ACTA ARITHMETICA
154.4 (2012)

Algebraic independence of real numbers with low density of nonzero digits

by

HAJIME KANEKO (Tokyo)

1. Introduction. Let $\alpha \geq 2$ be an integer. A *normal number* in base α is a positive number whose base-α digits show a uniform distribution, that is, all finite words with letters from the alphabet $\{0, 1, \ldots, \alpha - 1\}$ occur with the proper frequency. Borel [3] showed that almost all positive numbers are normal in each integral base. However, it is generally difficult to check whether a given number is normal or not.

Borel [4] conjectured that all algebraic irrational numbers are normal in every integral base. This conjecture is still open. No algebraic number has been proven to be normal yet. Moreover, no counterexample is known. If Borel's conjecture is true, then nonzero digits in base-α expansions of algebraic irrational numbers appear with average frequency tending to $(\alpha - 1)/\alpha$. Consequently, for any irrational ξ, if Borel's conjecture is true and if nonzero digits of ξ in base-α occur with average frequency tending to 0, then ξ is transcendental.

We now recall known results about transcendence and algebraic independence of positive numbers whose densities of nonzero digits are low. In this paper, $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{Z}_{\geq 1}$ the set of positive integers. We denote the integral part of a real number ξ by $[\xi]$, and use the Vinogradov symbols $\gg$ and $\ll$, as well as the Landau symbols O and o with their regular meanings. Recall that $f \ll g$, $g \gg f$ and $f = O(g)$ are all equivalent and mean that $|f| \leq c|g|$ with some positive constant c. Moreover, $f = o(g)$ (resp. $f \sim g$) means that the ratio f/g tends to zero (resp. 1). All implied constants may depend on the given data.

We consider the numbers

$$\xi = \sum_{n=1}^{\infty} \alpha^{-w(n)}, \tag{1.1}$$

2010 *Mathematics Subject Classification*: Primary 11J81; Secondary 11B13, 11K16.
Key words and phrases: algebraic independence, power series, sumsets.

DOI: 10.4064/aa154-4-1

where $\alpha \ge 2$ is an integer and $(w(n))_{n=1}^{\infty}$ is a strictly increasing sequence of nonnegative integers. Liouville [9, 10] was the first to show the existence of transcendental numbers in 1844. He obtained the transcendence of the number $\sum_{n=1}^{\infty} \alpha^{-n!}$ by proving what is nowadays called Liouville's inequality. Schmidt [16] generalized this inequality and showed that the numbers $\gamma_1, \gamma_2, \ldots$ defined by

$$\gamma_l = \sum_{n=1}^{\infty} \alpha^{-(ln)!} \qquad (l = 1, 2, \ldots)$$

are algebraically independent. Durand [7] verified for each real algebraic number z with $0 < z < 1$ that the uncountable set

$$\Big\{ \zeta_h = \sum_{n=0}^{\infty} z^{[hn!]} \,\Big|\, h > 0 \Big\}$$

is algebraically independent. Shiokawa [17] established algebraic independence of the values of gap series at algebraic points including those that appeared in [7] and [16]. However, we cannot apply Liouville's method in the case of

$$\limsup_{n\to\infty} \frac{w(n+1)}{w(n)} < \infty.$$

Let $k \ge 2$ be an integer. Mahler [11] verified that the number $\sum_{n=0}^{\infty} \alpha^{-k^n}$ is transcendental. More generally, he proved for each algebraic number z with $0 < |z| < 1$ that $\Phi_k(z) = \sum_{n=0}^{\infty} z^{k^n}$ is transcendental by using the functional equation

$$\Phi_k(z^k) = \Phi_k(z) - z. \tag{1.2}$$

Using the Schmidt Subspace Theorem, Corvaja and Zannier [5] generalized Mahler's results above as follows: Assume that $(w(n))_{n=1}^{\infty}$ is *lacunary*, that is,

$$\liminf_{n\to\infty} \frac{w(n+1)}{w(n)} > 1. \tag{1.3}$$

Then, for every algebraic z with $0 < |z| < 1$, the number $\sum_{n=1}^{\infty} z^{w(n)}$ is transcendental. Mahler's method is also applicable to algebraic independence theory. Using (1.2), Nishioka [12] showed that the values $\Phi_2(z), \Phi_3(z), \ldots$ are algebraically independent for each algebraic number z with $0 < |z| < 1$. For detailed information concerning Mahler's method, see [13].

Now we return to the base-α expansions of algebraic numbers. For positive numbers ξ and R, let $\lambda(\alpha, \xi, R)$ be the number of nonzero digits among the first $1 + [R]$ digits of the base-α expansion of ξ, that is

$$\lambda(\alpha, \xi, R) = \mathrm{Card}\{n \in \mathbb{N} \mid n \le [R],\ [\xi\alpha^n] - \alpha[\xi\alpha^{n-1}] \ne 0\},$$

where Card denotes cardinality. Assume that $\alpha = 2$. Bailey, Borwein, Crandall, and Pomerance [1] showed that for any algebraic irrational ξ there exists a positive computable constant $C(\xi)$ depending only on ξ satisfying

$$\lambda(2, \xi, N) \geq C(\xi) N^{1/\deg \xi} \tag{1.4}$$

for all sufficiently large N. With a suitable positive $C(\alpha, \xi)$ in place of $C(\xi)$ we will prove (1.4) for any integral base $\alpha \geq 2$ in the same way:

THEOREM 1.1. *Let α be an integer greater than 1 and $\xi > 0$ an algebraic irrational number. Then there exist effectively computable positive constants $C(\alpha, \xi)$ and $C'(\alpha, \xi)$ depending only on α and ξ such that, for any integer N with $N \geq C'(\alpha, \xi)$,*

$$\lambda(\alpha, \xi, N) \geq C(\alpha, \xi) N^{1/\deg \xi}. \tag{1.5}$$

The idea of the proof of Theorem 1.1 was inspired by the paper of Knight [8]. Let $A_D X^D + A_{D-1} X^{D-1} + \cdots + A_0 \in \mathbb{Z}[X]$ be the minimal polynomial of ξ, where $A_D > 0$. In the rest of this section, $C_1(\alpha, \xi)$ and $C_2(\alpha, \xi)$ denote effectively computable positive constants depending only on α and ξ. We have

$$A_D \xi^D + A_{D-1} \xi^{D-1} + \cdots + A_0 = 0. \tag{1.6}$$

We now explain the notion of nonzero islands introduced by Knight for another proof of the transcendence of $\xi_0 = \sum_{n=0}^{\infty} \alpha^{-2^n}$. Let $D', A'_0, A'_1, \ldots, A'_{D'}$ be integers with $D' \geq 1$ and $A'_{D'} \geq 1$. We will show that

$$\omega := \sum_{k=0}^{D'} A'_k \xi_0^k \neq 0.$$

For any k with $1 \leq k \leq D'$ we have

$$\xi_0^k = \sum_{m=0}^{\infty} \tau(m, k) \alpha^{-m},$$

where $\tau(m, k)$ denotes the number of ways that m can be written as a sum of k powers of 2. Let b be a sufficiently large integer. Put $N = (2^{D'} - 1) 2^b$. Let m be an integer with

$$N - 2^{b-1} + 1 \leq m \leq N + 2^b - 1.$$

Then Lemma 1 in [8] implies that

$$\tau(m, k) = \begin{cases} D'! & \text{if } m = N \text{ and } k = D', \\ 0 & \text{otherwise.} \end{cases}$$

Hence, considering the carries of the base-α expansion of $D'! A'_{D'} \alpha^{-N}$, we deduce the following: there exists an integer m with $N \leq m \leq N + O(1)$ such that the mth digit of the base-α expansion of ω is not zero. In particular,

$\omega \neq 0$. Knight used the term *nonzero islands* to refer to nonzero digits which occur from the carries of the base-α expansion of $D'!A'_{D'}\alpha^{-N}$.

In [1], the Thue–Siegel–Roth theorem [15] was used to find nonzero islands. However, this theorem is ineffective. In this paper, we use Liouville's inequality instead, and obtain the effective lower bounds $C'(\alpha,\xi)$ in Theorem 1.1.

We now give a sketch of the proof of Theorem 1.1 without technical details. For simplicity, assume that $1 \le \xi < 2$ and write the α-ary expansion of ξ as

$$\xi = \sum_{m=0}^{\infty} t(\xi,m)\alpha^{-m}.$$

Note that $t(\xi,0)=1$. For any k with $1 \le k \le D$,

$$\xi^k = \sum_{m=0}^{\infty} \alpha^{-m} \sum_{\substack{i_1,\ldots,i_k \ge 0\\ i_1+\cdots+i_k=m}} t(\xi,i_1)\cdots t(\xi,i_k) =: \sum_{m=0}^{\infty} \alpha^{-m}\rho(k,m). \tag{1.7}$$

Let $k \ge 2$. Then, putting $i_k=0$, we get

$$\rho(k,m) \ge \sum_{\substack{i_1,\ldots,i_{k-1} \ge 0\\ i_1+\cdots+i_{k-1}=m}} t(\xi,i_1)\cdots t(\xi,i_{k-1})t(\xi,0) = \rho(k-1,m). \tag{1.8}$$

Let N be a positive integer. In the same way as in the proof of Theorem 7.1 in [1], we can show that there exists an interval $I=[U_1,U_2) \subset [0,N)$ satisfying the following four conditions:

- $\rho(D-1,U_1) > 0$.
- If $U_2 < N$, then $\rho(D-1,U_2) > 0$.
- For any m with $U_1 < m < U_2$,

$$\rho(D-1,m) = 0. \tag{1.9}$$

- The length $|I| = U_2 - U_1$ of I satisfies

$$|I| \ge C_1(\alpha,\xi)N^{1/D}. \tag{1.10}$$

Using (1.8) and (1.9), we get

$$\rho(k,m) = 0, \tag{1.11}$$

where k and m are integers with $1 \le k \le D-1$ and $U_1 < m < U_2$. Liouville's inequality and (1.10) imply that if $N \ge C_2(\alpha,\xi)$, then there exists an m_0 satisfying $t(\xi,m_0) > 0$ and

$$\frac{1}{D+2}|I| \le m_0 \le \frac{D+1}{D+2}|I|.$$

Indeed, suppose that $t(\xi, m) = 0$ for any m with

$$\frac{1}{D+2}|I| \le m \le \frac{D+1}{D+2}|I|.$$

Put

$$m_1 := \max\left\{m \in \mathbb{N} \;\middle|\; m < \frac{1}{D+2}|I|,\, t(\xi, m) \neq 0\right\},$$
$$m_2 := \min\{m \in \mathbb{N} \mid m_2 > m_1,\, t(\xi, m) \neq 0\}.$$

Then we have

$$m_2 \ge (D+1)m_1.$$

Let

$$p := \sum_{m=0}^{m_1} t(\xi, m)\alpha^{m_1 - m}, \qquad q := \alpha^{m_1}.$$

Then p and q are integers. Thus,

$$\xi - \frac{p}{q} = \sum_{m=m_2}^{\infty} t(\xi, m)\alpha^{-m} \le \alpha^{1-m_2} \le \alpha^{1-(D+1)m_1} = \alpha q^{-D-1},$$

which contradicts Liouville's inequality in the case of $N \ge C_2(\alpha, \xi)$ because we have (1.10).

Hence, putting $U := U_1 + m_0$, we obtain

$$U_1 + \frac{1}{D+2}|I| \le U \le U_1 + \frac{D+1}{D+2}|I| \tag{1.12}$$

and

$$\rho(D, U) \ge \rho(D-1, U_1)t(\xi, m_0) > 0. \tag{1.13}$$

In what follows, we analyze the α-ary expansion of the left-hand side of (1.6), using (1.7). Note that (1.7) is not generally the α-ary expansion of ξ^k because $\alpha^{-m}\rho(k, m)$ causes $O(\log \rho(k, m))$ carries to higher digits. Recall that $A_D > 0$. Combining (1.10)–(1.13), we conclude that there are positive digits left in the α-ary expansion of (1.6), which is a contradiction. To explain the details of the remaining positive digits, Bailey, Borwein, Crandall, and Pomerance [1] introduced BBP tails, a concept defined in [2] to give rapid algorithms for the computation of the digits of certain transcendental numbers.

In the case of $\alpha = 2$, Rivoal [14] improved the constant $C(\xi)$ for certain classes of algebraic irrational ξ. For example, let ε be an arbitrary positive number and $\xi' = 0.558\ldots$ the unique positive zero of the polynomial $8X^3 - 2X^2 + 4X - 3$. Theorem 7.1 in [1] implies that for any sufficiently large N,

$$\lambda(2, \xi', N) \ge (1-\varepsilon)16^{-1/3}N^{1/3}.$$

On the other hand, using Corollary 2 of [14], we obtain

$$\lambda(2,\xi',N) \geq (1-\varepsilon)N^{1/3}$$

for all sufficiently large N.

Let us consider applications of Theorem 1.1. For each real $k>1$, put

$$\nu_k = \sum_{n=0}^{\infty} \alpha^{-[n^k]}.$$

Let d be a natural number with $2 \leq d < k$. Then (1.5) implies that ν_k is not an algebraic number of degree at most d. Moreover, using Theorem 1.1, we deduce criteria for transcendence.

Corollary 1.2. *Let α be an integer greater than* 1, *and ξ a positive irrational number. Assume that for every positive ε,*

$$\lambda(\alpha,\xi,N) = o(N^{\varepsilon}). \tag{1.14}$$

Then ξ is transcendental.

For instance, the numbers

$$\sum_{n=1}^{\infty} \alpha^{-n!}, \qquad \sum_{n=0}^{\infty} \alpha^{-2^n}$$

are transcendental by Corollary 1.2 because they satisfy (1.14). Moreover, for positive l and $x \geq 1$, let

$$f_l(x) = \exp((\log x)^{1+l}).$$

Then the number

$$\eta_l = \sum_{n=1}^{\infty} \alpha^{-[f_l(n)]}$$

is transcendental by Corollary 1.2 because it satisfies (1.14). Indeed, it is easily seen that, for any $\varepsilon > 0$,

$$\lambda(\alpha,\eta_l,R) \sim \exp((\log R)^{1/(1+l)}) = o(\exp(\varepsilon \log R)) = o(R^{\varepsilon})$$

as $R \to \infty$. Note that η_l does not satisfy (1.3), so we cannot prove its transcendence using the result of Corvaja and Zannier. Indeed, for $x>1$, we have

$$\log\left(\frac{f_l(x+1)}{f_l(x)}\right) = (\log(x+1))^{1+l} - (\log x)^{1+l}.$$

By the mean value theorem, there exists $\sigma = \sigma(l,x) \in (0,1)$ such that

$$\log\left(\frac{f_l(x+1)}{f_l(x)}\right) = (1+l)\frac{(\log(x+\sigma))^l}{x+\sigma}.$$

Since the right-hand side converges to zero as $x \to \infty$, we obtain

$$\lim_{x\to\infty} \frac{f_l(x+1)}{f_l(x)} = 1.$$

The main purpose of this paper is to deduce algebraic independence of certain classes of numbers ξ which satisfy (1.14). We will introduce criteria for algebraic independence in Theorem 2.1. We prove this theorem in Section 4. Our method is quite flexible because we do not use functional equations. As a consequence, we deduce algebraic independence of the values η_l for real $l \geq 1$:

THEOREM 1.3. *The uncountable set $\{\eta_l \mid l \geq 1\}$ is algebraically independent.*

We cannot prove, using Theorem 2.1, that $\{\eta_l \mid l > 0\}$ is algebraically independent. However, we deduce that any two elements of this set are algebraically independent:

THEOREM 1.4. *Let h and l be distinct positive real numbers. Then η_h and η_l are algebraically independent.*

We prove Theorems 1.3 and 1.4 in Section 3.

2. Criteria for algebraic independence. Let $\alpha \geq 2$ be an integer and ξ a positive number. For each integer m, put

$$t(\xi, m) = [\xi\alpha^m] - \alpha[\xi\alpha^{m-1}] \in \{0, 1, \ldots, \alpha - 1\}.$$

Note that $t(\xi, -m) = 0$ for all sufficiently large $m \in \mathbb{N}$. Then ξ is written as

$$\xi = \sum_{m=-\infty}^{\infty} t(\xi, m)\alpha^{-m},$$

which is the α-ary expansion of ξ. Set

$$S(\xi) = \{m \in \mathbb{N} \mid t(\xi, m) \neq 0\}.$$

Recall that for $R > 0$,

$$\lambda(\alpha, \xi, R) = \mathrm{Card}\{n \in S(\xi) \mid n \leq R\}.$$

Note that if $1 \leq \xi < \alpha$, then $0 \in S(\xi)$. For each positive integer a, let

$$aS(\xi) = \{n_1 + \cdots + n_a \mid n_1, \ldots, n_a \in S(\xi)\}.$$

For convenience, set $0S(\xi) = \{0\}$. Moreover, for any positive numbers $\xi_1, \ldots, \xi_r$ and nonnegative integers $a_1, \ldots, a_r$, let

$$\sum_{i=1}^{r} a_i S(\xi_i) = \{s_1 + \cdots + s_r \mid s_i \in a_i S(\xi_i) \text{ for } 1 \leq i \leq r\}.$$

For a nonempty subset $\mathcal{A}$ of $\mathbb{N}$ and $R > \min\{n \in \mathcal{A}\}$, define

$$\theta(R; \mathcal{A}) = \max\{n \in \mathcal{A} \mid n < R\}.$$

Assume that $1 \le \xi_1, \dots, \xi_r < \alpha$. Let $(a_1, \dots, a_r), (a'_1, \dots, a'_r) \in \mathbb{N}^r$, where $a_i \ge a'_i$ for every i with $1 \le i \le r$. Then we have

$$\sum_{i=1}^{r} a_i S(\xi_i) \supset \sum_{i=1}^{r} a'_i S(\xi_i)$$

because $S(\xi_1), \dots, S(\xi_r) \ni 0$.

We now state criteria for algebraic independence.

Theorem 2.1. *Let $\xi_1, \dots, \xi_r$ be positive irrational numbers. Suppose that:*

(1) *For every positive ε, we have, as $R \to \infty$,*

$$\lambda(\alpha, \xi_1, R) = o(R^{\varepsilon}),$$
$$\lambda(\alpha, \xi_h, R) = o(\lambda(\alpha, \xi_{h-1}, R)^{\varepsilon}) \quad \text{for } h = 2, \dots, r.$$

(2) *There exists a positive constant C_1 such that*

$$S(\xi_r) \cap [C_1 R, R] \ne \emptyset$$

for every sufficiently large real R.

(3) *Let $a_1, \dots, a_{r-1}, a_r$ be any nonnegative integers. If $r \ge 2$, then there exist a positive integer $\kappa = \kappa(a_1, \dots, a_{r-1})$ and a positive constant $C_2(a_1, \dots, a_r)$, both depending only on the indicated parameters, such that*

$$R - \theta\Big(R; \sum_{i=1}^{r-2} a_i S(\xi_i) + \kappa S(\xi_{r-1})\Big) < R \prod_{i=1}^{r} \lambda(\alpha, \xi_i, R)^{-a_i}$$

for each real $R \ge C_2(a_1, \dots, a_r)$.

Then $\xi_1, \dots, \xi_r$ are algebraically independent.

Remark 2.2. In the case of $r = 1$, Theorem 2.1 follows from Corollary 1.2.

We prove Theorem 2.1 in Section 4. In the rest of the present section we give a sketch of the proof without technical details for $r = 2$ and $\kappa(a_1) = 1 + a_1$ for all $a_1 \ge 0$, where $\kappa(a_1)$ is defined in condition (3). For simplicity, suppose that $1 \le \xi_1, \xi_2 < 2$. If ξ_1 and ξ_2 are algebraically dependent, then there exists a nonzero polynomial $P(X_1, X_2) \in \mathbb{Z}[X]$ such that

$$P(\xi_1, \xi_2) = 0. \tag{2.1}$$

Let

$$P(X_1, X_2) := \sum_{\mathbf{k}=(a_1, a_2) \in \Lambda} A_{\mathbf{k}} X_1^{a_1} X_2^{a_2},$$

where Λ is a nonempty finite subset of $\mathbb{N}^2$ and $A_{\mathbf{k}}$ a nonzero integer for each $\mathbf{k} \in \Lambda$. We search nonzero islands of the α-ary expansion of the left-hand side of (2.1). For any $\mathbf{k} = (a,b) \in \Lambda$, we get

$$\begin{aligned}(2.2)\qquad \xi_1^a\xi_2^b &= \Big(\sum_{x=0}^{\infty} t(\xi_1,x)\alpha^{-x}\Big)^a \Big(\sum_{y=0}^{\infty} t(\xi_2,y)\alpha^{-y}\Big)^b \\ &= \sum_{m=0}^{\infty} \sum_{\substack{i_1,\dots,i_a,j_1,\dots,j_b\ge 0\\ i_1+\dots+i_a+j_1+\dots+j_b=m}} t(\xi_1,i_1)\cdots t(\xi_1,i_a)t(\xi_2,j_1)\cdots t(\xi_2,j_b) \\ &=: \sum_{m=0}^{\infty} \alpha^{-m}\rho(\mathbf{k},m).\end{aligned}$$

Observe that $\rho(\mathbf{k},m) > 0$ if and only if $m \in aS(\xi_1) + bS(\xi_2)$. In the proof of Theorem 1.1, we used the relation

$$0 \in S(\xi) \subset 2S(\xi) \subset \cdots$$

to find nonzero islands. Indeed, (1.8) implies that $(k-1)S(\xi) \subset kS(\xi)$ (see also (1.13)). On the other hand, let $(a_1,a_2),(a_1',a_2') \in \Lambda$. Then, in general, neither $a_1S(\xi_1) + a_2S(\xi_2) \subset a_1'S(\xi_1) + a_2'S(\xi_2)$ nor $a_1'S(\xi_1) + a_2'S(\xi_2) \subset a_1S(\xi_1) + a_2S(\xi_2)$ holds. This is the main difference between the proofs of Theorems 1.1 and 2.1. Let $\succ$ be the lexicographical order in $\mathbb{N}^2$, that is, $(a_1,a_2) \succ (a_1',a_2')$ if either $a_1 > a_1'$, or $a_1 = a_1'$ and $a_2 > a_2'$. Let $\mathbf{g} = (g_1,g_2) \in \Lambda$ be the greatest element of Λ with respect to $\succ$. Without loss of generality, we may assume that $A_{\mathbf{g}} > 0$. For any $(a_1,a_2) \in \Lambda$, if $a_1 = g_1$, then $a_2 \le g_2$. Thus, $a_1S(\xi_1) + a_2S(\xi_2) \subset g_1S(\xi_1) + g_2S(\xi_2)$. If $a_1 < g_1$, then the relation above does not hold generally. However, by condition (3) of Theorem 2.1, the set $a_1S(\xi_1) + a_2S(\xi_2)$ is approximated by $(1+a_1)S(\xi_1)$ because $\kappa(a_1) = a_1 + 1$. Moreover, $(1+a_1)S(\xi_1) \subset g_1S(\xi_1)$.

Based on the observation above, we give nonzero islands in Section 4.4. Let N be a sufficiently large integer. We construct an interval $J = [T_1,T_2) \subset [0,N)$ such that:

- $T_1 \in p_1S(\xi_1) + p_2S(\xi_2)$ for some $(p_1,p_2) \in \Lambda$ with $p_1 < g_1$.
- If $T_2 < N$, then $T_2 \in q_1S(\xi_1) + q_2S(\xi_2)$ for some $(q_1,q_2) \in \Lambda$ with $q_1 < g_1$.
- Let m be any integer with $T_1 < m < T_2$ and let $(a_1,a_2) \in \Lambda$ with $a_1 < g_1$. Then $m \notin a_1S(\xi_1) + a_2S(\xi_2)$.

Since $p_1S(\xi_1)+p_2S(\xi_2)$ and $q_1S(\xi_1)+q_2S(\xi_2)$ are approximated by $g_1S(\xi_1)$, we get a subinterval $I = [R_1,R_2)$ of J satisfying:

- $R_1 \in g_1S(\xi_1) + (g_2-1)S(\xi_2)$.
- $R_2 \in g_1S(\xi_1) + (g_2-1)S(\xi_2)$.

- Let m be any integer with $R_1 < m < R_2$ and let $\mathbf{k} = (a_1, a_2) \in \Lambda$ with $\mathbf{g} \succ \mathbf{k}$. Then

$$m \notin a_1 S(\xi_1) + a_2 S(\xi_2). \tag{2.3}$$

Denote the length of I by $|I| = R_2 - R_1$. By condition (2) of Theorem 2.1, there is an $m_0 \in \mathbb{N}$ satisfying $m_0 \in S(\xi_2)$ and

$$\frac{C_1}{1+C_1}|I| \le m_0 \le \frac{1}{1+C_1}|I|.$$

Putting $U := R_1 + m_0$, we obtain $U \in g_1 S(\xi_1) + g_2 S(\xi_2)$ and

$$R_1 + \frac{C_1}{1+C_1}|I| \le U \le R_1 + \frac{1}{1+C_1}|I|. \tag{2.4}$$

In particular,

$$\rho(\mathbf{g}, U) > 0. \tag{2.5}$$

Now we analyze the α-ary expansion of the left-hand side of (2.1), using (2.2). Recall that $A_{\mathbf{g}} > 0$ and that $\alpha^{-m}\rho(\mathbf{k}, m)$ causes $O(\log \rho(\mathbf{k}, m))$ carries to higher digits. Hence, combining (2.3)–(2.5), we conclude that there are positive digits left in the α-ary expansion of (2.1), which is a contradiction. To explain the details of the remaining positive digits, we introduce BBP tails Y_R in Section 4.2.

3. Proofs of main results

Proof of Theorem 1.3. Let $\{\eta_{l_1}, \dots, \eta_{l_r}\}$ be any finite subset of $\{\eta_l \mid l \ge 1\}$. Without loss of generality, we may assume that $l_1 < \cdots < l_r$. Let

$$\xi_i = \eta_{l_i} - [\eta_{l_i}] + 1 \in (1, 2)$$

for $i = 1, \dots, r$. Then $S(\xi_i) \ni 0$ for $i = 1, \dots, r$. We check that $\xi_1, \dots, \xi_r$ satisfy the assumptions of Theorem 2.1. Let l be a positive number. In Section 1 we proved that, for any sufficiently large x,

$$f_l(x) \le f_l(x+1) \le 2 f_l(x). \tag{3.1}$$

Thus, we have verified condition (2) with $C_1 = 1/2$. For any positive numbers l and $x \ge 1$, put

$$g_l(x) = \exp((\log x)^{1/(1+l)}). \tag{3.2}$$

Note that g_l is the inverse function of f_l, that is, $f_l(g_l(x)) = x$ for any $x \ge 1$. Let i and j be integers with $1 \le i, j \le r$. As mentioned in Section 1, for any $\varepsilon > 0$,

$$\lambda(\alpha, \xi_i, R) \sim g_{l_i}(R) = o(R^{\varepsilon}) \tag{3.3}$$

as $R \to \infty$. If $i < j$, then, for any $\varepsilon > 0$,

$$g_{l_j}(R) = o(\exp(\varepsilon(\log R)^{1/(1+l_i)})) = o(g_{l_i}(R)^{\varepsilon}). \tag{3.4}$$

Thus condition (1) holds by (3.3) and (3.4). Finally we check (3). We invoke the results of Daniel [6]. Let $(\mu_n)_{n=1}^{\infty}$ be the strictly increasing sequence of those positive integers that can be represented as the sum of three cubes of positive integers. Then Daniel showed that

$$\mu_{n+1} - \mu_n = O(\mu_n^{8/27}).$$

In the same way as in the proof of the result above, we get the following:

LEMMA 3.1. *Let* $\mathbf{k} = (a_1, \ldots, a_r) \in \mathbb{N}^r \setminus \{(0, \ldots, 0)\}$. *Then, for* $R \geq 2$,

$$R - \theta\Big(R; \sum_{i=1}^{r} a_i S(\xi_i)\Big) \ll R(\log R)^{a_1+\cdots+a_r} \mathbf{g}(R)^{-\mathbf{k}}, \tag{3.5}$$

where

$$\mathbf{g}(R)^{-\mathbf{k}} = \prod_{i=1}^{r} g_{l_i}(R)^{-a_i}.$$

Proof. We argue by induction on $a_1 + \cdots + a_r$. Assume that $a_1 + \cdots + a_r = 1$. Then there exists an integer h with $1 \leq h \leq r$ and $a_h = 1$. We have

$$f'_{l_h}(x) = \frac{(1 + l_h) f_{l_h}(x) (\log f_{l_h}(x))^{l_h/(1+l_h)}}{g_{l_h}(f_{l_h}(x))}.$$

Let x be a sufficiently large real number. Then, by the mean value theorem, there exists $\rho = \rho(x) \in (0, 1)$ such that

$$\begin{aligned} f_{l_h}(x+1) - f_{l_h}(x) &= \frac{(1 + l_h) f_{l_h}(x+\rho) (\log f_{l_h}(x+\rho))^{l_h/(1+l_h)}}{g_{l_h}(f_{l_h}(x+\rho))} \\ &\leq \frac{(1 + l_h) f_{l_h}(x+1) (\log f_{l_h}(x+1))^{l_h/(1+l_h)}}{g_{l_h}(f_{l_h}(x))} \\ &\ll \frac{f_{l_h}(x) \log f_{l_h}(x)}{g_{l_h}(f_{l_h}(x))}, \end{aligned}$$

where for the last inequality we use (3.1). For $R > 1$, let

$$F(R) = \frac{R \log R}{g_{l_h}(R)}.$$

Taking the logarithm of $F(R)$, we deduce that $F(R)$ is increasing for large R. If R is sufficiently large, then there exists $m \in \mathbb{N}$ such that

$$[f_{l_h}(m)] < R \leq [f_{l_h}(1 + m)].$$

Thus, we get $\theta(R; S(\xi_h)) = [f_{l_h}(m)]$. Since

$$F(f_{l_h}(m)) \ll F([f_{l_h}(m)]) \leq F(R),$$

we obtain

$$0 < R - \theta(R; S(\xi_h)) \leq f_{l_h}(m+1) - f_{l_h}(m) + 1 \ll F(f_{l_h}(m)) \ll F(R),$$

which implies (3.5) in the case of $a_1 + \cdots + a_r = 1$. Next, assume that $a_1 + \cdots + a_r \geq 2$. Let

$$d = \max\{i \geq 1 \mid a_i \geq 1\}.$$

Put

$$\mathbf{k}' = (a'_1, \ldots, a'_r) := (a_1, \ldots, a_{d-1}, -1 + a_d, 0, \ldots, 0).$$

Then, using the case of $a_1 + \cdots + a_r = 1$, we deduce that there exists a positive constant C satisfying

$$R' := R - \theta(R; S(\xi_d)) \leq C \frac{R \log R}{g_{l_d}(R)}.$$

Note that

$$R - \theta\Big(R; \sum_{i=1}^{r} a_i S(\xi_i)\Big) \leq R'$$

because $\sum_{i=1}^{r} a_i S(\xi_i) \supset S(\xi_d)$. Thus, we may assume that $R' \geq 2$. By the induction hypothesis, we get

$$R' - \theta\Big(R'; \sum_{i=1}^{r} a'_i S(\xi_i)\Big) \ll R' (\log R')^{a'_1 + \cdots + a'_r} \mathbf{g}(R')^{-\mathbf{k}'} =: G(R').$$

Let

$$\gamma = \theta(R; S(\xi_d)) + \theta\Big(R'; \sum_{i=1}^{r} a'_i S(\xi_i)\Big).$$

Then since $\gamma \in \sum_{i=1}^{r} a_i S(\xi_i)$, we get

$$\begin{aligned} 0 < R - \theta\Big(R; \sum_{i=1}^{r} a_i S(\xi_i)\Big) &\leq R - \gamma \\ &= R' - \theta\Big(R'; \sum_{i=1}^{r} a'_i S(\xi_i)\Big) \ll G(R'). \end{aligned} \tag{3.6}$$

Taking the logarithm of $G(R)$, we deduce that the function $G(R)$ is increasing for sufficiently large R. Thus, we obtain

$$\begin{aligned} G(R') &\ll G\Big(C \frac{R \log R}{g_{l_d}(R)}\Big) \ll G\Big(\frac{R \log R}{g_{l_d}(R)}\Big) \\ &= \frac{R \log R}{g_{l_d}(R)} \Big(\log \frac{R \log R}{g_{l_d}(R)}\Big)^{a'_1 + \cdots + a'_r} \mathbf{g}\Big(\frac{R \log R}{g_{l_d}(R)}\Big)^{-\mathbf{k}'} \\ &\ll \frac{R}{g_{l_d}(R)} (\log R)^{a_1 + \cdots + a_r} \mathbf{g}\Big(\frac{R}{g_{l_d}(R)}\Big)^{-\mathbf{k}'}. \end{aligned} \tag{3.7}$$

Let $i \in \mathbb{N}$ with $1 \le i \le d$. Since $l_i \ge 1$, we observe that, for any sufficiently large R,

$$\begin{aligned}\left(\log\left(\frac{R}{g_{l_i}(R)}\right)\right)^{1/(1+l_i)} &= (\log R - (\log R)^{1/(1+l_i)})^{1/(1+l_i)} \\ &= (\log R)^{1/(1+l_i)}(1-(\log R)^{-l_i/(1+l_i)})^{1/(1+l_i)} \\ &\ge (\log R)^{1/(1+l_i)}\left(1-\frac{2}{1+l_i}(\log R)^{-l_i/(1+l_i)}\right) \\ &\ge (\log R)^{1/(1+l_i)} - 1\end{aligned}$$

and hence

$$g_{l_i}\left(\frac{R}{g_{l_d}(R)}\right) \gg g_{l_i}\left(\frac{R}{g_{l_i}(R)}\right) \gg g_{l_i}(R).$$

Therefore,

$$\mathbf{g}\left(\frac{R}{g_{l_d}(R)}\right)^{-\mathbf{k}'} \ll \mathbf{g}(R)^{-\mathbf{k}'}. \tag{3.8}$$

Combining the inequalities (3.6)–(3.8), we conclude that

$$0 < R - \theta\Big(R; \sum_{i=1}^{r} a_i S(\xi_i)\Big) \ll R(\log R)^{a_1+\cdots+a_r}\mathbf{g}(R)^{-\mathbf{k}},$$

which implies (3.5). ∎

Let $\mathbf{k} = (a_1, \ldots, a_r) \in \mathbb{N}^r$. Then, by (3.3), (3.4) and Lemma 3.1,

$$\begin{aligned}&R - \theta\Big(R; \sum_{i=1}^{r-2} a_i S(\xi_i) + (1+a_{r-1})S(\xi_{r-1})\Big) \\ &\qquad \le R g_{l_{r-1}}(R)^{-1/2}\prod_{i=1}^{r-1} g_{l_i}(R)^{-a_i} = o\Big(R\prod_{i=1}^{r}\lambda(\alpha,\xi_i,R)^{-a_i}\Big)\end{aligned}$$

as $R \to \infty$. Hence, condition (3) of Theorem 2.1 is satisfied with

$$\kappa = \kappa(a_1, \ldots, a_{r-1}) = 1 + a_{r-1}.$$

Thus we have proved Theorem 1.3. ∎

Proof of Theorem 1.4. Without loss of generality, we may assume that $h < l$. Let $\xi_1 = \eta_h - [\eta_h] + 1$ and $\xi_2 = \eta_l - [\eta_l] + 1$. Note that $1 < \xi_1, \xi_2 < 2$ and that $S(\xi_1), S(\xi_2) \ni 0$. As in the proof of Theorem 1.3, we can verify that ξ_1 and ξ_2 satisfy conditions (1) and (2) of Theorem 2.1 with $C_1 = 1/2$. In what follows, we prove that (3) is also satisfied. Let $g_l(x)$ be defined by (3.2).

LEMMA 3.2. *Let b be a positive integer. Then, for any $\varepsilon > 0$,*

$$R - \theta(R; bS(\xi_1)) \ll R g_h(R)^{-b+\varepsilon} \quad \textit{for } R \ge 2. \tag{3.9}$$

Proof. We show (3.9) by induction on b. Assume that $b = 1$. As in the proof of Lemma 3.1, we deduce that there exists a positive constant C satisfying

$$R' := R - \theta(R; S(\xi_1)) \leq C\frac{R\log R}{g_h(R)}, \tag{3.10}$$

which implies (3.9) because, for any positive ε,

$$\log R = o(g_h(R)^{\varepsilon})$$

as $R \to \infty$. Suppose that $b \geq 2$. Without loss of generality, we may assume that $R' \geq 2$ and $\varepsilon < 1$. In particular,

$$-b + 1 + \varepsilon < 0.$$

By the induction hypothesis,

$$R' - \theta(R'; (b-1)S(\xi_1)) \ll R' g_h(R')^{-b+1+\varepsilon/3} =: H(R').$$

Taking the logarithm of $H(R)$, we see that the function $H(R)$ is increasing for large R. Hence,

$$\begin{aligned}
0 &< R - \theta(R; bS(\xi_1)) \\
&\leq R - \theta(R; S(\xi_1)) - \theta(R'; (b-1)S(\xi_1)) \\
&= R' - \theta(R'; (b-1)S(\xi_1)) \ll H(R') \\
&\ll H\left(C\frac{R\log R}{g_h(R)}\right) \ll H\left(\frac{R\log R}{g_h(R)}\right) \\
&= \frac{R\log R}{g_h(R)} g_h\left(\frac{R\log R}{g_h(R)}\right)^{-b+1+\varepsilon/3} \\
&\ll R g_h(R)^{-1+\varepsilon/3} g_h\left(\frac{R}{g_h(R)}\right)^{-b+1+\varepsilon/3}.
\end{aligned} \tag{3.11}$$

Let

$$\varepsilon' := \frac{\varepsilon}{3b - 3 - \varepsilon} \in (0, 1).$$

Then

$$(1 - \varepsilon')\left(-b + 1 + \frac{\varepsilon}{3}\right) = -b + 1 + \frac{2}{3}\varepsilon.$$

For all sufficiently large R, we obtain

$$\begin{aligned}
\left(\log\left(\frac{R}{g_h(R)}\right)\right)^{1/(1+h)} &= (\log R - (\log R)^{1/(1+h)})^{1/(1+h)} \\
&\geq (1 - \varepsilon')(\log R)^{1/(1+h)}
\end{aligned}$$

and hence

$$g_h\left(\frac{R}{g_h(R)}\right)^{-b+1+\varepsilon/3} \leq g_h(R)^{(1-\varepsilon')(-b+1+\varepsilon/3)} = g_h(R)^{-b+1+2\varepsilon/3}.$$

Combining (3.11) and the inequality above yields (3.9). ∎

Let $(a_1, a_2) \in \mathbb{N}^2$. Then, applying Lemma 3.2 with $b = a_1+1$ and $\varepsilon = 1/2$, we get

$$\begin{aligned} R - \theta(R; (a_1+1)S(\xi_1)) &\ll R g_h(R)^{-a_1-1/2} \\ &= o\Big(R\prod_{i=1}^{2}\lambda(\alpha, \xi_i, R)^{-a_i}\Big) \end{aligned}$$

as $R \to \infty$. Thus we have checked condition (3) of Theorem 2.1 with $\kappa = \kappa(a_1) = a_1 + 1$ and hence verified Theorem 1.4. ∎

4. Proof of Theorem 2.1

4.1. Base-α expansions of powers of real numbers. We prove Theorem 2.1 by induction on r. Corollary 1.2 yields the case of $r = 1$. In what follows, suppose that $r \geq 2$. We may assume that $1 \leq \xi_1, \ldots, \xi_r < 2$. Indeed, $\xi_1, \ldots, \xi_r$ are algebraically independent if and only if $\xi_1', \ldots, \xi_r'$ are algebraically independent, where

$$\xi_i' = \xi_i - [\xi_i] + 1 \quad \text{for } i = 1, \ldots, r.$$

For simplicity, let

$$\lambda_i(R) = \lambda(\alpha, \xi_i, R) \quad \text{for } i = 1, \ldots, r \text{ and } R > 0.$$

For convenience, put $\mathbb{N}^0 = \{0\}$. Let $\xi > 0$, $b \in \mathbb{N}$, $\mathbf{x} = (x_1, \ldots, x_b) \in \mathbb{N}^b$, and $\mathbf{s} = (s_1, \ldots, s_b) \in \mathbb{Z}^b$. Put

$$|\mathbf{x}| = \begin{cases} 0 & (b = 0), \\ x_1 + \cdots + x_b & (b \geq 1), \end{cases}$$

$$t(\xi, \mathbf{x}) = \begin{cases} 1 & (b = 0), \\ t(\xi, x_1)\cdots t(\xi, x_b) & (b \geq 1), \end{cases}$$

$$\mathbf{x}^{\mathbf{s}} = \begin{cases} 1 & (b = 0), \\ x_1^{s_1}\cdots x_b^{s_b} & (b \geq 1). \end{cases}$$

Denote $(\xi_1, \ldots, \xi_r)$ and $(\lambda_1(R), \ldots, \lambda_r(R))$ by $\underline{\xi}$ and $\underline{\lambda}(R)$, respectively. Then, for each $\mathbf{k} = (a_1, \ldots, a_r) \in \mathbb{N}^r \setminus \{(0, \ldots, 0)\}$,

$$\underline{\xi}^{\mathbf{k}} = \prod_{i=1}^{r}\Big(\sum_{x=0}^{\infty} t(\xi_i, x)\alpha^{-x}\Big)^{a_i} = \prod_{i=1}^{r}\Big(\sum_{\mathbf{x}\in\mathbb{N}^{a_i}} t(\xi_i, \mathbf{x})\alpha^{-|\mathbf{x}|}\Big) \tag{4.1}$$
$$= \sum_{\mathbf{x}_1\in\mathbb{N}^{a_1},\ldots,\mathbf{x}_r\in\mathbb{N}^{a_r}} t(\xi_1, \mathbf{x}_1)\cdots t(\xi_r, \mathbf{x}_r)\alpha^{-|\mathbf{x}_1|-\cdots-|\mathbf{x}_r|}$$
$$= \sum_{m=0}^{\infty} \rho(\mathbf{k}, m)\alpha^{-m},$$

where

$$\rho(\mathbf{k}, m) := \sum_{\substack{\mathbf{x}_1\in\mathbb{N}^{a_1},\ldots,\mathbf{x}_r\in\mathbb{N}^{a_r}\\ |\mathbf{x}_1|+\cdots+|\mathbf{x}_r|=m}} t(\xi_1, \mathbf{x}_1)\cdots t(\xi_r, \mathbf{x}_r) \in \mathbb{N}.$$

Note that, for each $m \in \mathbb{N}$, $\rho(\mathbf{k}, m) > 0$ if and only if $m \in \sum_{i=1}^{r} a_i S(\xi_i)$. It is easily seen that

$$\rho(\mathbf{k}, m) \le \sum_{\substack{\mathbf{x}_1\in\mathbb{N}^{a_1},\ldots,\mathbf{x}_r\in\mathbb{N}^{a_r}\\ |\mathbf{x}_1|+\cdots+|\mathbf{x}_r|=m}} (\alpha-1)^{|\mathbf{k}|} = (\alpha-1)^{|\mathbf{k}|}\binom{m+|\mathbf{k}|-1}{|\mathbf{k}|-1}. \tag{4.2}$$

We now check the following:

LEMMA 4.1. *Let* $\mathbf{k} = (a_1, \ldots, a_r) \in \mathbb{N}^r \setminus \{(0, \ldots, 0)\}$ *and* $N \in \mathbb{N}$. *Then*

$$\sum_{m=0}^{N} \rho(\mathbf{k}, m) \le (\alpha-1)^{|\mathbf{k}|}\underline{\lambda}(N)^{\mathbf{k}},$$
$$\mathrm{Card}\{m \in \mathbb{N} \mid m \le N,\ \rho(\mathbf{k}, m) > 0\} \le (\alpha-1)^{|\mathbf{k}|}\underline{\lambda}(N)^{\mathbf{k}}.$$

Proof. Put, for $i = 1, \ldots, r$,

$$S_i = \{m \in S(\xi_i) \mid m \le N\}, \qquad S_i^0 = \{0\}.$$

Then

$$\sum_{m=0}^{N} \rho(\mathbf{k}, m) = \sum_{\substack{\mathbf{x}_1\in\mathbb{N}^{a_1},\ldots,\mathbf{x}_r\in\mathbb{N}^{a_r}\\ |\mathbf{x}_1|+\cdots+|\mathbf{x}_r|\le N}} t(\xi_1, \mathbf{x}_1)\cdots t(\xi_r, \mathbf{x}_r)$$
$$\le \sum_{\mathbf{x}_1\in S_1^{a_1},\ldots,\mathbf{x}_r\in S_r^{a_r}} (\alpha-1)^{|\mathbf{k}|} = (\alpha-1)^{|\mathbf{k}|}\underline{\lambda}(N)^{\mathbf{k}},$$

which implies the first statement. The second follows from the first because $\rho(\mathbf{k}, m) \in \mathbb{N}$ for each $m \in \mathbb{N}$. ∎

4.2. Auxiliary functions. We define the lexicographical order $\succ$ on $\mathbb{N}^r$ as follows. For any $\mathbf{k} = (a_1, \ldots, a_r)$, $\mathbf{k}' = (a_1', \ldots, a_r')$ with $\mathbf{k} \ne \mathbf{k}'$, there exists a positive l such that the first $l-1$ symbols in $\mathbf{k} = (a_1, \ldots, a_r)$ and $\mathbf{k}' = (a_1', \ldots, a_r')$ coincide, but their lth symbols are different. Then $\mathbf{k} = (a_1, \ldots, a_r) \succ \mathbf{k}' = (a_1', \ldots, a_r')$ if and only if $a_l > a_l'$.

Each nonzero polynomial $Q(\underline{X}) \in \mathbb{Z}[X_1, \ldots, X_r]$ is uniquely written as

$$Q(\underline{X}) = \sum_{\mathbf{k} \in \Lambda(Q)} B_{\mathbf{k}} \underline{X}^{\mathbf{k}},$$

where $\Lambda(Q)$ is a finite subset of $\mathbb{N}^r$ determined by Q, $B_{\mathbf{k}}$ a nonzero integer and $\underline{X} = (X_1, \ldots, X_r)$. Recall that $\underline{X}^{\mathbf{k}} = X_1^{a_1} \cdots X_r^{a_r}$ for $\mathbf{k} = (a_1, \ldots, a_r)$. Let $\mathbf{g}(Q) = (g_1(Q), \ldots, g_r(Q))$ be the greatest element of $\Lambda(Q)$ with respect to $\succ$. Moreover, put

$$\Lambda_1(Q) = \{\mathbf{k} \in \Lambda(Q) \mid a_1 = g_1(Q), \ldots, a_{r-1} = g_{r-1}(Q),\ a_r < g_r(Q)\},$$
$$\Lambda_2(Q) = \Lambda(Q) \setminus (\Lambda_1(Q) \cup \{\mathbf{g}(Q)\}),$$
$$\Lambda_3(Q) = \{\mathbf{k} \in \Lambda(Q) \mid a_1 = g_1(Q), \ldots, a_{r-2} = g_{r-2}(Q),\ a_{r-1} < g_{r-1}(Q)\},$$

where $\mathbf{k} = (a_1, \ldots, a_r)$. We define a number $e(Q)$ as follows. If $\Lambda_3(Q)$ is empty, then put $e(Q) = 0$. Otherwise, let

$$e(Q) = \max\{a_{r-1} \mid (a_1, \ldots, a_{r-1}, a_r) \in \Lambda_3(Q)\}.$$

Now assume that $\xi_1, \ldots, \xi_r$ are algebraically dependent. Then there exists a nonzero polynomial $P(\underline{X}) \in \mathbb{Z}[X_1, \ldots, X_r]$ such that

$$P(\underline{\xi}) = 0.$$

By the induction hypothesis $\xi_2, \ldots, \xi_r$ are algebraically independent. Thus, the degree of $P(\underline{X})$ in X_1 is positive, so $g_1(P) \geq 1$. Without loss of generality, we may assume that

$$X_r(X_r - 1) \mid P(\underline{X}). \tag{4.3}$$

Let

$$\kappa(n) := \kappa(g_1(P), \ldots, g_{r-2}(P), n),$$

where n is a nonnegative integer and the right-hand side is defined in condition (3) of Theorem 2.1. Let m and n be integers with $0 \leq m \leq n$. Then, for any positive number R,

$$R - \theta\Big(R; \sum_{i=1}^{r-2} g_i(P) S(\xi_i) + n S(\xi_{r-1})\Big) \leq R - \theta\Big(R; \sum_{i=1}^{r-2} g_i(P) S(\xi_i) + m S(\xi_{r-1})\Big)$$

because

$$\sum_{i=1}^{r-2} g_i(P) S(\xi_i) + n S(\xi_{r-1}) \supset \sum_{i=1}^{r-2} g_i(P) S(\xi_i) + m S(\xi_{r-1}).$$

So, by increasing $\kappa(n)$ if necessary, we may assume that $\kappa(n) \geq 1$ for any $n \in \mathbb{N}$ and that the sequence $(\kappa(n))_{n=0}^{\infty}$ is increasing.

Lemma 4.2. *There is a nonzero polynomial $F(X_{r-1}, X_r) \in \mathbb{Z}[X_{r-1}, X_r]$ such that*

$$g_{r-1}(FP) \geq \kappa(e(FP)).$$

Proof. We define a nonzero polynomial $\sigma(X_{r-1},X_r)\in\mathbb{Z}[X_{r-1},X_r]$ as follows. If $r=2$, then put

$$\sigma(X_1,X_2):=P(X_1,X_2). \tag{4.4}$$

If $r\geq 3$, then $P(\underline{X})$ is uniquely written as

$$P(\underline{X})=\sum_{\mathbf{k}=(a_1,\ldots,a_{r-2})\in\Gamma}\varphi_{\mathbf{k}}(X_{r-1},X_r)X_1^{a_1}\cdots X_{r-2}^{a_{r-2}}, \tag{4.5}$$

where Γ is a finite subset of $\mathbb{N}^{r-2}$ and $\varphi_{\mathbf{k}}(X_{r-1},X_r)\in\mathbb{Z}[X_{r-1},X_r]$ a nonzero polynomial. Note that $\mathbf{l}:=(g_1(P),\ldots,g_{r-2}(P))\in\Gamma$. Now put

$$\sigma(X_{r-1},X_r):=\varphi_{\mathbf{l}}(X_{r-1},X_r).$$

Let

$$\sigma(X_{r-1},X_r)=:\sum_{i=0}^{b}\sigma_i(X_r)X_{r-1}^i,$$

where $\sigma_i(X_r)\in\mathbb{Z}[X_r]$ with $\sigma_b(X_r)\neq 0$. We show that for any integer $n\geq b$ there is a nonzero polynomial $\psi^{(n)}(X_{r-1},X_r)\in\mathbb{Z}[X_{r-1},X_r]$ such that $\sigma(X_{r-1},X_r)\psi^{(n)}(X_{r-1},X_r)$ can be written as

$$\sigma(X_{r-1},X_r)\psi^{(n)}(X_{r-1},X_r)=\psi_b^{(n)}(X_r)X_{r-1}^n+\sum_{i=0}^{b-1}\psi_i^{(n)}(X_r)X_{r-1}^i, \tag{4.6}$$

where $\psi_i^{(n)}(X_r)\in\mathbb{Z}[X_r]$ for $i=0,1,\ldots,b$ with $\psi_b^{(n)}(X_r)\neq 0$. In the case of $b=0$, it is clear that $\psi^{(n)}(X_{r-1},X_r)=X_{r-1}^n$ satisfies (4.6). Suppose that $b\geq 1$. We check (4.6) by induction on n. If $n=b$, then putting $\psi^{(b)}(X_{r-1},X_r)=1$, we get (4.6). Assume that $n\geq b+1$. Then the induction hypothesis implies that

$$\psi^{(n)}(X_{r-1},X_r)=\sigma_b(X_r)X_{r-1}\psi^{(n-1)}(X_{r-1},X_r)-\psi_{-1+b}^{(n-1)}(X_r)$$

satisfies (4.6). Moreover, $\psi^{(n)}(X_{r-1},X_r)\neq 0$ since $\sigma_b(X_r)\psi^{(n-1)}(X_{r-1},X_r)\neq 0$. Let $w=\max\{0,b-1\}$. In what follows, we verify that

$$F(X_{r-1},X_r):=\psi^{(\kappa(w))}(X_{r-1},X_r)$$

satisfies the statement of Lemma 4.2. Using (4.5) and (4.6), we deduce that the first $r-2$ symbols of $\mathbf{g}(P)$ and $\mathbf{g}(FP)$ coincide in the case of $r\geq 3$. Moreover,

$$g_{r-1}(FP)=\kappa(w), \tag{4.7}$$

$$e(FP)\leq w. \tag{4.8}$$

In fact, if $\Lambda_3(FP)\neq\emptyset$, then by (4.4)–(4.6), we get $e(FP)\leq b-1$. Hence, combining (4.7) and (4.8), we conclude that $g_{r-1}(FP)\geq\kappa(e(FP))$ because the sequence $(\kappa(n))_{n=0}^{\infty}$ is increasing. ∎

For simplicity, put

$$\begin{aligned} \Lambda &= \Lambda(FP), \\ \Lambda_h &= \Lambda_h(FP) \quad \text{for } 1 \le h \le 3, \\ \mathbf{k}_0 &= (g_1, \ldots, g_r) := \mathbf{g}(FP). \end{aligned}$$

Recall that $g_i = g_i(FP) = g_i(P)$ for $i = 1, \ldots, r-2$, so

$$\kappa(n) = \kappa(g_1, \ldots, g_{r-2}, n)$$

for each $n \in \mathbb{N}$. Let

$$F(X_{r-1}, X_r)P(\underline{X}) = \sum_{\mathbf{k} \in \Lambda} A_{\mathbf{k}} \underline{X}^{\mathbf{k}},$$

where $A_{\mathbf{k}}$ is a nonzero integer. Then

$$\sum_{\mathbf{k} \in \Lambda} A_{\mathbf{k}} \underline{\xi}^{\mathbf{k}} = 0. \tag{4.9}$$

Note that $|\mathbf{k}| \ge 1$ for each $\mathbf{k} \in \Lambda$, because X_r divides $P(\underline{X})$. Without loss of generality, we may assume that $A_{\mathbf{k}_0} \ge 1$.

LEMMA 4.3. *Λ_1 and Λ_2 are not empty.*

Proof. First suppose that Λ_2 is empty. Then, for each $\mathbf{k} = (a_1, \ldots, a_r)$, we have $a_1 = g_1, \ldots, a_{r-1} = g_{r-1}$. Thus, (4.9) implies that ξ_r is an algebraic number, which contradicts the induction hypothesis.

Next, assume that Λ_1 is empty. Then we get

$$\sum_{\substack{\mathbf{k}=(a_1,\ldots,a_r)\in\Lambda \\ a_1=g_1,\ldots,a_{r-1}=g_{r-1}}} A_{\mathbf{k}} \underline{X}^{\mathbf{k}} = A_{\mathbf{k}_0} \underline{X}^{\mathbf{k}_0}. \tag{4.10}$$

Let $\Phi : \mathbb{Z}[X_1, \ldots, X_r] \to \mathbb{Z}[X_1, \ldots, X_{r-1}]$ be defined by

$$\Phi(Q(X_1, \ldots, X_r)) = Q(X_1, \ldots, X_{r-1}, 1).$$

By (4.10), the greatest element of $\Phi(F(X_{r-1}, X_r)P(\underline{X}))$ with respect to the lexicographical order on $\mathbb{N}^{r-1}$ is $(g_1, \ldots, g_{r-1})$. So, $\Phi(F(X_{r-1}, X_r)P(\underline{X}))$ is not zero, that is, $X_r - 1$ does not divide $F(X_{r-1}, X_r)P(\underline{X})$, which contradicts (4.3). ∎

Let

$$D = 1 + \max\{|\mathbf{k}| \mid \mathbf{k} \in \Lambda\}.$$

Denote the greatest elements of Λ_1 and Λ_2 with respect to $\succ$ by $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. Let

$$\mathbf{e} = (g_1, \ldots, g_{r-2}, e(FP), D).$$

Then $\mathbf{k} \prec \mathbf{e}$ for each $\mathbf{k} \in \Lambda_2$. Indeed, if Λ_3 is empty, then there exists a positive $l \le r-2$ such that the first $l-1$ symbols in $\mathbf{e}$ and $\mathbf{k}_2$ coincide, but the lth symbol of $\mathbf{e}$ is greater than that of $\mathbf{k}_2$. Otherwise,

$$\mathbf{k}_2 = (g_1, \dots, g_{r-2}, e(FP), a)$$

with $a < D$, and so $\mathbf{k}_2 \prec \mathbf{e}$. By condition (1) of Theorem 2.1, for any $\mathbf{k} \in \Lambda_2$,

$$\underline{\lambda}(n)^{\mathbf{k}} = o(\underline{\lambda}(n)^{\mathbf{e}}). \tag{4.11}$$

Lemma 4.2 implies

$$g_{r-1} \geq \kappa(e(FP)). \tag{4.12}$$

Let Ξ be the set of nonnegative integers N such that, for every integer n with $0 \leq n \leq N$,

$$n\underline{\lambda}(n)^{-\mathbf{e}} \leq N\underline{\lambda}(N)^{-\mathbf{e}}. \tag{4.13}$$

Note that Ξ is infinite. Indeed, by condition (1) of Theorem 2.1, we have

$$\lim_{N\to\infty} N\underline{\lambda}(N)^{-\mathbf{e}} = \infty.$$

If necessary, by increasing $C_2(\mathbf{e})$, we may assume that $\lambda_r(n) \geq 5$ for every $n \in \mathbb{N}$ with $n \geq C_2(\mathbf{e})$, where $C_2(\mathbf{e})$ is defined in condition (3) of Theorem 2.1. For simplicity, let

$$\theta(R) = \theta\Big(R; \sum_{i=1}^{r-1} g_i S(\xi_i)\Big).$$

Lemma 4.4. *Let M and E be any positive real numbers with*

$$M \geq C_2(\mathbf{e}) \quad \text{and} \quad E \geq 4M\underline{\lambda}(M)^{-\mathbf{e}}.$$

Then

$$M + E/2 < \theta(M + E).$$

Proof. Using

$$E/4 \geq M\underline{\lambda}(M)^{-\mathbf{e}}, \qquad E/4 > E\underline{\lambda}(M)^{-\mathbf{e}},$$

we get

$$E/2 > (M + E)\underline{\lambda}(M)^{-\mathbf{e}} \geq (M + E)\underline{\lambda}(M + E)^{-\mathbf{e}}.$$

Note that $M + E \geq C_2(\mathbf{e})$. Thus, using (4.12) and condition (3) of Theorem 2.1 with $(a_1, \dots, a_r) = \mathbf{e}$, $R = M + E$, we deduce that

$$M + E - \theta(M + E) < (M + E)\underline{\lambda}(M + E)^{-\mathbf{e}} < E/2,$$

which implies the conclusion. ∎

Using (4.1) and (4.9), we get, for each $R \in \mathbb{N}$,

$$0 = \alpha^R \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \underline{\xi}^{\mathbf{k}} = \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=-R}^{\infty} \rho(\mathbf{k}, m + R)\alpha^{-m},$$

so

$$Y_R := \sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=1}^{\infty} \rho(\mathbf{k}, m+R)\alpha^{-m} \in \mathbb{Z}.$$

Let $N \in \mathbb{N}$. In what follows, we estimate the number

$$y(N) = \mathrm{Card}\{R \in \mathbb{N} \mid R \le N,\ Y_R > 0\}.$$

4.3. Bounds for $y(N)$. First, we consider upper bounds for $y(N)$.

LEMMA 4.5. *We have*

$$y(N) = o(N)$$

as $N \to \infty$.

Proof. For $\mathbf{k} \in \Lambda$ and $R \in \mathbb{N}$, let

$$Y(\mathbf{k}, R) = \sum_{m=1}^{\infty} \rho(\mathbf{k}, m+R)\alpha^{-m} \ge 0.$$

Then by (4.2),

$$\begin{aligned} Y(\mathbf{k}, R) &\le \sum_{m=1}^{\infty} (\alpha-1)^{|\mathbf{k}|} \binom{m+R+|\mathbf{k}|-1}{|\mathbf{k}|-1} \alpha^{-m} \\ &\le (\alpha-1)^{|\mathbf{k}|} \sum_{m=1}^{\infty} \binom{m+R+|\mathbf{k}|-1}{|\mathbf{k}|-1} 2^{-m}. \end{aligned}$$

In the proof of Theorem 2.1 of [1], Bailey, Borwein, Crandall, and Pomerance showed that for $R \ge 0$ and $l \ge 1$,

$$\sum_{m=1}^{\infty} \binom{m+R+l-1}{l-1} 2^{-m} < \frac{(R+l)^l}{(l-1)!(R+1)}.$$

Since $|\mathbf{k}| \ge 1$, we get

$$Y(\mathbf{k}, R) < \frac{(\alpha-1)^{|\mathbf{k}|}(R+|\mathbf{k}|)^{|\mathbf{k}|}}{(|\mathbf{k}|-1)!(R+1)}. \tag{4.14}$$

In particular,

$$\begin{aligned} \sum_{R=0}^{N} Y(\mathbf{k}, R) &< \sum_{R=0}^{N} \frac{(\alpha-1)^{|\mathbf{k}|}(N+|\mathbf{k}|)^{|\mathbf{k}|}}{(|\mathbf{k}|-1)!} \\ &\le \frac{(\alpha-1)^{|\mathbf{k}|}(N+|\mathbf{k}|)^{|\mathbf{k}|+1}}{(|\mathbf{k}|-1)!}. \end{aligned} \tag{4.15}$$

By condition (1) of Theorem 2.1, we have

$$\underline{\lambda}(N)^{\mathbf{k}} = o(N). \tag{4.16}$$

Let $K = \lceil D \log_\alpha N \rceil$, where $\log_\alpha N = (\log N)/(\log \alpha)$ and $\lceil x \rceil$ is the smallest integer greater than or equal to x. Then by (4.15), (4.16) and the first inequality of Lemma 4.1, we get

$$\begin{aligned}\sum_{R=0}^{N-K} Y(\mathbf{k},R) &= \sum_{m=1}^{\infty}\alpha^{-m}\sum_{R=0}^{N-K}\rho(\mathbf{k},m+R)\\ &\le \sum_{m=1}^{K}\alpha^{-m}\sum_{R=0}^{N}\rho(\mathbf{k},R)+\alpha^{-K}\sum_{m=1+K}^{\infty}\alpha^{K-m}\sum_{R=0}^{N-K}\rho(\mathbf{k},m+R)\\ &\le (\alpha-1)^{|\mathbf{k}|}\underline{\lambda}(N)^{\mathbf{k}}+\alpha^{-K}\sum_{R=0}^{N-K}\sum_{m=1}^{\infty}\rho(\mathbf{k},m+R+K)\alpha^{-m}\\ &= o(N)+\alpha^{-K}\sum_{R=0}^{N-K}Y(\mathbf{k},R+K)\\ &\le o(N)+N^{-D}(\alpha-1)^{D}(N+D)^{D}=o(N).\end{aligned}$$

Since $Y(\mathbf{k},R)\ge 0$,

$$\sum_{R=0}^{N-K}|Y_R|\le\sum_{\mathbf{k}\in\Lambda}|A_{\mathbf{k}}|\sum_{R=0}^{N-K}Y(\mathbf{k},R)=o(N).$$

Using $Y_R\in\mathbb{Z}$, we obtain

$$y(N)\le K+\sum_{R=0}^{N-K}|Y_R|=o(N). \ \blacksquare$$

Next, we find lower bounds for $y(N)$.

LEMMA 4.6. *Let $N\in\mathbb{N}$ be sufficiently large and $I=[U_1,U_2)$ an interval with $I\subset[0,N)$. Suppose that $\rho(\mathbf{k},x)=0$ for any integer $x\in(U_1,U_2)$ and $\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}$. Moreover, assume that there exists $U\in\mathbb{N}$ satisfying*

$$U_1<U\le U_2-D\log_\alpha N \quad\textit{and}\quad \rho(\mathbf{k}_0,U)>0.$$

Then $Y_n>0$ for any $n\in[U_1,U)$.

Proof. We use induction on n. First we consider the case of $n=U-1$. Using (4.14), $A_{\mathbf{k}_0}\ge 1$, and the assumptions on I and U, we obtain

$$\begin{aligned}Y_{U-1}&=\sum_{\mathbf{k}\in\Lambda}A_{\mathbf{k}}\sum_{m=1}^{\infty}\rho(\mathbf{k},m+U-1)\alpha^{-m}\\ &\ge\frac{1}{\alpha}-\sum_{\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}}|A_{\mathbf{k}}|\sum_{m=1+U_2-U}^{\infty}\rho(\mathbf{k},m+U-1)\alpha^{-m}\end{aligned}$$

$$= \frac{1}{\alpha} - \sum_{\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}} |A_{\mathbf{k}}|\alpha^{U-U_2} Y(\mathbf{k}, U_2 - 1)$$
$$\geq \frac{1}{\alpha} - \sum_{\mathbf{k}\in\Lambda\setminus\{\mathbf{k}_0\}} |A_{\mathbf{k}}| N^{-D}(\alpha-1)^{D-1}(N+D-1)^{D-1} > 0$$

for all sufficiently large N.

Next, suppose that $Y_n > 0$ for some $n \in \mathbb{N}$ with $1 + U_1 \leq n \leq U - 1$. Then from $A_{\mathbf{k}_0}\rho(\mathbf{k}_0, n) \geq 0$ we get

$$Y_{n-1} = \frac{1}{\alpha}\sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}}\rho(\mathbf{k}, n) + \frac{1}{\alpha}\sum_{\mathbf{k}\in\Lambda} A_{\mathbf{k}} \sum_{m=2}^{\infty} \rho(\mathbf{k}, m+n-1)\alpha^{-m+1}$$
$$= \frac{1}{\alpha} A_{\mathbf{k}_0}\rho(\mathbf{k}_0, n) + \frac{1}{\alpha} Y_n > 0.$$

Hence we have verified Lemma 4.6. ∎

4.4. Completion of the proof of Theorem 2.1. We construct intervals $I = [U_1, U_2)$ satisfying the assumptions of Lemma 4.6. Using (4.11) and the second inequality of Lemma 4.1, we deduce the following: Let $N \in \Xi$ be sufficiently large. Then the number of nonnegative integers T with $T \leq N$ such that there exists a $\mathbf{k} \in \Lambda_2$ with $\rho(\mathbf{k}, T) > 0$ is at most

$$\sum_{\mathbf{k}\in\Lambda_2} (\alpha-1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}} \leq \frac{1}{32}\underline{\lambda}(N)^{\mathbf{e}}.$$

Say these T's are $0 = T_1 < \cdots < T_\tau$, where

$$\tau \leq \frac{1}{32}\underline{\lambda}(N)^{\mathbf{e}}. \tag{4.17}$$

Set $T_{1+\tau} = N$ and

$$\mathcal{J} = \{J = J(j) = [T_j, T_{1+j}) \mid 1 \leq j \leq \tau\}.$$

For any interval $I \subset \mathbb{R}$, let $|I|$ denote its length. Then

$$\sum_{J\in\mathcal{J}} |J| = N. \tag{4.18}$$

Moreover, put

$$\mathcal{J}_1 = \{J \in \mathcal{J} \mid |J| \geq 16N\underline{\lambda}(N)^{-\mathbf{e}}\}, \quad \mathcal{J}_2 = \{J \in \mathcal{J}_1 \mid J \subset [C_2(\mathbf{e}), N)\}.$$

Lemma 4.7. *Let $N \in \Xi$ be sufficiently large. Then*

$$\sum_{J\in\mathcal{J}_1} |J| \geq N/2, \quad \sum_{J\in\mathcal{J}_2} |J| \geq N/3.$$

Proof. By (4.17) and (4.18),

$$\sum_{J\in\mathcal{J}_1} |J| = \sum_{J\in\mathcal{J}} |J| - \sum_{J\in\mathcal{J}\setminus\mathcal{J}_1} |J| \geq N - \tau\cdot 16N\underline{\lambda}(N)^{-\mathbf{e}} \geq N/2,$$

which implies the first inequality. We now check the second. Take positive integers $N_0 < N_1$ satisfying $N_i > C_2(\mathbf{e})$ and $\rho(\mathbf{k}_2, N_i) > 0$ for $i = 0, 1$. If $N > N_1$, then there exists $j_0 = j_0(N)$ with

$$T_{j_0} = N_0, \quad T_{1+j_0} \le N_1$$

by the definition of $T_1, \ldots, T_{1+\tau}$. Let $J(j) \in \mathcal{J}_1 \setminus \mathcal{J}_2$. Then $j \le j_0$. Hence, for any $N \in \Xi$ with $N \ge 6N_1$,

$$\sum_{J \in \mathcal{J}_2} |J| \ge \sum_{J \in \mathcal{J}_1} |J| - \sum_{j=1}^{j_0} |J(j)| \ge N/2 - N_1 \ge N/3. \ \blacksquare$$

By Lemma 4.1 the number of nonnegative integers R with $R \le N$ such that there exists a $\mathbf{k} \in \Lambda_1$ with $\rho(\mathbf{k}, R) > 0$ is at most

$$\sum_{\mathbf{k} \in \Lambda_1} (\alpha - 1)^{|\mathbf{k}|} \underline{\lambda}(N)^{\mathbf{k}} \le C_3 \underline{\lambda}(N)^{\mathbf{k}_1},$$

where C_3 is a positive constant. Say these R's are $0 = R_1 < \cdots < R_\mu$, where

$$\mu \le C_3 \underline{\lambda}(N)^{\mathbf{k}_1}. \tag{4.19}$$

Let $R_{1+\mu} = N$ and

$$\mathcal{I} = \{I = [R_i, R_{1+i}) \mid 1 \le i \le \mu\}.$$

Then $\sum_{I \in \mathcal{I}} |I| = N$. Put

$$\mathcal{I}_1 = \{I \in \mathcal{I} \mid I \subset J \text{ for some } J \in \mathcal{J}\},$$
$$\mathcal{I}_2 = \left\{ I \in \mathcal{I}_1 \;\middle|\; |I| \ge \frac{1}{12C_3} N \underline{\lambda}(N)^{-\mathbf{k}_1} \right\}.$$

Lemma 4.8. *Let $N \in \Xi$ be sufficiently large. Then*

$$\sum_{I \in \mathcal{I}_1} |I| \ge N/6, \quad \sum_{I \in \mathcal{I}_2} |I| \ge N/12.$$

Proof. We check the first inequality. For any $J = [T_j, T_{1+j}) \in \mathcal{J}_2$, we have $C_2(\mathbf{e}) \le T_j < T_{1+j} \le N$. If $N \in \Xi$ is sufficiently large, then by (4.13),

$$|J|/4 \ge 4N\underline{\lambda}(N)^{-\mathbf{e}} \ge T_{1+j}\underline{\lambda}(T_{1+j})^{-\mathbf{e}}.$$

So, using (4.12) and condition (3) of Theorem 2.1 with $(a_1, \ldots, a_r) = \mathbf{e}$, $R = T_{1+j}$, we obtain

$$T_{1+j} > \theta(T_{1+j}) > T_{1+j} - T_{1+j}\underline{\lambda}(T_{1+j})^{-\mathbf{e}} \ge T_{1+j} - |J|/4.$$

Now, $\mathbf{k}_1$ is written as $\mathbf{k}_1 = (g_1, \ldots, g_{r-1}, u)$. Since

$$\theta(T_{1+j}) \in \sum_{h=1}^{r-1} g_h S(\xi_h) \subset \sum_{h=1}^{r-1} g_h S(\xi_h) + uS(\xi_r),$$

we get

$$\rho(\mathbf{k}_1, \theta(T_{1+j})) > 0.$$

Thus, by the definition of $R_1, \ldots, R_{1+\mu}$,

$$\theta(T_{1+j}) = R_i \tag{4.20}$$

for some $i \in \mathbb{N}$. Consequently, we put

$$\beta(J) = \min\{n \in \mathbb{N} \mid n > T_j,\ n = R_i \text{ for some } i \in \mathbb{N}\},$$
$$\gamma(J) = \max\{n \in \mathbb{N} \mid n < T_{1+j},\ n = R_i \text{ for some } i \in \mathbb{N}\}.$$

Then it is clear that

$$\sum_{I \in \mathcal{I},\, I \subset J} |I| = \gamma(J) - \beta(J) \tag{4.21}$$

and

$$\gamma(J) \ge \theta(T_{1+j}) > T_{1+j} - |J|/4. \tag{4.22}$$

Similarly, we have

$$|J|/4 \ge 4N\underline{\lambda}(N)^{-\mathbf{e}} \ge 4T_j\underline{\lambda}(T_j)^{-\mathbf{e}}.$$

Applying Lemma 4.4 with $M = T_j$, $E = |J|/4$, we get

$$T_j + |J|/8 < \theta(T_j + |J|/4) < T_j + |J|/4.$$

In the same way as in the proof of (4.20), we deduce that $\theta(T_j + |J|/4) = R_i$ for some $i \in \mathbb{N}$. Hence

$$\beta(J) \le \theta(T_j + |J|/4) < T_j + |J|/4. \tag{4.23}$$

Therefore, combining (4.21)–(4.23), we obtain

$$\sum_{I \in \mathcal{I},\, I \subset J} |I| \ge |J|/2.$$

Consequently, using Lemma 4.7, we conclude that

$$\sum_{I \in \mathcal{I}_1} |I| \ge \sum_{J \in \mathcal{J}_2} \sum_{I \in \mathcal{I},\, I \subset J} |I| \ge \frac{1}{2} \sum_{J \in \mathcal{J}_2} |J| \ge \frac{1}{6}N,$$

which is the first inequality of Lemma 4.8.

Using (4.19) and the first inequality, we get

$$\sum_{I \in \mathcal{I}_2} |I| = \sum_{I \in \mathcal{I}_1} |I| - \sum_{I \in \mathcal{I}_1 \setminus \mathcal{I}_2} |I| \ge \frac{1}{6}N - \mu\frac{1}{12C_3}N\underline{\lambda}(N)^{-\mathbf{k}_1} \ge \frac{1}{12}N.$$

Thus we have verified the second inequality. ∎

Now, we show that each interval $I \in \mathcal{I}_2$ satisfies the assumptions of Lemma 4.6. Condition (1) of Theorem 2.1 implies that, for any $\mathbf{k} \in \Lambda$,

$$\log_\alpha N = o(N\underline{\lambda}(N)^{-\mathbf{k}}). \tag{4.24}$$

By condition (2), there exists $C_4 > 0$ such that, for any real $R \ge C_4$,

$$S(\xi_r) \cap [C_1R, R] \ne \emptyset.$$

Moreover, by (4.24) there is $C_5 > 0$ such that, for each natural $N \geq C_5$,

$$\frac{1}{12C_3} N\underline{\lambda}(N)^{-\mathbf{k}_1} - D\log_\alpha N \geq C_4. \tag{4.25}$$

Let $N \in \Xi$ and $I = [R_i, R_{1+i}) \in \mathcal{I}_2$. Suppose that N is sufficiently large. Then

$$|I| \geq \frac{1}{12C_3} N\underline{\lambda}(N)^{-\mathbf{k}_1}. \tag{4.26}$$

If $N \geq C_5$, then by (4.25) and (4.26), there exists $V \in S(\xi_r)$ with

$$C_1(|I| - D\log_\alpha N) \leq V \leq |I| - D\log_\alpha N. \tag{4.27}$$

Using (4.24) and (4.26), we get

$$C_1(|I| - D\log_\alpha N) \geq 1 + [C_1|I|/2] \tag{4.28}$$

because N is sufficiently large. Let $U = R_i + V$. Then there exists $\mathbf{k} = (g_1, \ldots, g_{r-1}, b) \in \Lambda_1$ $(b < g_r)$ such that

$$U \in \sum_{i=1}^{r-1} g_i S(\xi_i) + (1+b)S(\xi_r) \subset \sum_{i=1}^{r} g_i S(\xi_i),$$

so $\rho(\mathbf{k}_0, U) > 0$. Moreover, by (4.27) and (4.28),

$$R_i + 1 + [C_1|I|/2] \leq U \leq R_{i+1} - D\log_\alpha N.$$

By the definition of $\mathcal{I}_2$, there exists a positive integer j such that

$$I = [R_i, R_{i+1}) \subset [T_j, T_{j+1}).$$

Hence, for any integer x with $x \in (R_i, R_{i+1})$ and $\mathbf{k} \in \Lambda \setminus \{\mathbf{k}_0\}$, we have $\rho(\mathbf{k}, x) = 0$ because $\Lambda \setminus \{\mathbf{k}_0\} = \Lambda_1 \cup \Lambda_2$. Thus, by Lemma 4.6, $Y_n > 0$ for any $n \in \mathbb{N}$ with

$$R_i \leq n \leq R_i + [C_1|I|/2].$$

Hence, using Lemma 4.8, we conclude that

$$y(N) \geq \sum_{I \in \mathcal{I}_2} (1 + [C_1|I|/2]) \geq \frac{1}{24} C_1 N,$$

which contradicts the statement of Lemma 4.5. Thus we have proved Theorem 2.1.

Acknowledgements. I am deeply grateful to Prof. Masayoshi Hata for many suggestions and for improving the language of this paper. I would like to thank Prof. Yann Bugeaud for his careful reading of the manuscript and for giving useful advice. I would like to express my gratitude to Prof. Masanori Katsurada and Prof. Takaaki Tanaka for their careful reading and correcting the manuscript. I would like to thank the referee for the constructive and fruitful comments and suggestions on the manuscript.

This work was supported by the JSPS fellowship.

References

[1] D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, *On the binary expansions of algebraic numbers*, J. Théor. Nombres Bordeaux 16 (2004), 487–518.

[2] D. H. Bailey, P. B. Borwein and S. Plouffe, *On the rapid computation of various polylogarithmic constants*, Math. Comput. 66 (1997), 903–913.

[3] É. Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rend. Circ. Mat. Palermo 27 (1909), 247–271.

[4] É. Borel, *Sur les chiffres décimaux de* $\sqrt{2}$ *et divers problèmes de probabilités en chaîne*, C. R. Acad. Sci. Paris 230 (1950), 591–593.

[5] P. Corvaja and U. Zannier, *Some new applications of the subspace theorem*, Compos. Math. 131 (2002), 319–340.

[6] S. Daniel, *On gaps between numbers that are sums of three cubes*, Mathematika 44 (1997), 1–13.

[7] A. Durand, *Indépendance algébrique de nombres complexes et critère de transcendance*, Compos. Math. 35 (1977), 259–267.

[8] M. J. Knight, *An 'ocean of zeros' proof that a certain non-Liouville number is transcendental*, Amer. Math. Monthly 98 (1991), 947–949.

[9] J. Liouville, *Remarques relatives* 1° *à des classes très-étendues de quantités dont la valeur n'est ni rationnelle ni même réducible à des irrationnelles algébriques;* 2° *à un passage du livre des Principes où Newton calcule l'action exercée par une sphère sur un point extérieur*, C. R. Acad. Sci. Paris 18 (1844), 883–885.

[10] J. Liouville, *Nouvelle démonstration d'un théorème sur les irrationnelles algébriques*, C. R. Acad. Sci. Paris 18 (1844), 910–911.

[11] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366.

[12] K. Nishioka, *Algebraic independence by Mahler's method and* S*-unit equations*, Compos. Math. 92 (1994), 87–110.

[13] K. Nishioka, *Mahler Functions and Transcendence*, Lecture Notes in Math. 1631, Springer, 1996.

[14] T. Rivoal, *On the bits counting function of real numbers*, J. Austral. Math. Soc. 85 (2008), 95–111.

[15] K. Roth, *Rational approximations to algebraic numbers*, Mathematika 2 (1955), 1–20; Corrigendum, p. 168.

[16] W. M. Schmidt, *Simultaneous approximation and algebraic independence of numbers*, Bull. Amer. Math. Soc. 68 (1962), 475–478.

[17] I. Shiokawa, *Algebraic independence of certain gap series*, Arch. Math. (Basel) 38 (1982), 438–442.

Hajime Kaneko
Department of Mathematics
College of Science and Technology
Nihon University
1-8-14 Kanda-Surugadai, Chiyoda-ku
Tokyo 101-8308, Japan
E-mail: kanekoha@math.cst.nihon-u.ac.jp

Received on 2.12.2009
and in revised form on 21.4.2012 (6234)