int_{1}^{\infty} x^{-1}\,d\Pi_0(x).$$

Hence

$$I_1 \leq c_2 + \sum_{\ell\geq 2}\frac{1}{\ell!}c_2c_4^{\ell-1} \leq c_2(1+e^{c_4})/2. \tag{6.4}$$

From (6.4) and (6.3), (6.2) follows. ∎

It remains only to establish property (2) of the theorem. The relations (4.2), (4.3), (4.6), and (6.1), along with the inequality $k(x) \ll f(x)$ of Lemma 2.1, give

$$\frac{|N(x)-Ax|}{x} = |E(x)| \leq \frac{1}{x} + |E_1(x)| + |E_2(x)| \ll \frac{f(x)}{\log x}.$$

Also, by (4.7) and (6.2),

$$\int_1^\infty x^{-2}|N(x) - Ax|\, dx \le \int_1^\infty x^{-1}(x^{-1} + |E_1(x)| + |E_2(x)|)\, dx < \infty.$$

These estimates complete the proof of Theorem 1.1.

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Harold G. Diamond (corresponding author)
Department of Mathematics
University of Illinois
Urbana, IL 61801, U.S.A.
E-mail: diamond@math.uiuc.edu

Wen-Bin Zhang
920 West Lawrence Ave. #1112
Chicago, IL 60640, U.S.A.
E-mail: cheungmanping@yahoo.com

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