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On a diophantine problem with one prime, two squares of primes and s powers of two

by

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1. Introduction. In this paper we are interested in the values of the form

$$\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s}, \tag{1.1}$$

where p_1, p_2, p_3 are prime numbers, $m_1, \ldots, m_s$ are positive integers, and the coefficients λ_1, λ_2, λ_3 and $\mu_1, \ldots, \mu_s$ are real numbers satisfying suitable relations.

This problem can be seen as a variation of the Waring–Goldbach and the Goldbach–Linnik problems. A huge literature exists for both problems and so we will mention just some of the most important results.

Concerning the Goldbach–Linnik problem, the first result was established by Linnik himself [23, 24] who proved that every sufficiently large even integer is a sum of two primes and a suitable number s of powers of two; he gave no explicit estimate of s. Other results were proved by Gallagher [6], J. Liu–M.-C. Liu–Wang [26, 27, 28], Wang [47] and H. Li [17, 18]. Now the best conditional result is due to Pintz–Ruzsa [37] and Heath-Brown–Puchta [11] ($s = 7$ suffices under the assumption of the Generalized Riemann Hypothesis), while, unconditionally, it is due to Heath-Brown–Puchta [11] ($s = 13$ suffices). Elsholtz, in unpublished work, improved it to $s = 12$. We should also remark that Pintz–Ruzsa announced a proof for the case $s = 8$ in their paper [37]. Looking for the size of the exceptional set of the Goldbach problem we recall the fundamental paper by Montgomery–Vaughan [34] in which they showed that the number of even integers up to X that are not the sum of two primes is $\ll X^{1-\delta}$. Pintz [36] announced that $\delta = 1/3$ is admissible in the previous estimate. Concerning the exceptional set for the Goldbach–Linnik problem, Languasco–Pintz–Zaccagnini [15] proved that for

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every $s \geq 1$, there are $\ll X^{3/5}(\log X)^{10}$ even integers in $[1, X]$ that are not the sum of two primes and s powers of two.

In diophantine approximation several results were proved concerning linear forms with primes that, in some sense, can be considered as the real analogues of the binary and ternary Goldbach problems. On this topic we recall the papers by Vaughan [43, 44, 45], Harman [9], Brüdern–Cook–Perelli [1], and Cook–Harman [3]. A diophantine problem with two primes and powers of two was solved by Parsell [35]; his estimate on the needed powers of two was recently improved by Languasco–Zaccagnini [16].

The problem of representing an integer using a suitable number of prime powers is usually called the Waring–Goldbach problem. We refer to the beautiful Vaughan–Wooley survey paper [46] for the literature on this problem. Here we just mention that in 1938 Hua [12] proved that almost all the integers $n \equiv 3 \bmod 24$ and $n \not\equiv 0 \bmod 5$ are representable as sums of three squares of primes, and all sufficiently large $n \equiv 5 \bmod 24$ are representable as sums of five squares of primes. Also several results were obtained about the size of the exceptional set for this problem. On this topic we just recall a recent result of J. Liu, Wooley and Yu [30].

Concerning mixed problems with powers of primes and powers of two, we recall the results by H. Li [19], [20], J. Liu and Lü [29], J. Liu and M.-C. Liu [25], Lü and Sun [33], Z. Liu and Lü [32].

Replacing one of the prime summands in the problem in Parsell [35] with the sum of two squares of primes, we obtain the problem in (1.1); the only result we know about it is by W. P. Li and Wang [21]. We improve their estimate on s with the following result whose quality depends on rational approximations to λ_2/λ_3.

THEOREM. *Suppose that $\lambda_1 < 0$ and $\lambda_2, \lambda_3 > 0$ with λ_2/λ_3 irrational. Further suppose that $\mu_1, \dots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$ for $i \in \{1, 2, 3\}$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(|\lambda_1/a_1|; \lambda_2/a_2; \lambda_3/a_3)$. Finally let*

(1.2)
$$s_0 = 3 + \left\lceil \frac{\log(4C(q_1, q_2, q_3, \epsilon)(|\lambda_1| + |\lambda_2| + |\lambda_3|)) - \log((3 - 2\sqrt{2} - \epsilon)\eta)}{-\log 0.8844472132} \right\rceil,$$

where $\epsilon > 0$ is an arbitrarily small constant, $C(q_1, q_2, q_3, \epsilon)$ satisfies

(1.3)
$$C(q_1, q_2, q_3, \epsilon) = (1 + \epsilon)(\log 2 + C \cdot \mathfrak{S}'(q_1))^{1/2} \times ((\log 2)^2 + D \cdot \mathfrak{S}''(q_2))^{1/4}((\log 2)^2 + D \cdot \mathfrak{S}''(q_3))^{1/4},$$

$C = 10.0219168340$, $D = 17{,}646{,}979.6536361512$, *and*

$$\mathfrak{S}'(n) = \prod_{\substack{p \mid n \\ p>2}} \frac{p-1}{p-2}, \qquad \mathfrak{S}''(n) = \prod_{\substack{p \mid n \\ p>2}} \frac{p+1}{p}. \tag{1.4}$$

Then for every real number ϖ and every integer $s \ge s_0$ the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \varpi| < \eta \tag{1.5}$$

has infinitely many solutions in primes p_1, p_2, p_3 and positive integers $m_1, \dots, m_s$.

Arguing analogously we can prove the case $\lambda_1, \lambda_2 < 0$, $\lambda_3 > 0$ (see the argument at the end of §4).

Our value in (1.2) largely improves W. P. Li–Wang's [21] one given by

(1.6)
$$s_0 = 3 + \left\lceil \frac{\log(2^9 C_1(q_1, q_2, q_3, \epsilon)(|\lambda_1| + |\lambda_2| + |\lambda_3|)^2) - \log((1-\epsilon)|\lambda_1|\eta)}{-\log 0.995} \right\rceil,$$

where

$$C_1(q_1, q_2, q_3, \epsilon) = 5(1+\epsilon)\left(\frac{11^4 \cdot 43 \cdot \pi^{26}}{2^{27} \cdot 25} + (\log 2)^2\right)^{1/2} \times (\log 2q_1)^{1/2} (\log 2q_2)^{1/4} (\log 2q_3)^{1/4}. \tag{1.7}$$

Comparing only denominators in (1.2) and in (1.6), we see that our gain is about 95.9%. Moreover the numerical constants involved in the definition (1.3) are better than the ones in (1.7) (see the remark after Lemma 3.6 below).

In practice, the following example shows that the gain is actually slightly larger. For instance, taking $\lambda_1 = -\sqrt{5} = \mu_1^{-1}$, $\lambda_2 = \sqrt{3} = \mu_2^{-1}$, $\lambda_3 = \sqrt{2} = \mu_3^{-1}$, $\eta = 1$ and $\epsilon = 10^{-20}$, we get $s_0 = 120$, while W. P. Li–Wang's estimate (1.6) gives $s_0 = 4120$.

Moreover we remark that the works of Rosser–Schoenfeld [39] on $n/\varphi(n)$ and of Solé–Planat [41] on the Dedekind Ψ function (see Lemmas 3.1 and 3.2 below) give for $\mathfrak{S}'(q)$ and $\mathfrak{S}''(q)$ a sharper estimate than $2\log(2q)$, used in (1.7), for large values of q.

With respect to [21], our main gain comes from enlarging the size of the major arc since this lets us use sharper estimates on the minor arc. In particular, on the major arc we replaced the technique used in [21] with an argument involving an L^2-estimate of the exponential sum over prime squares ($S_2(\alpha)$). This is a standard tool when working on primes (see, e.g., [16] for an application to a similar problem) but it seems that it is the first time that a similar technique is used for prime squares so we inserted a detailed proof of the relevant lemmas (Lemmas 3.12 and 3.13 below) since they could be of some independent interest.

On the minor arc we use the Ghosh estimate [7] to deal with the exponential sum on primes squares while to treat the exponential sum on primes ($S_1(\alpha)$) we follow the argument in [16]. To work with the exponential sum over powers of two ($G(\alpha)$), we applied Pintz–Ruzsa's [37] algorithm to estimate the measure of the subset of the minor arc on which $|G(\alpha)|$ is "large". These ingredients lead to a sharper estimate on the minor arc and let us improve the size of the denominators in (1.2).

A second, less important, gain arises from our Lemmas 3.3 and 3.6 below, which improves the numerical values in (1.3) compared with the ones in (1.7) (see also Parsell [35, Lemma 3]).

Using the notation $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, as a consequence of the Theorem we have

Corollary. *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are nonzero real numbers, not all of the same sign, such that λ_2/λ_3 is irrational. Further suppose $\mu_1, \dots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$ for $i \in \{1, 2, 3\}$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(|\lambda_1/a_1|; |\lambda_2/a_2|; |\lambda_3/a_3|)$ and $\tau \geq \eta > 0$. Finally let $s_0 = s_0(\boldsymbol{\lambda}, \boldsymbol{\mu}, \eta, \epsilon)$ as defined in* (1.2), *where $\epsilon > 0$ is arbitrarily small. Then for every real number ϖ and every integer $s \geq s_0$ the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \varpi| < \tau$$

has infinitely many solutions in primes p_1, p_2, p_3 and positive integers $m_1, \dots, m_s$.

This Corollary immediately follows from the Theorem by rearranging the λ's. Hence the Theorem ensures that (1.5) has infinitely many solutions and the Corollary immediately follows from the condition $\tau \geq \eta$.

2. Definitions. Let ϵ be a sufficiently small positive constant (not necessarily the same at each occurrence), X be a large parameter, $M = |\mu_1| + \cdots + |\mu_s|$ and $L = \log_2(\epsilon X/(2M))$, where $\log_2 v$ is the base 2 logarithm of v. We will use the Davenport–Heilbronn variation of the Hardy–Littlewood method to count the number $\mathfrak{N}(X)$ of solutions of the inequality (1.5) with $\epsilon X \leq p_1, p_2^2, p_3^2 \leq X$ and $1 \leq m_1, \dots, m_s \leq L$. Let now $e(u) = \exp(2\pi i u)$ and

$$S_1(\alpha) = \sum_{\epsilon X \leq p \leq X} \log p \, e(p\alpha), \quad S_2(\alpha) = \sum_{\epsilon X \leq p^2 \leq X} \log p \, e(p^2\alpha),$$

$$G(\alpha) = \sum_{1 \leq m \leq L} e(2^m \alpha).$$

For $\alpha \neq 0$, we also define

$$K(\alpha,\eta)=\left(\frac{\sin \pi\eta\alpha}{\pi\alpha}\right)^2.$$

It is well-known that

$$\widehat{K}(t,\eta)=\int_{\mathbb{R}} K(\alpha,\eta)e(t\alpha)\,d\alpha=\max(0;\eta-|t|) \tag{2.1}$$

and

$$K(\alpha,\eta)\ll \min(\eta^2;\alpha^{-2}). \tag{2.2}$$

Letting

$$I(X;\mathbb{R})=\int_{\mathbb{R}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\varpi\alpha)K(\alpha,\eta)\,d\alpha,$$

it follows from (2.1) that

$$I(X;\mathbb{R})\ll \eta(\log X)^3\cdot \mathfrak{N}(X).$$

We will prove, for $X\to+\infty$ running over a suitable integral sequence, that

$$I(X;\mathbb{R})\gg_{s,\boldsymbol{\lambda},\epsilon} \eta^2 X(\log X)^s \tag{2.3}$$

thus obtaining

$$\mathfrak{N}(X)\gg_{s,\boldsymbol{\lambda},\epsilon} \eta X(\log X)^{s-3}$$

and hence the Theorem follows.

To prove (2.3) we first dissect the real line into the major, minor and trivial arcs, by choosing $P=X^{2/5}/\log X$ and letting

$$\mathfrak{M}=\{\alpha\in\mathbb{R}:|\alpha|\le P/X\},\quad \mathfrak{m}=\{\alpha\in\mathbb{R}:P/X<|\alpha|\le L^2\}, \tag{2.4}$$

and $\mathfrak{t}=\mathbb{R}\setminus(\mathfrak{M}\cup\mathfrak{m})$. Accordingly, we write

$$I(X;\mathbb{R})=I(X;\mathfrak{M})+I(X;\mathfrak{m})+I(X;\mathfrak{t}). \tag{2.5}$$

We will prove that

$$I(X;\mathfrak{M})\ge c_1\eta^2 XL^s, \tag{2.6}$$

$$|I(X;\mathfrak{t})|=o(XL^s) \tag{2.7}$$

both hold for all sufficiently large X, and

$$|I(X;\mathfrak{m})|\le c_2(s)\eta XL^s \tag{2.8}$$

holds for $X\to+\infty$ running over a suitable integral sequence, where $c_2(s)>0$ depends on s, $c_2(s)\to 0$ as $s\to+\infty$, and $c_1=c_1(\epsilon,\boldsymbol{\lambda})>0$ is a constant such that

$$c_1\eta-c_2(s)\ge c_3\eta \tag{2.9}$$

for some absolute positive constant c_3 and $s\ge s_0$. Inserting (2.6)–(2.9) into (2.5), we finally conclude that (2.3) holds, thus proving the Theorem.

3. Lemmas. Let n be a positive integer. We denote by $\mathfrak{S}(n)$ the singular series and set $\mathfrak{S}(n) = 2c_0\mathfrak{S}'(n)$ where $\mathfrak{S}'(n)$ is defined in (1.4) and

$$c_0 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right).$$

Notice that $\mathfrak{S}'(n)$ is a multiplicative function. According to Gourdon–Sebah [8], we have $0.66016181584 < c_0 < 0.66016181585$.

The first lemma is an upper bound for the multiplicative part of the singular series.

Lemma 3.1 (Languasco–Zaccagnini [16, Lemma 2]). *For $n \in \mathbb{N}$, $n \geq 3$, we have*

$$\mathfrak{S}'(n) < \frac{n}{c_0\varphi(n)} < \frac{e^{\gamma}\log\log n}{c_0} + \frac{2.50637}{c_0 \log\log n},$$

where $\gamma = 0.5772156649\ldots$ is the Euler constant.

Letting $f(1) = f(2) = 1$ and $f(n) = n/(c_0\varphi(n))$ for $n \geq 3$, we can see that the inequality $\mathfrak{S}'(n) \leq f(n)$ is sharper than Parsell's estimate $\mathfrak{S}'(n) \leq 2\log(2n)$ (see [35, p. 369]) for every $n \geq 1$. Since it is clear that computing the exact value of $f(n)$ for large values of n is not easy (it requires the knowledge of every prime factor of n), we also remark that the second estimate in Lemma 3.1 leads to a sharper bound than $\mathfrak{S}'(n) \leq 2\log(2n)$ for every $n \geq 14$.

Let now $\mathfrak{S}''(n)$ be defined as in (1.4). We first remark that it is connected with the Dedekind Ψ function defined by

$$\Psi(n) = n\prod_{p\mid n}\frac{p+1}{p}$$

since $\mathfrak{S}''(n) = \Psi(n)/n$ for n odd and $\mathfrak{S}''(n) = (2/3)\Psi(n)/n$ for n even. We also have

Lemma 3.2. *For $n \in \mathbb{N}$, $n \geq 31$, we have*

$$\mathfrak{S}''(n) < e^{\gamma}\log\log n,$$

where γ is the Euler constant.

Proof. This follows immediately from Corollary 2 of Solé–Planat [41] and the previous remarks. ∎

The estimate in Lemma 3.2 is sharper than W. P. Li–Wang's one $\mathfrak{S}''(n) \leq 2\log(2n)$ (see [22, p. 171]) for every $n \geq 31$. We also remark that $\mathfrak{S}''(1) = \mathfrak{S}''(2) = 1$, and that the computation of $\mathfrak{S}''(n)$ in the remaining interval $3 \leq n \leq 30$ is an easy task.

Now we state some lemmas we need to estimate $I(X;\mathfrak{m})$.

LEMMA 3.3 (Languasco–Zaccagnini [16, Lemma 4]). *Let X be a sufficiently large parameter and let $\lambda, \mu \neq 0$ be two real numbers such that $\lambda/\mu \in \mathbb{Q}$. Let $a, q \in \mathbb{Z}\setminus\{0\}$ with $q > 0$ and $(a, q) = 1$ be such that $\lambda/\mu = a/q$. Let further $0 < \eta < |\lambda/a|$. Then*

$$\int_{\mathbb{R}} |S_1(\lambda\alpha)G(\mu\alpha)|^2 K(\alpha, \eta)\, d\alpha < \eta X L^2((1-\epsilon)\log 2 + C \cdot \mathfrak{S}'(q)) + \mathcal{O}_{M,\epsilon}(\eta X L),$$

where $C = 10.0219168340$.

LEMMA 3.4. *Let ϵ be an arbitrarily small positive constant. Let $n \in \mathbb{Z}$, $n \neq 0$, $|n| \leq X$, $n \equiv 0 \bmod 24$ and*

$$r(n) = |\{n = p_1^2 + p_2^2 - p_3^2 - p_4^2, \text{ where } p_j \leq X^{1/2},\ j = 1, \dots, 4\}|.$$

Then

$$r(n) \leq (1+\epsilon) c_4 \frac{\pi^2}{16} \mathfrak{S}_-(n) \frac{X}{(\log X)^4},$$

where

$$\mathfrak{S}_-(n) = \left(2 - \frac{1}{2^{\beta_0 - 1}} - \frac{1}{2^{\beta_0}}\right) \prod_{\substack{p>2 \\ p^\beta \| n \\ \beta \geq 0}} \left(1 + \frac{1}{p} - \frac{1}{p^{\beta+1}} - \frac{1}{p^{\beta+2}}\right),$$

$c_4 = 101 \cdot 2^{20}$ and β_0 is such that $2^{\beta_0} \parallel n$.

Lemma 3.4 follows by inserting the remark of H. Li [19, p. 385] into the proof of Lemma 2.2 of J. Liu–Lü [29]. We immediately remark that $\mathfrak{S}_-(n) \leq 2\mathfrak{S}''(n)$.

We will also need the following

LEMMA 3.5 (H. Li [19]). *Let d be a positive odd integer and $\xi(d)$ be the quantity $\min\{\mu\colon 2^\mu \equiv 1 \pmod d\}$. Then the series*

$$\sum_{\substack{d=1 \\ 2\nmid d}}^{+\infty} \frac{\mu^2(d)}{d\xi(d)}$$

is convergent and its value c_5 satisfies $c_5 < 1.620767$.

The next lemma is the analogue of Lemma 3.3 for exponential sums over prime squares.

LEMMA 3.6. *Let X be a sufficiently large parameter and let $\lambda, \mu \neq 0$ be two real numbers such that $\lambda/\mu \in \mathbb{Q}$. Let $a, q \in \mathbb{Z} \setminus \{0\}$ with $q > 0$ and $(a, q) = 1$ be such that $\lambda/\mu = a/q$. Let further $0 < \eta < |\lambda/a|$. Then*

$$\int_{\mathbb{R}} |S_2(\lambda\alpha)G(\mu\alpha)|^4 K(\alpha, \eta)\, d\alpha < (1+\epsilon)\eta X L^4((\log 2)^2 + D \cdot \mathfrak{S}''(q)),$$

where $D = c_4 c_5 \pi^2/96$, c_4, c_5 *are as in Lemmas* 3.4–3.5 *respectively and* ϵ *is an arbitrarily small positive constant.*

This should be compared with Lemma 4.3 of W. P. Li–Wang [22] (see also Lemma 4.2 of [21]) in which the value $D_1 = 2^{-27} \cdot 11^4 \cdot 43 \cdot \pi^{26}/25$ plays the role of D. Using the values $c_4 = 101 \cdot 2^{20}$ and $c_5 < 1.620767$ as in Lemmas 3.4–3.5, we see that $D < 17{,}646{,}979.6536361512$ while $D_1 = 1{,}581{,}925{,}383.0798448770$. We remark that $D < 0.0112 \cdot D_1$ and so the reduction factor here is close to 98.8%. With an abuse of notation, in the statement of the Theorem we will set $D = 17{,}646{,}979.6536361512$.

Proof of Lemma 3.6. Letting now

$$I = \int_{\mathbb{R}} |S_2(\lambda\alpha)G(\mu\alpha)|^4 K(\alpha,\eta)\, d\alpha,$$

by (2.1) we immediately have

$$I = \sum_{\epsilon X \le p_1^2, p_2^2, p_3^2, p_4^2 \le X} \log p_1 \log p_2 \log p_3 \log p_4 \times \sum_{1 \le m_1, m_2, m_3, m_4 \le L} \max\bigl(0; \eta - |\lambda(p_1^2 + p_2^2 - p_3^2 - p_4^2) + \mu(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4})|\bigr). \tag{3.1}$$

Let $\delta = \lambda(p_1^2 + p_2^2 - p_3^2 - p_4^2) + \mu(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4})$. For a sufficiently small $\eta > 0$, we claim that

$$|\delta| < \eta \quad \text{is equivalent to} \quad \delta = 0. \tag{3.2}$$

Recall our hypothesis on a and q, and assume that $\delta \neq 0$ in (3.2). For $\eta < |\lambda/a|$ this leads to a contradiction. In fact we have

$$\begin{aligned}\frac{1}{|a|} > \frac{\eta}{|\lambda|} &> \left| p_1^2 + p_2^2 - p_3^2 - p_4^2 + \frac{q}{a}(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}) \right| \\ &= \left| \frac{a(p_1^2 + p_2^2 - p_3^2 - p_4^2) + q(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4})}{a} \right| \ge \frac{1}{|a|},\end{aligned}$$

since $a(p_1^2 + p_2^2 - p_3^2 - p_4^2) + q(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}) \neq 0$ is a linear integral combination. Inserting (3.2) in (3.1), for $\eta < |\lambda/a|$ we can write

$$I = \eta \sum_{\substack{\epsilon X \le p_1^2, p_2^2, p_3^2, p_4^2 \le X \\ \lambda(p_1^2 + p_2^2 - p_3^2 - p_4^2) + \mu(2^{m_1} + 2^{m_2} - 2^{m_3} - 2^{m_4}) = 0}} \sum_{1 \le m_1, m_2, m_3, m_4 \le L} \log p_1 \log p_2 \log p_3 \log p_4. \tag{3.3}$$

The diagonal contribution in (3.3) is equal to

$$\eta \sum_{\substack{\epsilon X \le p_1^2, p_2^2, p_3^2, p_4^2 \le X \\ p_1^2 + p_2^2 = p_3^2 + p_4^2}} \log p_1 \log p_2 \log p_3 \log p_4 \sum_{\substack{1 \le m_1, m_2, m_3, m_4 \le L \\ 2^{m_1} + 2^{m_2} = 2^{m_3} + 2^{m_4}}} 1. \tag{3.4}$$

The number of solutions of $p_1^2+p_2^2=p_3^2+p_4^2$ when $p_1p_2\neq p_3p_4$ can be estimated using Satz 3, p. 94 of Rieger [38] and it is $\ll X(\log X)^{-3}$. This gives a contribution to the first sum which is $\ll X\log X$. In the remaining case $p_1p_2=p_3p_4$ the first sum becomes

$$2\sum_{\epsilon X\le p_1^2,p_2^2\le X}(\log p_1)^2(\log p_2)^2=2\Big(\sum_{\sqrt{\epsilon X}\le p\le\sqrt{X}}(\log p)^2\Big)^2<(1-\epsilon)\frac{X}{2}(\log X)^2,$$

where we used the Prime Number Theorem and the fact that ϵ is a sufficiently small positive constant. The sum over the powers of two in (3.4) can be evaluated by fixing first $m_1=m_3$ (thus getting exactly L^2 solutions) and then fixing $m_1\neq m_3$ (which gives other L^2-L solutions). Hence the contribution of the second sum in (3.4) is $2L^2-L$.

Combining these results we see that the total contribution of (3.4) is

$$<(1-\epsilon)\eta XL^2(\log X)^2<\eta XL^4(\log 2)^2. \tag{3.5}$$

Now we have to estimate the contribution I' of the nondiagonal solutions of $\delta=0$ and we will achieve this by connecting I' with the singular series of Lemma 3.4. First, we remark that if $p_j>3$ for every $j=1,\dots,4$, then $n=p_1^2+p_2^2-p_3^2-p_4^2\equiv 0 \bmod 24$. So if $n=p_1^2+p_2^2-p_3^2-p_4^2\not\equiv 0\bmod 24$ then at least one of the p_j must be equal to 2 or 3, and hence $r(n)$, defined as in the statement of Lemma 3.4, satisfies $r(n)\ll X^{1/2+\epsilon}$. Recalling that $\lambda/\mu=a/q\neq 0$, $(a,q)=1$, if $2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2}\neq 0$ and $(q/a)(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})\not\equiv 0\bmod 24$, we have

$$|\{(p_1,\dots,p_4):p_1^2+p_2^2-p_3^2-p_4^2=(q/a)(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})\}|\ll X^{1/2+\epsilon}.$$

Otherwise, by Lemma 3.4, $\mathfrak{S}_-(n)\le 2\mathfrak{S}''(n)$, $r((q/a)(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2}))\neq 0$ if and only if $a\,|\,(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})$, $\log p_j\le(1/2)\log X$ and $|(q/a)(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})|\le|q/a|2\epsilon X/M\le 2\epsilon X/|\lambda|<X$ for ϵ sufficiently small, we have

$$\begin{aligned}I'&\le\frac{\eta}{16}(\log X)^4\sum_{1\le m_1,m_2,m_3,m_4\le L}r\Big(\frac{q}{a}(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})\Big)\\&<(1+\epsilon)c_4\frac{\pi^2}{128}\eta X\sum_{1\le m_1,m_2,m_3,m_4\le L}\mathfrak{S}''\Big(\frac{q}{a}(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})\Big).\end{aligned}\tag{3.6}$$

Using the multiplicativity of $\mathfrak{S}''(n)$ (defined in (1.4)), we get

$$\begin{aligned}\mathfrak{S}''\Big(\frac{q}{a}(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})\Big)&\le\mathfrak{S}''(q)\mathfrak{S}''\Big(\frac{2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2}}{a}\Big)\\&\le\mathfrak{S}''(q)\mathfrak{S}''(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2})\end{aligned}$$

and so, by (3.6), we can write, for every sufficiently large X,

$$I' \le (1+\epsilon)c_4\frac{\pi^2}{128}\mathfrak{S}''(q)\eta X \sum_{1\le m_1,m_2,m_3,m_4\le L} \mathfrak{S}''(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2}).$$

Arguing now as at pages 63–64 of J. Liu–Lü [29] we have

$$\sum_{1\le m_1,m_2,m_3,m_4\le L} \mathfrak{S}''(2^{m_3}+2^{m_4}-2^{m_1}-2^{m_2}) \le \frac{4}{3}c_5(1+\epsilon)L^4,$$

thus getting

$$I' \le (1+\epsilon)c_4c_5\frac{\pi^2}{96}\mathfrak{S}''(q)\eta X L^4, \tag{3.7}$$

for a sufficiently small ϵ. Hence, by (3.3)–(3.5) and (3.7), we finally get

$$I < (1+\epsilon)\eta X L^4\bigg((\log 2)^2 + c_4c_5\frac{\pi^2}{96}\mathfrak{S}''(q)\bigg),$$

this way proving Lemma 3.6. ■

We now recall a famous result by Ghosh about $S_2(\alpha)$.

Lemma 3.7 (Ghosh [7, Theorem 2]). *Let α be a real number and a, q be positive integers satisfying $(a,q)=1$ and $|\alpha - a/q| < q^{-2}$. Let moreover $\epsilon > 0$. Then*

$$S_2(\alpha) \ll_\epsilon X^{1/2+\epsilon}\bigg(\frac{1}{q}+\frac{1}{X^{1/4}}+\frac{q}{X}\bigg)^{1/4}.$$

As an application of the previous lemma, we get the following result.

Lemma 3.8. *Suppose that λ_2/λ_3 is irrational, and let $X = q^2$ where q is the denominator of a convergent of the continued fraction for λ_2/λ_3. Let $V(\alpha) = \min(|S_2(\lambda_2\alpha)|; |S_2(\lambda_3\alpha)|)$. Then for arbitrarily small ϵ we have*

$$\sup_{\alpha\in\mathfrak{m}} V(\alpha) \ll X^{7/16+\epsilon}.$$

Proof. Let $\alpha \in \mathfrak{m}$ and $Q = X^{1/4}/(\log X)^2 \le P$. By the Dirichlet Theorem, there exist integers a_i, q_i with $1 \le q_i \le X/Q$ and $(a_i,q_i)=1$ such that $|\lambda_i\alpha q_i - a_i| \le Q/X$, $i=2,3$. We remark that $a_2a_3 \ne 0$, otherwise we would have $\alpha \in \mathfrak{M}$. Now suppose that $q_i \le Q$, $i = 2,3$. In this case we get

$$a_3q_2\frac{\lambda_2}{\lambda_3} - a_2q_3 = (\lambda_2\alpha q_2 - a_2)\frac{a_3}{\lambda_3\alpha} - (\lambda_3\alpha q_3 - a_3)\frac{a_2}{\lambda_3\alpha}$$

and hence

$$\bigg|a_3q_2\frac{\lambda_2}{\lambda_3} - a_2q_3\bigg| \le 2\bigg(1+\bigg|\frac{\lambda_2}{\lambda_3}\bigg|\bigg)\frac{Q^2}{X} < \frac{1}{2q}$$

for a sufficiently large X. Then, from the law of best approximation and the definition of $\mathfrak{m}$, we obtain

$$X^{1/2} = q \le |a_3 q_2| \ll q_2 q_3 (\log X)^2 \le Q^2 (\log X)^2 \le X^{1/2} (\log X)^{-2}.$$

Hence either $q_2 > Q$ or $q_3 > Q$. Assume, without loss of generality, that $q_2 > Q$. Using Lemma 3.7 for $S_2(\lambda_2 \alpha)$, we have

$$\begin{aligned} V(\alpha) \le |S_2(\lambda_2\alpha)| &\ll_\epsilon X^{1/2+\epsilon} \sup_{Q < q_2 \le X/Q} \left(\frac{1}{q_2} + \frac{1}{X^{1/4}} + \frac{q_2}{X} \right)^{1/4} \\ &\ll_\epsilon X^{7/16+\epsilon} (\log X)^{1/2}, \end{aligned}$$

thus proving Lemma 3.8. ■

To estimate the contribution of $G(\alpha)$ on the minor arc we use Pintz–Ruzsa's method as developed in [37, §3–7].

Lemma 3.9 (Pintz–Ruzsa [37, §7]). *Let $0 < c < 1$. Then there exists $\nu = \nu(c) \in (0,1)$ such that*

$$|E(\nu)| := |\{\alpha \in (0,1) : |G(\alpha)| > \nu L\}| \ll_{M,\epsilon} X^{-c}.$$

To obtain explicit values for ν we used the version of the Pintz–Ruzsa algorithm already implemented to get the results used in Languasco–Zaccagnini [16]. We used the PARI/GP [42] language and the gp2c compiling tool to compute fifty decimal digits (but we write here just ten) of the constant involved in the previous lemma. If we run the program in our case, Lemma 3.9 gives the following result:

$$|G(\alpha)| \le 0.8844472132 \cdot L \tag{3.8}$$

if $\alpha \in [0,1] \setminus E$ where $|E| \ll_{M,\epsilon} X^{-3/4-10^{-20}}$. The computing time to get (3.8) on an Apple MacBook Pro was 26 minutes and 28 seconds (but to get 30 correct digits just 3 minutes and 31 seconds suffice). You can download the PARI/GP source code of our program together with the cited numerical values at www.math.unipd.it/˜languasc/PintzRuzsaMethod.html.

Now we state some lemmas we will use to work on the major arc. Let $\theta(x) = \sum_{p \le x} \log p$,

$$J(X,h) = \int_{\epsilon X}^{X} (\theta(x+h) - \theta(x) - h)^2 \, dx \tag{3.9}$$

and

$$J^*(X,h) = \int_{\epsilon X}^{X} \left(\theta(\sqrt{x+h}) - \theta(\sqrt{x}) - (\sqrt{x+h} - \sqrt{x})\right)^2 dx \tag{3.10}$$

be two different versions of the Selberg integral, and

$$U_1(\alpha) = \sum_{\epsilon X \le n \le X} e(\alpha n) \quad \text{and} \quad U_2(\alpha) = \sum_{\epsilon X \le n^2 \le X} e(\alpha n^2).$$

Applying Gallagher's famous lemma on the truncated L^2-norm of exponential sums to $S_1(\alpha) - U_1(\alpha)$, one gets the following well-known statement which we quote from Brüdern–Cook–Perelli [1, Lemma 1].

LEMMA 3.10. *For $1/X \le Y \le 1/2$ we have*

$$\int_{-Y}^{Y} |S_1(\alpha) - U_1(\alpha)|^2 \, d\alpha \ll_\epsilon \frac{(\log X)^2}{Y} + Y^2 X + Y^2 J\Big(X, \frac{1}{2Y}\Big),$$

where $J(X,h)$ is defined in (3.9).

To estimate the Selberg integral, we use the next result.

LEMMA 3.11 (Saffari–Vaughan [40, §6]). *Let ϵ be an arbitrarily small positive constant. There exists a positive constant $c_6(\epsilon)$ such that*

$$J(X,h) \ll_\epsilon h^2 X \exp\bigg(-c_6\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\bigg)$$

uniformly for $X^{1/6+\epsilon} \le h \le X$.

In a similar way we can also prove

LEMMA 3.12. *For $1/X \le Y \le 1/2$ we have*

$$\int_{-Y}^{Y} |S_2(\alpha) - U_2(\alpha)|^2 \, d\alpha \ll_\epsilon \frac{(\log X)^2}{YX} + Y^2 X + Y^2 J^*\Big(X, \frac{1}{2Y}\Big),$$

where $J^*(X,h)$ is defined in (3.10).

Proof. Letting $\mathcal{I} := \int_{-Y}^{Y} |S_2(\alpha) - U_2(\alpha)|^2 \, d\alpha$, we can write

$$\begin{aligned}\mathcal{I} &= \int_{-Y}^{Y} \Big| \sum_{\epsilon X \le p^2 \le X} \log p \; e(p^2\alpha) - \sum_{\epsilon X \le n^2 \le X} e(\alpha n^2)\Big|^2 \, d\alpha \\ &= \int_{-Y}^{Y} \Big| \sum_{\epsilon X \le n^2 \le X} (k(n) - 1) e(n^2\alpha)\Big|^2 \, d\alpha,\end{aligned}$$

where $k(n) = \log p$ if $n = p$ prime and $k(n) = 0$ otherwise. By Gallagher's lemma (Lemma 1 of [5]) we obtain

$$\mathcal{I} \ll Y^2 \int_{-\infty}^{\infty} \Big(\sum_{\substack{x \le n^2 \le x+H \\ \epsilon X \le n^2 \le X}} (k(n) - 1)\Big)^2 \, dx$$

where we defined $H = 1/(2Y)$. We can restrict the integration range to $E = [\epsilon X - H, X]$ since otherwise the inner sum is empty. Moreover we split E as $E = E_1 \sqcup E_2 \sqcup E_3$ where $\sqcup$ represents disjoint union and $E_1 = [\epsilon X - H, \epsilon X]$, $E_2 = [\epsilon X, X - H]$, $E_3 = [X - H, X]$. Accordingly,

(3.11)
$$\mathcal{I} \ll Y^2 \Big(\int_{E_1} + \int_{E_2} + \int_{E_3} \Big) \Big(\sum_{\substack{x \le n^2 \le x+H \\ \epsilon X \le n^2 \le X}} (k(n) - 1) \Big)^2 dx = Y^2 (I_1 + I_2 + I_3),$$

say. We now proceed to estimate I_i for $i = 1, 2, 3$.

Estimation of I_1. By trivial estimates we have

(3.12)
$$\begin{aligned} I_1 &= \int_{E_1} \Big(\sum_{\epsilon X \le n^2 \le x+H} (k(n) - 1) \Big)^2 dx \\ &= \int_{\epsilon X - H}^{\epsilon X} \big(\theta\big(\sqrt{x+H}\big) - \theta\big(\sqrt{\epsilon X}\big) - \big(\sqrt{x+H} - \sqrt{\epsilon X}\big) + \mathcal{O}(1)\big)^2 dx \\ &\ll \int_{\epsilon X - H}^{\epsilon X} \big(\theta\big(\sqrt{x+H}\big) - \theta\big(\sqrt{\epsilon X}\big) - \big(\sqrt{x+H} - \sqrt{\epsilon X}\big)\big)^2 dx + H. \end{aligned}$$

Using a trivial estimate in (3.12) we have

(3.13)
$$I_1 \ll (\log X)^2 \int_{\epsilon X - H}^{\epsilon X} \big(\sqrt{x+H} - \sqrt{\epsilon X}\big)^2 dx + H \ll_\epsilon \frac{H^3 (\log X)^2}{X} + H,$$

where the last step follows by applying the Mean Value Theorem to the integrand.

Estimation of I_3. The estimation of I_3 is similar to the one of I_1. We have

$$\begin{aligned} I_3 &= \int_{E_3} \Big(\sum_{x \le n^2 \le X} (k(n) - 1) \Big)^2 dx \\ &\ll \int_{X-H}^{X} \big(\theta\big(\sqrt{X}\big) - \theta\big(\sqrt{x}\big) - \big(\sqrt{X} - \sqrt{x}\big)\big)^2 dx + H. \end{aligned}$$

Again using a trivial estimate and the Mean Value Theorem we get

$$I_3 \ll (\log X)^2 \int_{X-H}^{X} \big(\sqrt{X} - \sqrt{x}\big)^2 dx + H \ll_\epsilon \frac{H^3 (\log X)^2}{X} + H. \tag{3.14}$$

Estimation of I_2. We have

$$I_2 = \int_{E_2} \Big(\sum_{x \le n^2 \le x+H} (k(n)-1) \Big)^2 \, dx \tag{3.15}$$
$$\ll \int_{\epsilon X}^{X} \big(\theta(\sqrt{x+H}) - \theta(\sqrt{x}) - (\sqrt{x+H} - \sqrt{x})\big)^2 \, dx + X$$
$$= J^*(X,H) + X,$$

where we used the definition (3.10). Therefore, by (3.11), (3.13)–(3.15) and $Y \ge 1/X$, and recalling $H = 1/(2Y)$, we have

$$\mathcal{I} \ll_\epsilon \frac{(\log X)^2}{XY} + XY^2 + Y^2 J^*\Big(X, \frac{1}{2Y}\Big),$$

and this proves Lemma 3.12. ∎

To estimate $J^*(X,h)$, we use the next result.

LEMMA 3.13. *Let ϵ be an arbitrarily small positive constant. There exists a positive constant $c_6(\epsilon)$ such that*

$$J^*(X,h) \ll_\epsilon h^2 \exp\Big(-c_6\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big)$$

uniformly for $X^{7/12+\epsilon} \le h \le X$.

Proof. We reduce our problem to estimating

$$J^*_\psi(X,h) := \int_{\epsilon X}^{X} \big(\psi(\sqrt{x+h}) - \psi(\sqrt{x}) - (\sqrt{x+h} - \sqrt{x})\big)^2 \, dx \tag{3.16}$$

since, using $|a+b|^2 \le 2|a|^2 + 2|b|^2$, it is easy to see that

$$J^*(X,h) \ll J^*_\psi(X,h) + \int_{\epsilon X}^{X} \big(\psi(\sqrt{x+h}) - \psi(\sqrt{x}) - \theta(\sqrt{x+h}) + \theta(\sqrt{x})\big)^2 \, dx.$$

By a trivial estimate and the Mean Value Theorem we obtain

$$J^*(X,h) \ll_\epsilon J^*_\psi(X,h) + \int_{\epsilon X}^{X} \frac{h^2}{X^{3/2}} (\log X)^4 \, dx \ll_\epsilon J^*_\psi(X,h) + h^2 \frac{(\log X)^4}{X^{1/2}}. \tag{3.17}$$

To estimate the right hand side of (3.17), we use the following result we will prove later.

LEMMA 3.14. *Let ϵ be an arbitrarily small positive constant. There exists a positive constant $c_6(\epsilon)$ such that*

$$J^*_\psi(X,h) \ll_\epsilon h^2 \exp\biggl(-c_6\biggl(\frac{\log X}{\log\log X}\biggr)^{1/3}\biggr)$$

uniformly for $X^{7/12+\epsilon} \le h \le X$, where $J^*_\psi(X,h)$ is defined in (3.16).

Therefore, by (3.17) and Lemma 3.14, we obtain

$$J^*(X,h) \ll_\epsilon h^2 \exp\biggl(-c_6\biggl(\frac{\log X}{\log\log X}\biggr)^{1/3}\biggr),$$

thus proving Lemma 3.13. ∎

Lemma 3.14 will follow from the following lemma.

LEMMA 3.15. *Let ϵ be an arbitrarily small positive constant. There exists a positive constant $c_6(\epsilon)$ such that*

$$\begin{aligned}\widetilde{J}^*_\psi(X,\delta) &:= \int_{\epsilon X}^{X} \bigl(\psi(\sqrt{x+\delta x}) - \psi(\sqrt{x}) - (\sqrt{x+\delta x}-\sqrt{x})\bigr)^2\,dx \\ &\ll_\epsilon \delta^2 X^2 \exp\biggl(-c_6\biggl(\frac{\log X}{\log\log X}\biggr)^{1/3}\biggr)\end{aligned}$$

uniformly for $X^{-5/12+\epsilon} \le \delta \le 1$.

Proof. We follow the argument of Saffari–Vaughan [40, §5]. To estimate $\widetilde{J}^*_\psi(X,\delta)$, we use the truncated explicit formula for $\psi(x)$ (see, e.g., Davenport [4, eq. (9)–(10) of §17]):

$$\psi(x) = x - \sum_{|\gamma|\le T} \frac{x^\rho}{\rho} + \mathcal{O}\biggl(\frac{x}{T}(\log(xT))^2 + \log x\biggr)$$

uniformly in $T \ge 2$ and for $\rho = \beta + i\gamma$ nontrivial zeros of $\zeta(s)$. So

$$\widetilde{J}^*_\psi(X,\delta) \ll \int_{\epsilon X}^{X} \biggl|\sum_{\substack{|\gamma|\le T\\ \beta\ge 1/2}} x^{\rho/2}\frac{((1+\delta)^{\rho/2}-1)}{\rho}\biggr|^2 dx + \frac{X^2}{T^2}(\log(XT))^4 + X\log^2 X. \tag{3.18}$$

As in Ivić [13, p. 316], we define $c(\delta,\rho) = ((1+\delta)^\rho - 1)/\rho$, and remark

$$|c(\delta,\rho/2)| \ll \min(1/|\gamma|;\delta). \tag{3.19}$$

Assuming $T \ge 1/\delta$, we can split the summation in (3.18) into two cases defined according to (3.19). We obtain

$$\widetilde{J}^*_\psi(X,\delta) \ll A_{[0,1/\delta)} + A_{[1/\delta,T]} + \frac{X^2}{T^2}(\log(XT))^4 + X(\log X)^2, \tag{3.20}$$

with

(3.21)
$$\begin{aligned}A_I &= \int_{\epsilon X}^{X} \Big| \sum_{\substack{|\gamma|\in I\\ \beta\ge 1/2}} x^{\rho/2} c\Big(\delta, \frac{\rho}{2}\Big)\Big|^2 \, dx\\ &= \sum_{\substack{|\gamma_1|\in I\\ \beta_1\ge 1/2}} \sum_{\substack{|\gamma_2|\in I\\ \beta_2\ge 1/2}} c\Big(\delta, \frac{\rho_1}{2}\Big) c\Big(\delta, \frac{\overline{\rho}_2}{2}\Big) \frac{2X^{(\rho_1+\overline{\rho}_2)/2+1}(1-\epsilon^{(\rho_1+\overline{\rho}_2)/2+1})}{\rho_1+\overline{\rho}_2+2}\\ &\ll \sum_{\substack{|\gamma_1|\in I\\ \beta_1\ge 1/2}} \sum_{\substack{|\gamma_2|\in I\\ 1/2\le\beta_2\le\beta_1}} \Big|c\Big(\delta, \frac{\rho_1}{2}\Big)\Big|\Big|c\Big(\delta, \frac{\overline{\rho}_2}{2}\Big)\Big| \frac{X^{\beta_1+1}}{1+|\gamma_1-\gamma_2|}.\end{aligned}$$

Now we deal separately with $A_{[0,1/\delta)}$ and $A_{[1/\delta,T]}$.

Estimation of $A_{[0,1/\delta)}$. From (3.19) and (3.21) we can write

$$\begin{aligned}A_{[0,1/\delta)} &\ll \delta^2 X \sum_{\substack{|\gamma_1|<\delta^{-1}\\ \beta_1\ge 1/2}} X^{\beta_1} \sum_{\substack{|\gamma_2|<\delta^{-1}\\ 1/2\le\beta_2\le\beta_1}} \frac{1}{1+|\gamma_1-\gamma_2|}\\ &\ll \delta^2 X (\log X)^2 \sum_{\substack{|\gamma_1|<\delta^{-1}\\ \beta_1\ge 1/2}} X^{\beta_1},\end{aligned} \tag{3.22}$$

where the last inequality follows from

(3.23)
$$\sum_{\substack{|\gamma_2|<\delta^{-1}\\ 1/2\le\beta_2\le\beta_1}} \frac{1}{1+|\gamma_1-\gamma_2|} \ll \sum_{n=0}^{2/\delta} \frac{\log(\gamma_1+n)}{1+n} \ll \Big(\log\Big(\frac{3}{\delta}\Big)\Big)^2 \ll (\log X)^2$$

in which we used the Riemann–von Mangoldt formula and $\delta > X^{-1}$. Denoting by $S_{[0,1/\delta)}$ the sum on the right hand side of (3.22), we get

$$S_{[0,1/\delta)} := \sum_{\substack{|\gamma|<1/\delta\\ \beta\ge 1/2}} X^{\beta} \ll \log X \max_{1/2\le u\le 1} X^u N(u, 1/\delta).$$

We recall the Ingham–Huxley zero-density estimate: for $1/2 \le \sigma \le 1$ we have $N(\sigma, t) \ll t^{(12/5)(1-\sigma)}(\log t)^B$, and the Vinogradov–Korobov zero-free region: there are no zeros $\beta + i\gamma$ of the Riemann zeta function having

$$\beta \ge 1 - \frac{c_7}{(\log(|\gamma|+2))^{2/3}(\log\log(|\gamma|+2))^{1/3}},$$

where $c_7 > 0$ is an absolute constant. In the following c_7 will not necessarily be the same at each occurrence. Here we have $|\gamma| \le T$, and so $N(u, t) = 0$

for every $t \le T$ and $u \ge 1-K$ with

$$K = \frac{c_7}{(\log T)^{2/3}(\log\log T)^{1/3}}.$$

From the previous remarks, we obtain

$$\begin{aligned} S_{[0,1/\delta)} &\ll \log X \max_{1/2\le u\le 1-K} (\delta^{-1})^{(12/5)(1-u)}(\log(\delta^{-1}))^B X^u \\ &\ll (\log X)^{B+1}\delta^{-12/5} \max_{1/2\le u\le 1-K} (\delta^{12/5}X)^u, \end{aligned}$$

since $\delta > X^{-1}$. The maximum is attained at $u = 1-K$ and so

$$S_{[0,1/\delta)} \ll (\log X)^{B+1}\delta^{-12/5}\delta^{(12/5)(1-K)}X^{1-K} = X(\log X)^{B+1}(\delta^{12/5}X)^{-K}.$$

Inserting the last estimate into (3.22), we can write

$$A_{[0,1/\delta)} \ll \delta^2 X^2(\log X)^{B+3}(\delta^{12/5}X)^{-K}. \tag{3.24}$$

Estimation of $A_{[1/\delta,T]}$. From (3.19) and (3.21) we get

$$\begin{aligned} A_{[1/\delta,T]} &\ll X \sum_{\substack{1/\delta\le|\gamma_1|\le T\\ \beta_1\ge 1/2}} \frac{X^{\beta_1}}{|\gamma_1|} \sum_{\substack{1/\delta\le|\gamma_2|\le T\\ 1/2\le\beta_2\le\beta_1}} \frac{1}{|\gamma_2|(1+|\gamma_1-\gamma_2|)} \\ &\ll X \sum_{\substack{1/\delta\le|\gamma_1|\le T\\ \beta_1\ge 1/2}} \frac{X^{\beta_1}}{|\gamma_1|^2} \sum_{\substack{|\gamma_1|\le|\gamma_2|\le T\\ 1/2\le\beta_2\le\beta_1}} \frac{1}{1+|\gamma_1-\gamma_2|} \\ &\ll X(\log T)^2 \sum_{\substack{1/\delta\le|\gamma_1|\le T\\ \beta_1\ge 1/2}} \frac{X^{\beta_1}}{|\gamma_1|^2}, \end{aligned}$$

where the last step follows from (3.23) with T instead of $1/\delta$. By a simple trick, we can rewrite the previous inequality as

$$A_{[1/\delta,T]} \ll X(\log T)^2(S'_{[1/\delta,T]} + S''_{[1/\delta,T]}) \tag{3.25}$$

with

$$S'_{[1/\delta,T]} = \sum_{\substack{1/\delta\le|\gamma|\le T\\ \beta\ge 1/2}} X^\beta\left(\frac{1}{|\gamma|^2} - \frac{1}{T^2}\right) \quad \text{and} \quad S''_{[1/\delta,T]} = \frac{1}{T^2}\sum_{\substack{1/\delta\le|\gamma|\le T\\ \beta\ge 1/2}} X^\beta.$$

For $S''_{[1/\delta,T]}$ we can argue as we did for $S_{[0,1/\delta)}$, just keeping in mind that this time $1/\delta \le |\gamma| \le T$. Hence

$$S''_{[1/\delta,T]} \ll \frac{\log X}{T^2} \max_{1/2\le u\le 1-K} X^u[N(u,T) - N(u,1/\delta)].$$

Concerning $S'_{[1/\delta,T]}$ we immediately obtain

$$S'_{[1/\delta,T]} = \sum_{\substack{1/\delta\le|\gamma|\le T\\ \beta\ge 1/2}} X^\beta \int_{|\gamma|}^{T} \frac{2}{t^3}\,dt = 2\int_{1/\delta}^{T} \Big(\sum_{\substack{1/\delta\le|\gamma|\le t\\ \beta\ge 1/2}} X^\beta \Big) \frac{dt}{t^3}.$$

Using $t \le T$, we can write

$$S'_{[1/\delta,T]} \ll \log X \int_{1/\delta}^{T} \max_{1/2\le u\le 1-K} X^u [N(u,t) - N(u,1/\delta)]\, \frac{dt}{t^3}.$$

Therefore

$$S'_{[1/\delta,T]} + S''_{[1/\delta,T]} \ll \log X \log(T\delta) \times \max_{1/\delta\le t\le T} \Big(\frac{1}{t^2} \max_{1/2\le u\le 1-K} X^u t^{(12/5)(1-u)} (\log t)^B \Big),$$

by the Ingham–Huxley zero-density estimate. So, by (3.25), this estimate and $t \le T$, we get

$$A_{[1/\delta,T]} \ll X(\log T)^{B+2} \log X \log(T\delta) \max_{1/2\le u\le 1-K} \Big(X^u \max_{1/\delta\le t\le T} t^{(12/5)(1-u)-2} \Big).$$

To compute the inner maximum above, we just remark that $(12/5)(1-u) - 2 < 0$ (which holds for $u > 1/6$), and hence it is attained at $t = 1/\delta$. So

$$\begin{aligned} A_{[1/\delta,T]} &\ll X(\log T)^{B+2} \log X \log(T\delta) \max_{1/2\le u\le 1-K} X^u (\delta^{-1})^{(12/5)(1-u)-2} \\ &= \delta^{-2/5} X(\log T)^{B+2} \log X \log(T\delta) \max_{1/2\le u\le 1-K} (X\delta^{12/5})^u. \end{aligned}$$

The maximum is attained at $u = 1 - K$, thus

$$\begin{aligned} A_{[1/\delta,T]} &\ll \delta^{-2/5} X(\log T)^{B+2} \log X \log(T\delta)(X\delta^{12/5})^{1-K} \\ &= \delta^2 X^2 (\log T)^{B+2} \log X \log(T\delta)(X\delta^{12/5})^{-K}. \end{aligned} \tag{3.26}$$

Conclusion of the proof. Inserting (3.24) and (3.26) into (3.20), we get

$$\begin{aligned} \widetilde{J}^*_\psi(X,\delta) \ll \delta^2 X^2 (X\delta^{12/5})^{-K} \log X ((\log X)^{B+2} \\ + (\log T)^{B+2} \log(T\delta)) + \frac{X^2}{T^2} (\log(XT))^4 + X(\log X)^2. \end{aligned} \tag{3.27}$$

Choosing $T \le X^{1/2}$ we have

$$K = \frac{c_7}{(\log T)^{2/3} (\log\log T)^{1/3}} \ge \frac{c_8}{(\log X)^{2/3} (\log\log X)^{1/3}},$$

for a suitable positive constant c_8. If we now take $T \geq X^{5/12-\epsilon}(X\delta^{12/5})^{K/2} \times(\log X)^{-B/2}$ and recall $\delta > X^{-5/12+\epsilon}$, estimate (3.27) becomes

$$\widetilde{J}_{\psi}^{*}(X,\delta) \ll \delta^2 X^2 (X\delta^{12/5})^{-K}(\log X)^{B+4}$$

since the conditions on T are compatible. Hence we immediately obtain

$$\begin{aligned}\widetilde{J}_{\psi}^{*}(X,\delta) &\ll \delta^2 X^2 (\log X)^{B+4} \exp\biggl(-\frac{c_8(\log X + (12/5)\log\delta)}{(\log X)^{2/3}(\log\log X)^{1/3}}\biggr)\\ &\ll \delta^2 X^2 \exp\biggl(-c_9\Bigl(\frac{\log X}{\log\log X}\Bigr)^{1/3}\biggr)\end{aligned}$$

for a sufficiently large X and $c_9 = c_9(\epsilon)$. Hence Lemma 3.15 is proved. ∎

Proof of Lemma 3.14. We follow the argument of [40, §6]. Let now $2h \leq v \leq 3h$. To estimate $J_{\psi}^{*}(X,h)$ (defined in (3.16)), we first remark

$$\begin{aligned}(3.28)\qquad hJ_{\psi}^{*}(X,h) &\ll \int_{\epsilon X}^{X}\int_{2h}^{3h}\bigl(\psi(\sqrt{x+v})-\psi(\sqrt{x})-(\sqrt{x+v}-\sqrt{x})\bigr)^2\,dv\,dx\\ &\quad+\int_{\epsilon X}^{X}\int_{2h}^{3h}\bigl(\psi(\sqrt{x+v})-\psi(\sqrt{x+h})-(\sqrt{x+v}-\sqrt{x+h})\bigr)^2\,dv\,dx.\end{aligned}$$

We set $z = v-h, y = x+h$ and change variables in the last integration, so that the right hand side of (3.28) becomes

$$\begin{aligned}&\ll \int_{\epsilon X}^{X}\int_{2h}^{3h}\bigl(\psi(\sqrt{x+v})-\psi(\sqrt{x})-(\sqrt{x+v}-\sqrt{x})\bigr)^2\,dv\,dx\\ &\quad+\int_{\epsilon X+h}^{X+h}\int_{h}^{2h}\bigl(\psi(\sqrt{y+z})-\psi(\sqrt{y})-(\sqrt{y+z}-\sqrt{y})\bigr)^2\,dz\,dy.\end{aligned}$$

Since both the integrands are nonnegative, we can extend the integration ranges merging x with y and v with z. Hence

$$\begin{aligned}hJ_{\psi}^{*}(X,h) &\ll \int_{\epsilon X}^{X+h}\int_{h}^{3h}\bigl(\psi(\sqrt{x+v})-\psi(\sqrt{x})-(\sqrt{x+v}-\sqrt{x})\bigr)^2\,dv\,dx\\ &= \int_{\epsilon X}^{X+h} x\int_{h/x}^{3h/x}\bigl(\psi(\sqrt{x+x\delta})-\psi(\sqrt{x})-(\sqrt{x+x\delta}-\sqrt{x})\bigr)^2\,d\delta\,dx,\end{aligned}$$

where in the last step we made the change of variable $\delta = v/x$, thus getting $\delta \geq h/x \geq X^{-5/12+\epsilon}$ as in the hypothesis of Lemma 3.15. Interchanging the integration order we obtain

$$hJ^*_\psi(X,h) \ll (X+h) \times \int_{h/(X+h)}^{3h/(\epsilon X)} \int_{\epsilon X}^{X+h} \big(\psi(\sqrt{x+x\delta}) - \psi(\sqrt{x}) - (\sqrt{x+x\delta} - \sqrt{x})\big)^2 \, dx \, d\delta.$$

Finally, using Lemma 3.15, we get

$$\begin{aligned} J^*_\psi(X,h) &\ll_\epsilon \frac{X+h}{h} \int_{h/(X+h)}^{3h/(\epsilon X)} \delta^2 X^2 \exp\bigg(-c_6\bigg(\frac{\log X}{\log\log X}\bigg)^{1/3}\bigg) \, d\delta \\ &\ll_\epsilon h^2 \exp\bigg(-c_6\bigg(\frac{\log X}{\log\log X}\bigg)^{1/3}\bigg). \end{aligned}$$

This concludes the proof of Lemma 3.14. ∎

4. The major arc. Letting

$$T_1(\alpha) = \int_{\epsilon X}^{X} e(t\alpha)\, dt \ll_\epsilon \min(X; 1/|\alpha|) \tag{4.1}$$

and

$$T_2(\alpha) = \int_{(\epsilon X)^{1/2}}^{X^{1/2}} e(t^2\alpha)\, dt = \frac{1}{2}\int_{\epsilon X}^{X} v^{-1/2} e(v\alpha)\, dv \ll_\epsilon X^{-1/2} \min(X; 1/|\alpha|), \tag{4.2}$$

we first write

$$\begin{aligned} I(X;\mathfrak{M}) &= \int_{\mathfrak{M}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha) \prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\, d\alpha \\ &+ \int_{\mathfrak{M}} (S_1(\lambda_1\alpha) - T_1(\lambda_1\alpha))T_2(\lambda_2\alpha)T_2(\lambda_3\alpha) \prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\, d\alpha \\ &+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha)(S_2(\lambda_2\alpha) - T_2(\lambda_2\alpha))T_2(\lambda_3\alpha) \prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\, d\alpha \\ &+ \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha) - T_2(\lambda_3\alpha)) \prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\, d\alpha \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{4.3}$$

say. In what follows we will prove that

$$J_1 \ge \frac{(3-2\sqrt{2})\eta^2 X L^s}{4|\lambda_1|+|\lambda_2|+|\lambda_3|} + \mathcal{O}_\epsilon(\eta^2 X^{1/5} L^{s+2}) \tag{4.4}$$

and

$$J_2 + J_3 + J_4 = o(\eta^2 X L^s), \tag{4.5}$$

thus obtaining, by (4.3)–(4.5),

$$I(X;\mathfrak{M}) \ge \frac{3-2\sqrt{2}-\epsilon}{4\,(|\lambda_1|+|\lambda_2|+|\lambda_3|)}\eta^2 X L^s,$$

proving that (2.6) holds with $c_1 = 2^{-2}(3-2\sqrt{2}-\epsilon)(|\lambda_1|+|\lambda_2|+|\lambda_3|)^{-1}$ and $\epsilon > 0$ an arbitrarily small constant.

We will need the following estimates. The first one is a consequence of the Prime Number Theorem:

$$\int_0^1 |S_1(\alpha)|^2 \, d\alpha \ll_\epsilon X \log X, \tag{4.6}$$

while the second one is based on Satz 3 of Rieger [38, p. 94] (see also the estimate of H_{12} of T. Liu [31, p. 106]):

$$\int_0^1 |S_2(\alpha)|^4 \, d\alpha \ll_\epsilon X (\log X)^2. \tag{4.7}$$

Estimation of J_2, J_3 and J_4. We first estimate J_4. We remark that, by Euler's summation formula,

$$T_i(\alpha) - U_i(\alpha) \ll 1 + X|\alpha| \quad \text{for } i = 1,2. \tag{4.8}$$

So, by (2.4), the Cauchy–Schwarz inequality, and (4.6)–(4.8), we get

$$\begin{aligned}
&\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha)|\,|T_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|\,d\alpha \\
&\quad \ll_{\boldsymbol{\lambda}} \int_{-1/X}^{1/X} |S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha)|\,d\alpha + X \int_{1/X}^{P/X} |\alpha|\,|S_1(\lambda_1\alpha)|\,|S_2(\lambda_2\alpha)|\,d\alpha \\
&\quad \ll_{\boldsymbol{\lambda}} X^{-1/4} \Big(\int_0^1 |S_1(\alpha)|^2 \, d\alpha \Big)^{1/2} \Big(\int_0^1 |S_2(\alpha)|^4 \, d\alpha \Big)^{1/4} \\
&\qquad + X \Big(\int_{1/X}^{P/X} \alpha^4 \, d\alpha \Big)^{1/4} \Big(\int_0^1 |S_2(\alpha)|^4 \, d\alpha \Big)^{1/4} \Big(\int_0^1 |S_1(\alpha)|^2 \, d\alpha \Big)^{1/2} \\
&\quad \ll_{\boldsymbol{\lambda},\epsilon} X^{1/2} \log X + P^{5/4} X^{1/2} \log X = o(X)
\end{aligned}$$

since $P = X^{2/5}/\log X$. Hence, using the trivial estimates $|G(\mu_i\alpha)| \le L$, $K(\alpha,\eta) \ll \eta^2$, we can write

$$J_4 = \int_{\mathfrak{M}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)(S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)) \prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\,d\alpha + o_{\boldsymbol{\lambda},M,\epsilon}(\eta^2 X L^s).$$

Now using (2.4), $|S_2(\lambda_2\alpha)| \ll X^{1/2}$, the Cauchy–Schwarz inequality, (4.6), Lemmas 3.12–3.13 with $Y = P/X$, and again the trivial estimates $|G(\mu_i\alpha)| \le L$, $K(\alpha,\eta) \ll \eta^2$, we have

$$\begin{aligned} J_4 &\ll \eta^2 L^s X^{1/2} \Big(\int_{\mathfrak{M}} |S_2(\lambda_3\alpha) - U_2(\lambda_3\alpha)|^2\,d\alpha\Big)^{1/2} \Big(\int_{\mathfrak{M}} |S_1(\lambda_1\alpha)|^2\,d\alpha\Big)^{1/2} \\ &\quad + o_{\boldsymbol{\lambda},M,\epsilon}(\eta^2 X L^s) \\ &\ll_{\boldsymbol{\lambda},M,\epsilon} \eta^2 L^s X^{1/2} \Big(\int_0^1 |S_1(\alpha)|^2\,d\alpha\Big)^{1/2} \exp\Big(-\frac{c_6(\epsilon)}{2}\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big) \\ &\quad + o_{\boldsymbol{\lambda},M}(\eta^2 X L^s) \\ &\ll_{\boldsymbol{\lambda},M,\epsilon} \eta^2 X L^{s+1/2} \exp\Big(-\frac{c_6(\epsilon)}{2}\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big) = o(\eta^2 X L^s). \end{aligned}$$

The integral J_3 can be estimated analogously using (4.2) instead of $|S_2(\lambda_3\alpha)| \ll X^{1/2}$.

For J_2 we argue as follows. First of all, using again (4.8) and (4.2) for $i = 2,3$, we get

$$\begin{aligned} &\int_{\mathfrak{M}} |T_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)|\,|T_2(\lambda_2\alpha)|\,|T_2(\lambda_3\alpha)|\,d\alpha \\ &\qquad\qquad \ll_{\boldsymbol{\lambda}} X \int_{-1/X}^{1/X} d\alpha + \int_{1/X}^{P/X} \frac{X|\alpha|}{X\alpha^2}\,d\alpha \ll_{\boldsymbol{\lambda}} 1 + \log P = o(X) \end{aligned}$$

since $P = X^{2/5}/\log X$. Hence, using the trivial estimates $|G(\mu_i\alpha)| \le L$, $K(\alpha,\eta) \ll \eta^2$, we can write

$$J_2 = \int_{\mathfrak{M}} \big(S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)\big)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha) \prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\,d\alpha + o_{\boldsymbol{\lambda},M}(\eta^2 X L^s).$$

Using (2.4), the Cauchy–Schwarz inequality, Lemmas 3.10–3.11 with $Y = P/X$, and the trivial estimates $|G(\mu_i\alpha)| \le L$, $K(\alpha,\eta) \ll \eta^2$, we have

$$J_2 \ll \eta^2 L^s\Big(\int_{\mathfrak{M}} |S_1(\lambda_1\alpha) - U_1(\lambda_1\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{M}} |T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)|^2\,d\alpha\Big)^{1/2}$$
$$+ o_{\boldsymbol{\lambda},M}(\eta^2 X L^s)$$
$$\ll_{\boldsymbol{\lambda},M,\epsilon} \eta^2 X L^s \exp\Big(-\frac{c_6(\epsilon)}{2}\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big) + o_{\boldsymbol{\lambda},M}(\eta^2 X L^s)$$
$$= o(\eta^2 X L^s),$$

since, by (4.2), $\int_{\mathfrak{M}} |T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)|^2\,d\alpha \ll_{\boldsymbol{\lambda}} X$. Hence (4.5) holds.

Estimation of J_1. Recalling that $P = X^{2/5}/\log X$, using (2.4) and (4.1)–(4.3) we obtain

(4.9)
$$J_1 = \sum_{1\le m_1\le L} \cdots \sum_{1\le m_s\le L} \mathcal{J}(\mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \varpi, \eta) + \mathcal{O}_\epsilon(\eta^2 X^{1/5} L^{s+2}),$$

where $\mathcal{J}(u,\eta)$ is defined by

$$\mathcal{J}(u,\eta) := \int_{\mathbb{R}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_2(\lambda_3\alpha)e(u\alpha)K(\alpha,\eta)\,d\alpha$$
$$= \frac{1}{4}\int_{\epsilon X}^{X}\int_{\epsilon X}^{X}\int_{\epsilon X}^{X} \widehat{K}(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + u, \eta) u_2^{-1/2} u_3^{-1/2}\,du_1\,du_2\,du_3$$

and the second relation follows by (4.1)–(4.2) and interchanging the integration order. We recall that $\lambda_1 < 0$ and $\lambda_2, \lambda_3 > 0$. If $|u| \le \epsilon X$, for

$$\frac{X|\lambda_1|}{2(|\lambda_1| + \lambda_2 + \lambda_3)} \le u_2, u_3 \le \frac{X|\lambda_1|}{|\lambda_1| + \lambda_2 + \lambda_3},$$

sufficiently large X and sufficiently small ϵ, we get

$$-\frac{\eta}{2} - (\lambda_2 u_2 + \lambda_3 u_3 + u) \le |\lambda_1| u_1 \le \eta/2 - (\lambda_2 u_2 + \lambda_3 u_3 + u).$$

Hence there exists an interval for u_1 of length $\eta|\lambda_1|^{-1}$ and contained in $[\epsilon X, X]$ such that $\widehat{K}(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + u, \eta) \ge \eta/2$. So, letting $b = X|\lambda_1|/(|\lambda_1| + \lambda_2 + \lambda_3)$, we can write

$$\mathcal{J}(u,\eta) \ge \frac{\eta^2}{8|\lambda_1|}\Big(\int_{b/2}^{b} v^{-1/2}\,dv\Big)^2 = \frac{(3-2\sqrt{2})\eta^2 X}{4(|\lambda_1| + \lambda_2 + \lambda_3)}.$$

By the definition of L, we have $|\mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \varpi| \le \epsilon X$ for X sufficiently large. Hence by (4.9) we obtain

$$J_1 \ge \frac{(3-2\sqrt{2})\eta^2 X L^s}{4(|\lambda_1| + \lambda_2 + \lambda_3)} + \mathcal{O}_\epsilon(\eta^2 X^{1/5} L^{s+2}),$$

thus proving (4.4). Arguing analogously we can prove the case $\lambda_1, \lambda_2 < 0$, $\lambda_3 > 0$.

5. The trivial arc. Recalling (2.4), the trivial estimate $|G(\mu_i\alpha)| \le L$ and using twice the Cauchy–Schwarz inequality, we get

$$|I(X;\mathfrak{t})| \ll L^s\Big(\int_{L^2}^{+\infty} |S_1(\lambda_1\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2} \times \Big(\int_{L^2}^{+\infty} |S_2(\lambda_2\alpha)|^4 K(\alpha,\eta)\,d\alpha\Big)^{1/4}\Big(\int_{L^2}^{+\infty} |S_2(\lambda_3\alpha)|^4 K(\alpha,\eta)\,d\alpha\Big)^{1/4}.$$

By (2.2) and making a change of variable, we have, for $i = 2, 3$,

$$\int_{L^2}^{+\infty} |S_2(\lambda_i\alpha)|^4 K(\alpha,\eta)\,d\alpha \ll_{\boldsymbol{\lambda}} \int_{\lambda_i L^2}^{+\infty} \frac{|S_2(\alpha)|^4}{\alpha^2}\,d\alpha \ll \sum_{n\ge\lambda_i L^2} \frac{1}{(n-1)^2}\int_{n-1}^{n} |S_2(\alpha)|^4\,d\alpha \ll_{\boldsymbol{\lambda}} L^{-2}\int_0^1 |S_2(\alpha)|^4\,d\alpha \ll_{\boldsymbol{\lambda},M,\epsilon} X,$$

by (4.7). Moreover, arguing analogously,

$$\int_{L^2}^{+\infty} |S_1(\lambda_1\alpha)|^2 K(\alpha,\eta)\,d\alpha \ll_{\boldsymbol{\lambda}} \int_{|\lambda_1| L^2}^{+\infty} \frac{|S_1(\alpha)|^2}{\alpha^2}\,d\alpha \ll \sum_{n\ge|\lambda_1| L^2} \frac{1}{(n-1)^2}\int_{n-1}^{n} |S_1(\alpha)|^2\,d\alpha \ll_{\boldsymbol{\lambda}} L^{-2}\int_0^1 |S_1(\alpha)|^2\,d\alpha \ll_{\boldsymbol{\lambda},M,\epsilon} \frac{X}{\log X},$$

by (4.6). Hence (2.7) holds.

6. The minor arc. Recalling first

$$I(X;\mathfrak{m}) = \int_{\mathfrak{m}} S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_2(\lambda_3\alpha)\prod_{i=1}^{s} G(\mu_i\alpha)e(\varpi\alpha)K(\alpha,\eta)\,d\alpha,$$

and letting $c \in (0,1)$ be chosen later, we first split $\mathfrak{m}$ as $\mathfrak{m}_1 \sqcup \mathfrak{m}_2$, where $\mathfrak{m}_2$ is the set of $\alpha \in \mathfrak{m}$ such that $|G(\mu_i\alpha)| > \nu(c)L$ for some $i \in \{1,\dots,s\}$, and $\nu(c)$ is defined in Lemma 3.9. We will choose c to get $|I(X;\mathfrak{m}_2)| = o(\eta X)$, since, again by Lemma 3.9, we know that $|\mathfrak{m}_2| \ll_{M,\epsilon} sL^2X^{-c}$.

To this end, we first use the trivial estimates $|G(\mu_i\alpha)| \le L$ and $K(\alpha,\eta) \ll \eta^2$ and Lemma 3.8 (assuming, without any loss of generality, that $V(\alpha) = |S_2(\lambda_2\alpha)|$). Then, using twice the Cauchy–Schwarz inequality and (4.6)–(4.7), we get

$$
\begin{aligned}
|I(X;\mathfrak{m}_2)| &\le \eta^2 L^s \Big(\sup_{\alpha\in\mathfrak{m}} |V(\alpha)|\Big)\Big(\int_{\mathfrak{m}_2} |S_1(\lambda_1\alpha)S_2(\lambda_3\alpha)|\,d\alpha\Big) \\
&\ll \eta^2 L^s X^{7/16+\epsilon}|\mathfrak{m}_2|^{1/4}\Big(\int_{\mathfrak{m}_2} |S_1(\lambda_1\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{m}_2} |S_2(\lambda_3\alpha)|^4\,d\alpha\Big)^{1/4} \\
&\ll_{\boldsymbol{\lambda}} \eta^2 L^s X^{7/16+\epsilon}|\mathfrak{m}_2|^{1/4}\Big(L^2\int_0^1 |S_1(\alpha)|^2\,d\alpha\Big)^{1/2}\Big(L^2\int_0^1 |S_2(\alpha)|^4\,d\alpha\Big)^{1/4} \\
&\ll_{\boldsymbol{\lambda},M,\epsilon} s^{1/4}\eta^2 L^{s+3} X^{19/16+\epsilon-c/4},
\end{aligned}
$$

where $X = q^2$ and q is the denominator of a convergent of the continued fraction for λ_2/λ_3. Taking $c = 3/4 + 10^{-20}$ and using (3.8), we get, for $\nu = 0.8844472132$ and a sufficiently small $\epsilon > 0$,

$$
|I(X;\mathfrak{m}_2)| = o(\eta X). \tag{6.1}
$$

We remark that neither the result of Kumchev [14] nor the approach of Cook, Fox and Harman (see [2], [3], [10]) seem to give any improvement of the previous estimates.

Now we evaluate the contribution of $\mathfrak{m}_1$. Using the Cauchy–Schwarz inequality, and Lemmas 3.3 and 3.6, we have

$$
\begin{aligned}
|I(X;\mathfrak{m}_1)| &\le (\nu L)^{s-3}\Big(\int_{\mathfrak{m}} |S_1(\lambda_1\alpha)G(\mu_1\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2} \\
&\quad\times\Big(\int_{\mathfrak{m}} |S_2(\lambda_2\alpha)G(\mu_2\alpha)|^4 K(\alpha,\eta)\,d\alpha\Big)^{1/4} \\
&\quad\times\Big(\int_{\mathfrak{m}} |S_2(\lambda_3\alpha)G(\mu_3\alpha)|^4 K(\alpha,\eta)\,d\alpha\Big)^{1/4} \\
&< \nu^{s-3} C(q_1,q_2,q_3,\epsilon)\eta X L^s,
\end{aligned} \tag{6.2}
$$

where $C(q_1,q_2,q_3,\epsilon)$ is defined as in (1.3).

Hence, by (6.1)–(6.2), for X sufficiently large we finally get

$$
|I(X;\mathfrak{m})| < (0.8844472132)^{s-3} C(q_1,q_2,q_3,\epsilon)\eta X L^s.
$$

This means that (2.8) holds with $c_2(s) = (0.8844472132)^{s-3}\, C(q_1,q_2,q_3,\epsilon)$.

7. Proof of the Theorem. We have to verify that there is an $s_0 \in \mathbb{N}$ such that (2.9) holds for X sufficiently large, where $X = q^2$ and q is the denominator of a convergent of the continued fraction for λ_2/λ_3. Combining the inequalities (2.6)–(2.8), where $c_2(s) = (0.8844472132)^{s-3} C(q_1,q_2,q_3,\epsilon)$, we conclude that (2.9) holds for $s \ge s_0$ where s_0 defined in (1.2).

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