Inhomogeneous approximation with coprime integers and lattice orbits

by

MICHEL LAURENT and ARNALDO NOGUEIRA (Marseille)

1. Introduction and results. Minkowski proved that for every real irrational $\xi$ and every real $y \notin \mathbb{Z}\xi + \mathbb{Z}$, there exist infinitely many pairs of integers $p,q$ such that

$$|q\xi + p - y| \leq \frac{1}{4|q|}$$

(see for instance Theorem II in Chapter 3 of Cassels’ monograph [4]). The statement is optimal in the sense that the approximating function $\ell \mapsto (4\ell)^{-1}$ cannot be decreased. Note that the restriction $y \notin \mathbb{Z}\xi + \mathbb{Z}$ can be dropped at the cost of replacing the upper bound $(4|q|)^{-1}$ by $c|q|^{-1}$ for any constant $c$ greater than $1/\sqrt{5}$. When $y = 0$, the primitive point $(p/gcd(p,q), q/gcd(p,q))$ remains a solution to the above inequality, therefore we may moreover require that the integers $p,q$ be coprime. However, for a non-zero $y$, this extra requirement is far from being obvious to satisfy. In this direction, Chalk and Erdős [6] obtained the following result:

**Theorem (Chalk–Erdős).** Let $\xi$ be an irrational real number and let $y$ be a real number. There exists an absolute constant $c$ such that the inequality

$$|q\xi + p - y| \leq \frac{c(\log q)^2}{q(\log \log q)^2}$$

holds for infinitely many pairs of coprime integers $(p,q)$ with $q$ positive.

We study more generally the diophantine inequality

$$|q\xi + p - y| \leq \psi(|q|)$$

for coprime integers $p$ and $q$, where $\psi : \mathbb{N} \to \mathbb{R}^+$ is a given function. Two types of questions naturally arise. First, finding unconditional results which are valid for every real pair $(\xi, y)$ with $\xi$ irrational as in (1), and secondly...
getting metrical results valid for almost all points \((\xi, y)\). Here is an example of the first kind.

**Theorem 1.** Let \(\xi\) be an irrational real number and let \(y\) be a non-zero real number. There exist infinitely many integer quadruples \((p_1, q_1, p_2, q_2)\) satisfying
\[ q_1p_2 - p_1q_2 = 1 \]
and
\[ |q_i\xi + p_i - y| \leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}} \leq \frac{c}{\sqrt{|q_i|}} \quad (i = 1, 2), \]
with \(c = 2\sqrt{3}\max(1, |\xi|)^{1/2}|y|^{1/2}\).

Theorem 1 will be deduced in Section 2 from our results [10] on effective density for \(\operatorname{SL}(2, \mathbb{Z})\)-orbits in \(\mathbb{R}^2\). The estimate (2) is best possible, up to the value of the constant \(c\). However, the optimality of (1) remains unclear. We address the following

**Problem.** Can we replace the function \(\psi(\ell) = c(\log \ell)^2/\ell(\log \log \ell)^2\) occurring in (1) by a smaller one, possibly \(\psi(\ell) = c\ell^{-1}\) ?

We shall further discuss this problem in Section 4 for the function \(\psi(\ell) = 2\ell^{-1}\), offering some hints and indicating the difficulties which then arise. It turns out that the approximating function \(\psi(\ell) = \ell^{-1}\) is permitted for almost all pairs \((\xi, y)\) of real numbers relative to Lebesgue measure. The last assertion follows from the following metrical statement:

**Theorem 2.** Let \(\psi : \mathbb{N} \to \mathbb{R}^+\) be a function. Assume that \(\psi\) is non-increasing, tends to 0 at infinity and that for every positive integer \(c\) there exists a positive real number \(c_1\) satisfying
\[ \psi(c\ell) \geq c_1\psi(\ell), \quad \forall \ell \geq 1. \]
Furthermore assume that
\[ \sum_{\ell \geq 1} \psi(\ell) = +\infty. \]
Then for almost all pairs \((\xi, y)\) of real numbers there exist infinitely many primitive points \((p, q)\) such that
\[ q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q). \]
If \(\sum_{\ell \geq 1} \psi(\ell)\) converges, the pairs \((\xi, y)\) satisfying (4) for infinitely many primitive points \((p, q)\) form a set of zero Lebesgue measure.

Note that we could have equivalently required in (4) that \(q\) be negative. Such a refinement could as well be achieved in the setting of Theorem 1, with a weaker approximating function of the form \(\psi(\ell) = \ell^{-\mu}\) for any given real \(\mu < 1/3\), by employing alternatively Theorem 5 in Section 9 of [10]. We
leave the details of the proof, obtained by arguing as in Section 2, to the
interested reader. For questions of density involving signs, see also [7].

The proof of Theorem 2 is given in Section 3. It combines standard tools
from metrical number theory with the ergodic properties of the linear action
of $\text{SL}(2, \mathbb{Z})$ on $\mathbb{R}^2$ (see [13]). We refer to Harman’s book [8] for closely related
results. See also the recent overview [1] and the monographs [14], [15].

Theorem 2 is a metrical statement about pairs $(\xi, y)$ of real numbers. A
natural question is to understand what happens on each fiber when we fix
either $\xi$ or $y$. In this direction, here is a partial result which will be deduced
from the explicit construction in Section 4.

**THEOREM 3.** Let $\xi$ be an irrational number and let $(p_k/q_k)_{k \geq 0}$ be the
sequence of its convergents. Assume that the series

\begin{equation}
\sum_{k \geq 0} \frac{1}{\max(1, \log q_k)}
\end{equation}

diverges. Then for almost every real number $y$ there exist infinitely many
primitive points $(p,q)$ satisfying

$$|q\xi + p - y| \leq 2/|q|.$$  

Moreover the series (5) diverges for almost every real $\xi$.

We now turn to the second part of the paper devoted to density ex-
ponents for lattice orbits in $\mathbb{R}^2$. As already mentioned, the approximating
function $\psi(\ell) = c\ell^{-1/2}$ occurring in Theorem 1 is directly connected to
the density exponent $1/2$ for $\text{SL}(2, \mathbb{Z})$-orbits. We intend to show that this
exponent $1/2$ is best possible in general.

We work in the more general setting of lattices $\Gamma$ in $\text{SL}(2, \mathbb{R})$. Recall
that a lattice $\Gamma$ in $\text{SL}(2, \mathbb{R})$ is a discrete subgroup for which the quotient
$\Gamma \backslash \text{SL}(2, \mathbb{R})$ has finite Haar measure. We view $\mathbb{R}^2$ as a space of column vectors
on which the group of matrices $\Gamma$ acts by left multiplication. We equip $\mathbb{R}^2$
with the supremum norm $||$, and for any matrix $\gamma \in \Gamma$, we denote also by $|\gamma|$ the maximum of the absolute values of the entries of $\gamma$. Let us first give

**DEFINITION.** Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x}, \mathbf{y})$
the supremum, possibly infinite, of the exponents $\mu$ such that the inequality

\begin{equation}
|\gamma \mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu}
\end{equation}

has infinitely many solutions $\gamma \in \Gamma$.

Note that for a fixed $\mathbf{x} \in \mathbb{R}^2$, the function $\mathbf{y} \mapsto \mu_\Gamma(\mathbf{x}, \mathbf{y})$ is $\Gamma$-invariant.
By the ergodicity of the action of $\Gamma$ on $\mathbb{R}^2$ (see [13]), this function is therefore
constant almost everywhere on $\mathbb{R}^2$. We denote by $\mu_\Gamma(\mathbf{x})$ its generic value, called the *generic density exponent* of the orbit $\Gamma \mathbf{x}$.
Theorem 4. The upper bound $\mu_\Gamma(x) \leq 1/2$ holds true for any point $x \in \mathbb{R}^2$ such that the orbit $\Gamma x$ is dense in $\mathbb{R}^2$.

Equivalently Theorem 4 asserts that $\mu(x, y) \leq 1/2$ for almost all $y \in \mathbb{R}^2$. This bound was already known in the case of $\Gamma = \text{SL}(2, \mathbb{Z})$ as a consequence of Theorem 3 in [10].

One may optimistically conjecture that $\mu_\Gamma(x) = 1/2$ for every $x$ such that $\Gamma x$ is dense in $\mathbb{R}^2$, or at least for almost every $x \in \mathbb{R}^2$. In this direction, it follows from [10] that $\mu_{\text{SL}(2, \mathbb{Z})}(x) \geq 1/3$ for all points $x$ in $\mathbb{R}^2 \setminus \{0\}$ with irrational slope. Weaker lower bounds valid for any lattice $\Gamma \subset \text{SL}(2, \mathbb{R})$ can also be deduced from [12]. Note that the function $x \mapsto \mu_\Gamma(x)$ is $\Gamma$-invariant since $\mu_\Gamma(x)$ obviously depends only on the orbit $\Gamma x$. Thus, the generic density exponent $\mu_\Gamma(x)$ takes the same value for almost all $x \in \mathbb{R}^2$.

2. Proof of Theorem 1. We first state a result obtained in [10]. In this section, we denote by $\Gamma$ the lattice $\text{SL}(2, \mathbb{Z})$. For any point $x = (x_1 x_2)$ in $\mathbb{R}^2$ with irrational slope $x_1/x_2$, the orbit $\Gamma x$ is dense in $\mathbb{R}^2$. We have obtained in [10] effective results concerning the density of such an orbit. In particular, our estimates are essentially optimal when the target point $y$ has rational slope.

Lemma 1. Let $x$ be a point in $\mathbb{R}^2$ with irrational slope and $y = (y_1 y_2)$ a point on the diagonal with $y \neq 0$. Then there exist infinitely many matrices $\gamma \in \Gamma$ such that

$$ |\gamma x - y| \leq c/|\gamma|^{1/2} \quad \text{with} \quad c = 2\sqrt{3}|x|^{1/2}|y|^{1/2}. $$

Proof. The point $y$ has rational slope 1. Apply Theorem 1(ii) of [10] with $a = b = 1$. \[\blacksquare\]

Put $x = (\xi)$. The point $x$ has irrational slope $\xi$ so that Lemma 1 may be applied. Write $\gamma = (q_1 p_1 q_2 p_2)$ for a matrix provided by Lemma 1. Then (7) gives

$$ \max(|q_1 \xi + p_1 - y|, |q_2 \xi + p_2 - y|) \leq \frac{c}{\max(|p_1|, |p_2|, |q_1|, |q_2|)^{1/2}} \leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}}. $$

Therefore, both $(p_1, q_1)$ and $(p_2, q_2)$ satisfy (2), and since the determinant $q_1 p_2 - q_2 p_1$ is 1, these two integer points are primitive. As there exist infinitely many matrices $\gamma$ satisfying (7), we thus find infinitely many solutions to (2).

Assume now that the irrational number $\xi$ has bounded partial quotients. Then Theorem 4 in [10] gives us in the opposite direction a lower bound of
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the form

\[ |\gamma x - y| \geq c'/|\gamma|^{1/2} \]

for every \( \gamma \in \Gamma \), where the positive constant \( c' \) depends only upon \((\xi, y)\). Since we have \( |\gamma| \leq c'' \max(|q_1|, |q_2|) \) when (2) holds, the estimate (2) is optimal up to the value of \( c \).

Remark. The single inequality \( |q_1 \xi + p_1 - y| \leq \psi(|q_1|) \) geometrically means that the point \( \gamma x \) falls inside a neighborhood of the vertical line \( x_1 = y \). A better understanding of the shrinking target problem for the dense orbit \( \Gamma x \), not to a point \( y \) as in [10] but to a line in \( \mathbb{R}^2 \), may possibly lead to a refinement of (1).

3. Proof of Theorem 2. It is convenient to view the pairs \((\xi, y)\) occurring in Theorem 2 as column vectors \((\xi, y)\) in \( \mathbb{R}^2 \). We are concerned with the set \( \mathcal{E}(\psi) \) of vectors \((\xi, y)\) in \( \mathbb{R}^2 \) for which there exist infinitely many primitive integer points \((p, q)\) such that

\[ q \geq 1 \quad \text{and} \quad |q \xi + p - y| \leq \psi(q). \tag{8} \]

For fixed \( p, q \), denote by \( \mathcal{E}_{p,q}(\psi) \) the strip

\[ \mathcal{E}_{p,q}(\psi) := \left\{ \left( \begin{array}{c} \xi \\ y \end{array} \right) \in \mathbb{R}^2; \ |q \xi + p - y| \leq \psi(q) \right\}, \]

and for every positive integer \( q \), let

\[ \mathcal{E}_q(\psi) := \bigcup_{p \in \mathbb{Z} \atop \gcd(p, q) = 1} \mathcal{E}_{p,q}(\psi) \]

be the union of all relevant strips involved in (8) for fixed \( q \). Without loss of generality, we shall assume that \( \psi(q) \leq 1/2 \), so that the above union is disjoint. Then \( \mathcal{E}(\psi) \) is equal to the limsup set

\[ \mathcal{E}(\psi) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \mathcal{E}_q(\psi). \]

As usual when dealing with limsup sets in metrical theory, we first estimate the Lebesgue measure of pairwise intersections of the subsets \( \mathcal{E}_q(\psi) \), \( q \geq 1 \). We next establish a new kind of zero-one law.

3.1. Measuring intersections. In this section, we restrict our attention to points in the unit square \([0, 1]^2\). We denote by \( \varphi \) the Euler totient function and by \( \lambda \) the Lebesgue measure on \( \mathbb{R}^2 \).

Lemma 2. Let \( \psi : \mathbb{N} \to [0, 1/2] \) be a function.

(i) For every positive integer \( q \), we have

\[ \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = 2\varphi(q)\psi(q)/q. \]
(ii) Let $q$ and $s$ be distinct positive integers. Then

$$\lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0,1]^2) \leq 4\psi(q)\psi(s).$$

**Proof.** Denote by $\chi_q$ the characteristic function of $[-\psi(q), \psi(q)]$. Then the characteristic function $\chi_{\mathcal{E}_q(\psi)}$ of the subset $\mathcal{E}_q(\psi) \subset \mathbb{R}^2$ is equal to

$$\chi_{\mathcal{E}_q(\psi)}(\xi, y) = \sum_{p \in \mathbb{Z}} \chi_q(q\xi + p - y) = \sum_{p \in \mathbb{Z}} \chi_q(q\xi - p - y).$$

Observe that if $\left(\frac{\xi}{y}\right) \in [0,1]^2$, the indices $p$ of non-vanishing terms occurring in the last sum satisfy $-1 \leq p \leq q$. Integrating first with respect to $x$, we find

$$\lambda(\mathcal{E}_q(\psi) \cap [0,1]^2) = \int_{0}^{1} \int_{0}^{1} \chi_{\mathcal{E}_q(\psi)}(x, y) \, dx \, dy$$

$$= \sum_{-1 \leq p \leq q, \gcd(p,q)=1} \int_{0}^{1} \int_{0}^{1} \chi_q(qx - p - y) \, dx \, dy$$

$$= \int_{1-\psi(q)}^{1} \frac{-1 + y + \psi(q)}{q} dy + \sum_{1 \leq p \leq q-2, \gcd(p,q)=1} \int_{0}^{1} \frac{2\psi(q)}{q} dy$$

$$+ \int_{0}^{1-\psi(q)} \frac{2\psi(q)}{q} dy + \int_{1-\psi(q)}^{1} \frac{1 - y + \psi(q)}{q} dy$$

$$= \frac{2\varphi(q)\psi(q)}{q}.$$

The first term appearing in the third equality of the above formula corresponds to the summation index $p = -1$ and the last two to $p = q - 1$. We have thus proved (i).

For the second assertion, we majorize

$$\lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0,1]^2) = \int_{0}^{1} \int_{0}^{1} \chi_{\mathcal{E}_q(\psi)}(x, y) \chi_{\mathcal{E}_s(\psi)}(x, y) \, dx \, dy$$

$$\leq \int_{0}^{1} \int_{0}^{1} \left( \sum_{p \in \mathbb{Z}} \chi_q(qx + p - y) \right) \left( \sum_{r \in \mathbb{Z}} \chi_s(sx + r - y) \right) \, dx \, dy$$

$$= \int_{0}^{1} \int_{0}^{1} \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) \, dx \, dy,$$

where $\| \cdot \|$ stands as usual for the distance to the nearest integer. Now, (ii)
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follows from the probabilistic independence formula
\[
\int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) \, dx \, dy = 4\psi(q)\psi(s),
\]
on obtained by Cassels [4, p. 124, Proof (ii)].

### 3.2. A zero-one law.
We say that a subset of \( \mathbb{R}^2 \) is a **null set** if it has Lebesgue measure 0. A set whose complement is a null set is called a full set. The goal of this section is to prove

**Proposition.** Let \( \psi \) be an approximating function as in Theorem 2. Then \( \mathcal{E}(\psi) \) is either a null set or a full set.

To prove the proposition, it is convenient to introduce the larger set
\[
\mathcal{E}'(\psi) = \bigcup_{k \geq 1} \mathcal{E}(k\psi).
\]
In other words, \( \mathcal{E}'(\psi) \) is the set of all points \( (\xi, y) \) in \( \mathbb{R}^2 \) for which there exist a positive real \( \kappa \), depending possibly on \( (\xi, y) \), and infinitely many primitive points \( (p, q) \) satisfying
\[
q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \kappa \psi(q).
\]
Observe that \( \mathcal{E}(k\psi) \subseteq \mathcal{E}(k'\psi) \) if \( 1 \leq k \leq k' \). In particular, \( \mathcal{E}(\psi) \subseteq \mathcal{E}'(\psi) \).

**Lemma 3.** Assume that the approximating function \( \psi : \mathbb{N} \to \mathbb{R}^+ \) tends to zero at infinity. Then \( \mathcal{E}'(\psi) \setminus \mathcal{E}(\psi) \) is a null set.

**Proof.** We show that all sets \( \mathcal{E}(k\psi), \ k \geq 1 \), have the same Lebesgue measure. For every real \( y \), denote by \( \mathcal{E}(\psi, y) \subseteq \mathbb{R} \) the section of \( \mathcal{E}(\psi) \) on the horizontal line \( \mathbb{R} \times \{y\} \), i.e.
\[
\mathcal{E}(\psi, y) = \left\{ \xi \in \mathbb{R}; \left(\frac{\xi}{y}\right) \in \mathcal{E}(\psi) \right\}.
\]
Then, using (8), we can express
\[
\mathcal{E}(\psi, y) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{p \in \mathbb{Z}} \left[ \frac{-p + y - \psi(q)}{q}, \frac{-p + y + \psi(q)}{q} \right]
\]
as a limsup set of intervals. If we restrict ourselves to a bounded part of \( \mathcal{E}(\psi, y) \), the above union over \( p \) reduces to a finite one. Observe that the centers \( (-p + y)/q \) of these intervals do not depend on \( \psi \), and that their length is multiplied by the constant factor \( k \) when replacing \( \psi \) by \( k\psi \). Appealing now to a result due to Cassels [5], we infer that all limsup sets \( \mathcal{E}(k\psi, y), \ k \geq 1 \), have the same Lebesgue measure. See also [8, Corollary of Lemma 2.1, p. 30]. Notice that for fixed \( k \), the length \( 2k\psi(q)/q \) of the
relevant intervals tends to 0 as $q$ tends to infinity, as required by Lemma 2.1. By Fubini, the fibered sets

$$
\mathcal{E}(k\psi) = \prod_{y \in \mathbb{R}} \left( \mathcal{E}(k\psi, y) \times \{y\} \right), \quad k \geq 1,
$$

all have the same Lebesgue measure in $\mathbb{R}^2$ as well.

**Lemma 4.** Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a non-increasing function satisfying (3). Then $\mathcal{E}'(\psi)$ is either a null set or a full set.

**Proof.** The proof is based on the following observation. Let $\left( \begin{array}{c} \xi \\ y \end{array} \right) \in \mathcal{E}'(\psi)$ and let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})$ be such that $c\xi + d > 0$. Then the point $\left( \begin{array}{c} \xi' \\ y' \end{array} \right)$ with coordinates

$$
\xi' = \frac{a\xi + b}{c\xi + d} \quad \text{and} \quad y' = \frac{y}{c\xi + d}
$$

belongs to $\mathcal{E}'(\psi)$. Indeed, substituting

$$
q = aq' + cp', \quad p = bq' + dp'
$$

in (9) and dividing by $c\xi + d$, we obtain

$$
q' \geq 1 \quad \text{and} \quad |q'\xi' + p' - y'| \leq \frac{\kappa}{c\xi + d} \psi(q) \leq \kappa' \psi(q'),
$$

for some $\kappa' > 0$ independent of $q'$. The positivity of $q'$ is proved as follows. Note that (9) implies the estimate

$$
p = -q\xi + O_{\xi,y}(1).
$$

Then, inverting the linear substitution (10), we find

$$
q' = dq - cp = q(c\xi + d) + O_{\gamma,\xi,y}(1).
$$

Since we have assumed that $c\xi + d > 0$, the term $q(c\xi + d)$ is arbitrarily large when $q$ is large enough. The condition (3) now shows that $\psi(q) \asymp \psi(q')$. Thus (11) is satisfied for infinitely many primitive points $(p', q')$, since the linear substitution (10) is unimodular. We have shown that $\left( \begin{array}{c} \xi' \\ y' \end{array} \right) \in \mathcal{E}'(\psi)$.

We now prove that $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ is either a full subset or a null subset of the half-plane $\mathbb{R} \times \mathbb{R}^+$. To that end, we consider the map

$$
\Phi : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+ \quad \text{defined by} \quad \Phi \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x/y \\ 1/y \end{array} \right).
$$

Clearly $\Phi$ is a continuous involution of $\mathbb{R} \times \mathbb{R}^+$. The image

$$
\Omega := \Phi(\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+))
$$

is formed by all points of the type

$$
\left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} \xi/y \\ 1/y \end{array} \right),
$$

for some $\xi, y \in \mathbb{R}$. Since $\mathcal{E}'(\psi)$ is a measure-zero subset of $\mathbb{R} \times \mathbb{R}^+$, it follows that $\Omega$ is also a measure-zero subset of $\mathbb{R} \times \mathbb{R}^+$. Therefore, $\Omega$ is either a full subset or a null subset of the half-plane $\mathbb{R} \times \mathbb{R}^+$. This completes the proof of Lemma 4.
where \((\xi, y)\) ranges over \(E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\). Now, the above condition \(c\xi + d > 0\) is obviously equivalent to \(cu + dv > 0\) since \(y\) is positive. Then the point
\[
\Phi\left( \frac{au + bv}{cu + dv} \right) = \left( \frac{a\xi + b}{c\xi + d} \right) \left( \frac{y}{c\xi + d} \right)
\]
belongs to \(E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\), by the preceding observation. Applying the involution \(\Phi\), we find that
\[
\Phi\left( \left( \frac{a\xi + b}{c\xi + d} \right) \left( \frac{u}{c\xi + d} \right) \right) = \left( \frac{au + bv}{cu + dv} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{c} u \\ v \end{array} \right)
\]
belongs to \(\Omega\). In other words, setting \(\Gamma = \text{SL}(2, \mathbb{Z})\), we have established the inclusion
\[
(\Gamma \Omega) \cap (\mathbb{R} \times \mathbb{R}^+) \subseteq \Omega.
\]
Since the reverse inclusion is obvious, we have \(\Omega = (\Gamma \Omega) \cap (\mathbb{R} \times \mathbb{R}^+)\).

Assuming that \(\Omega\) is not a null set, the ergodicity of the linear action of \(\Gamma\) on \(\mathbb{R}^2\) shows that \(\Gamma \Omega\) is a full set in \(\mathbb{R}^2\). Hence \(\Omega\) is a full set in the half-plane \(\mathbb{R} \times \mathbb{R}^+\). Transforming now \(\Omega\) by \(\Phi\), we find that
\[
\Phi(\Omega) = E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)
\]
is also a full set in \(\mathbb{R} \times \mathbb{R}^+\), thus proving the claim.

We finally use another transformation to carry the zero-one law from the positive half-plane \(\mathbb{R} \times \mathbb{R}^+\) to the negative one \(\mathbb{R} \times \mathbb{R}^-\). Writing (9) in the equivalent form
\[
q \geq 1 \quad \text{and} \quad |q(-\xi) + (-p) - (-y)| \leq \kappa \psi(q)
\]
shows that \(E'(\psi)\) is invariant under the symmetry \((\xi, y) \mapsto (-\xi, -y)\) which maps \(\mathbb{R} \times \mathbb{R}^+\) onto \(\mathbb{R} \times \mathbb{R}^-\). Therefore \(E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^-)\) is a null set or a full set in \(\mathbb{R} \times \mathbb{R}^-\) whenever \(E'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\) is, respectively, a null set or a full set in \(\mathbb{R} \times \mathbb{R}^+\).

Now, the combination of Lemmas 3 and 4 obviously yields our proposition.

**3.3. Concluding the proof of Theorem 2.** Assume first that \(\sum \psi(\ell)\) converges. We have to show that the set
\[
\mathcal{E}(\psi) = \limsup_{q \to \infty} \mathcal{E}_q(\psi)
\]
has zero Lebesgue measure. Lemma 2 shows that the partial sums
\[
\sum_{q=1}^Q \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = 2 \sum_{q=1}^Q \frac{\varphi(q) \psi(q)}{q} \leq 2 \sum_{q=1}^Q \psi(q)
\]
has zero Lebesgue measure.
converge \(^{(1)}\). Then the Borel–Cantelli Lemma ensures that the limsup set \(E(ψ) \cap [0, 1]^2\) is a null set. Thus \(E(ψ)\) cannot be a full set. Now, the above proposition tells us that \(E(ψ)\) is a null set.

We now consider the case of a divergent series \(∑ ψ(ℓ)\). Observe that

\[
\frac{1}{2} \sum_{q=1}^{Q} ψ(q) \leq \sum_{q=1}^{Q} \frac{φ(q)ψ(q)}{q} \leq \sum_{q=1}^{Q} ψ(q)
\]

for any large integer \(Q\), since the sequence \((ψ(ℓ))_{ℓ≥1}\) is non-increasing. The right inequality is obvious, while the left one easily follows by Abel summation. See for instance Chapter 2 of [8], where full details are provided. By Lemma 2 and (12), the sums

\[
\sum_{q=1}^{Q} λ(E_q(ψ) \cap [0, 1]^2) = 2 \sum_{q=1}^{Q} \frac{φ(q)ψ(q)}{q} \geq \sum_{q=1}^{Q} ψ(q)
\]

are then unbounded. Then, using a classical converse to the Borel–Cantelli Lemma, we have the lower bound

\[
λ(E(ψ) \cap [0, 1]^2) = λ\left(\limsup_{q→∞}(E_q(ψ) \cap [0, 1]^2)\right) \geq \limsup_{Q→∞} \frac{(\sum_{q=1}^{Q} λ(E_q(ψ) \cap [0, 1]^2))^2}{\sum_{q=1}^{Q} \sum_{s=1}^{Q} λ(E_q(ψ) \cap E_s(ψ) \cap [0, 1]^2)}.
\]

See for instance Lemma 2.3 in [8]. Lemma 2 and (12) now show that the numerator on the right hand side of (13) equals

\[
4 \left(\sum_{q=1}^{Q} \frac{φ(q)ψ(q)}{q}\right)^2 \geq \left(\sum_{q=1}^{Q} ψ(q)\right)^2
\]

when \(Q\) is large, while the denominator is bounded from above by

\[
4 \sum_{q=1, s=1}^{Q} ψ(q)ψ(s) + 2 \sum_{q=1}^{Q} ψ(q) \leq 4 \left(\sum_{q=1}^{Q} ψ(q)\right)^2 + 2 \sum_{q=1}^{Q} ψ(q).
\]

Thus (13) yields the lower bound

\[
λ(E(ψ) \cap [0, 1]^2) \geq 1/4.
\]

Hence \(E(ψ)\) is not a null set; it is thus a full set according to our proposition.

4. An approach to our problem. In this section, we apply a transference principle between homogeneous and inhomogeneous approximation,

\(^{(1)}\) Here again we assume without loss of generality that \(ψ(q) \leq 1/2\) for every \(q ≥ 1\), so that Lemma 2 may be applied.
as displayed in Chapter V of [4] and in [3], for constructing explicit integer solutions of the inequality
\[(14) \qquad |q\xi + p - y| \leq 2/|q|.
\]

Let \((p_k/q_k)_{k\geq 0}\) be the sequence of convergents to the irrational number \(\xi\). The theory of continued fractions (see for instance the monograph [9]) tells us that
\[(15) \qquad |q_k\xi - p_k| \leq 1/q_{k+1} \quad \text{and} \quad p_kq_{k+1} - p_{k+1}q_k = (-1)^{k+1}
\]
for any \(k \geq 0\). Setting \(\nu_k = (-1)^{k+1}q_ky\), we thus have the relations
\[(16) \qquad \nu_kq_{k+1} + \nu_{k+1}q_k = 0 \quad \text{and} \quad \nu_k(q_{k+1}\xi - p_{k+1}) + \nu_{k+1}(q_k\xi - p_k) = y.
\]
Now, let \(n_k\) be either \(\lfloor \nu_k \rfloor\) or \(\lceil \nu_k \rceil\). Then
\[(17) \qquad |\nu_k - n_k| < 1,
\]
and \(n_k\) is either equal to \((-1)^{k+1}\lfloor yq_k \rfloor\) or to \((-1)^{k+1}\lceil yq_k \rceil\). Setting
\[(18) \qquad p = -n_kp_{k+1} - n_{k+1}p_k \quad \text{and} \quad q = n_kq_{k+1} + n_{k+1}q_k,
\]
we deduce from (16) the expressions
\[(19) \qquad q\xi + p - y = n_k(q_{k+1}\xi - p_{k+1}) + n_{k+1}(q_k\xi - p_k) - y
\]
\[= (n_k - \nu_k)(q_{k+1}\xi - p_{k+1}) + (n_{k+1} - \nu_{k+1})(q_k\xi - p_k)
\]
and
\[(20) \qquad q = (n_k - \nu_k)q_{k+1} + (n_{k+1} - \nu_{k+1})q_k.
\]
Recall that \(q_k\xi - p_k\) and \(q_{k+1}\xi - p_{k+1}\) have opposite signs. Assuming that \(n_k - \nu_k\) and \(n_{k+1} - \nu_{k+1}\) have the same sign, we infer from (19), (20) and (15), (17) that
\[(21) \qquad |q\xi + p - y| < 1/q_{k+1} \quad \text{and} \quad |q| < 2q_{k+1}.
\]
Otherwise, we have
\[(22) \qquad |q\xi + p - y| < 2/q_{k+1} \quad \text{and} \quad |q| < q_{k+1}.
\]
The inequalities (21) and (22) obviously imply (14).

Since the linear substitution (18) is unimodular, the integers \(p\) and \(q\) are coprime if and only if \(n_k\) and \(n_{k+1}\) are coprime. Recall that the two choices \(n_k = \lfloor \nu_k \rfloor\) and \(n_k = \lceil \nu_k \rceil\) are admissible, both for \(n_k\) and \(n_{k+1}\). It thus remains to find indices \(k\) for which at least one of the coprimality conditions
\[(23) \qquad \gcd(\lfloor yq_k \rfloor, \lfloor yq_{k+1} \rfloor) = 1, \quad \gcd(\lceil yq_k \rceil, \lceil yq_{k+1} \rceil) = 1,
\]
\[\gcd(\lfloor yq_k \rfloor, \lceil yq_{k+1} \rceil) = 1, \quad \gcd(\lceil yq_k \rceil, \lfloor yq_{k+1} \rfloor) = 1.
\]

\((2)\) As usual \(\lfloor x \rfloor\) and \(\lceil x \rceil\) stand respectively for the floor and the ceiling of the real number \(x\). Then \(\lfloor x \rfloor = \lceil x \rceil - 1\), unless \(x\) is an integer in which case \(\lfloor x \rfloor = \lceil x \rceil = x\).
is satisfied. Note that obviously there is no such \( k \geq 0 \) when \( y \) is an integer not equal to 1 or to \(-1\). Otherwise, the existence of infinitely many indices \( k \) satisfying (23) is a non-trivial problem that we leave hanging.

Let us mention that the proof of (1) in [6] follows the same idea, finding a primitive integer point inside the square centered at the point \((\nu_k, \nu_{k+1})\) \( \in \mathbb{R}^2 \) with side \( C \log |\nu_k|/\log \log |\nu_k| \) for some suitable large absolute constant \( C \).

4.1. Proof of Theorem 3. We quote the following metrical result due to Harman (Theorem 8.3 in [8]). Assume that the series (5) diverges. Then for almost all positive real numbers \( y \), there exist infinitely many indices \( k \) such that the integer part \([yq_k]\) is a prime number. These indices \( k \) satisfy (23) since, assuming for simplicity that \( y \) is irrational, either \([yq_{k+1}]\) or \([yq_{k+1}] + 1\) is not divisible by \([yq_k]\) and is thus relatively prime to \([yq_k]\). Hence (14) has infinitely many coprime solutions \((p, q)\) for almost every positive real number \( y \). Writing now (14) in the equivalent form

\[ |(-q)\xi + (-p) - (-y)| \leq 2/|q| \]

shows that, given \( \xi \), the set of all real numbers \( y \) for which (14) has infinitely many coprime solutions is invariant under the symmetry \( y \mapsto -y \). The first assertion is thus established. To complete the proof, note that

\[
\lim_{k \to \infty} \frac{\log q_k}{k} = \frac{\pi^2}{12 \log 2}
\]

for almost every \( \xi \) by the Khintchine–Lévy Theorem (see equation (4.18) in [2]). Thus the series (5) diverges for almost every \( \xi \).

5. Generic density exponents. In this section we prove Theorem 4, as a consequence of the Borel–Cantelli Lemma combined with the following counting result.

**Lemma 5.** Let \( x \) be a point in \( \mathbb{R}^2 \) whose orbit \( \Gamma x \) is dense in \( \mathbb{R}^2 \). For every symmetric compact set \( \Omega \) in \( \mathbb{R}^2 \setminus \{0\} \) there exists \( c > 0 \) such that

\[
\text{Card}\{\gamma \in \Gamma; \gamma x \in \Omega, |\gamma| \leq T\} \leq cT
\]

for any real \( T \geq 1 \).

**Proof.** Ledrappier [11] has shown that the limit formula

\[
\lim_{T \to \infty} \frac{1}{T} \sum_{\gamma \in \Gamma, |\gamma| \leq T} f(\gamma x) = \frac{4}{|x| \text{vol}(\Gamma \setminus \text{SL}(2, \mathbb{R}))} \int_{\mathbb{R}^2 \setminus \{0\}} \frac{f(y)}{|y|} dy
\]

holds for any even continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) having compact support on \( \mathbb{R}^2 \setminus \{0\} \), with a suitable normalisation of Haar measure on \( \text{SL}(2, \mathbb{R}) \). Approximating uniformly from above and from below the characteristic func-
tion of $\Omega$ by even continuous functions, we deduce that
\[
\lim_{T \to \infty} \frac{\text{Card}\{\gamma \in \Gamma; \gamma \mathbf{x} \in \Omega, |\gamma| \leq T\}}{T} = \frac{4}{|\mathbf{x}| \text{vol}(\Gamma \setminus \text{SL}(2, \mathbb{R}))} \int_{\Omega} \frac{dy}{|y|}.
\]
Lemma 5 immediately follows. ■

For any $\mathbf{y} \in \mathbb{R}^2$ and any positive real number $r$, we denote by
\[
B(\mathbf{y}, r) = \{\mathbf{z} \in \mathbb{R}^2; |\mathbf{z} - \mathbf{y}| \leq r\}
\]
the closed disc centered at $\mathbf{y}$ with radius $r$.

**Lemma 6.** Let $\mathbf{x}$ be a point in $\mathbb{R}^2$ whose orbit $\Gamma \mathbf{x}$ is dense in $\mathbb{R}^2$, $\Omega$ a symmetric compact set in $\mathbb{R}^2 \setminus \{0\}$ and $\mu$ a real number $> 1/2$. For every integer $n \geq 1$, put
\[
B_n = \bigcup_{\gamma \in \Gamma, |\gamma| = n, \gamma \mathbf{x} \in \Omega} B(\gamma \mathbf{x}, n^{-\mu}).
\]
Then
\[
\mathcal{B} := \limsup_{n \to \infty} B_n = \bigcap_{N \geq 1} \bigcup_{n \geq N} B_n = \bigcap_{N \geq 1} \bigcup_{\gamma \in \Gamma, |\gamma| \geq N, \gamma \mathbf{x} \in \Omega} B(\gamma \mathbf{x}, |\gamma|^{-\mu})
\]
is a null set.

**Proof.** We apply the Borel–Cantelli Lemma to prove that the series
\[
\sum_{n \geq 1} \lambda(B_n)
\]
converges if $\mu > 1/2$.

For every positive integer $n$, set
\[
M_n = \text{Card}\{\gamma \in \Gamma; \gamma \mathbf{x} \in \Omega, |\gamma| = n\}.
\]
Lemma 5 gives us the upper bound
\[
M_1 + \cdots + M_n = \text{Card}\{\gamma \in \Gamma; \gamma \mathbf{x} \in \Omega, |\gamma| \leq n\} \leq cn
\]
for some $c > 0$ independent of $n \geq 1$. Since a ball of radius $r$ has Lebesgue measure $4r^2$, we trivially bound from above
\[
\lambda(B_n) \leq \sum_{|\gamma| = n, \gamma \mathbf{x} \in \Omega} 4n^{-2\mu} = 4M_n n^{-2\mu}.
\]
Summing by parts, we deduce from (24) that
\[
\sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} = \sum_{n=1}^{N-1} (M_1 + \cdots + M_n) \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{M_1 + \cdots + M_N}{N^{2\mu}}
\]
\[
\leq c \sum_{n=1}^{N-1} n \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{cN}{N^{2\mu}} = c \sum_{n=1}^{N} \frac{1}{n^{2\mu}}.
\]
The partial sums

\[ \sum_{n=1}^{N} \lambda(B_n) \leq 4 \sum_{n=1}^{N} \frac{M_n}{n^{2\mu}} \leq 4c \sum_{n=1}^{N} \frac{1}{n^{2\mu}} \]

thus converge if \( \mu > 1/2 \).

5.1. Proof of Theorem 4. Suppose on the contrary that \( \mu_\Gamma(x) > 1/2 \). Fix a real \( \mu \) with \( 1/2 < \mu < \mu_\Gamma(x) \). Then for almost all \( y \in \mathbb{R}^2 \), we have \( \mu(x, y) > \mu \). This means that there exist infinitely many \( \gamma \in \Gamma \) satisfying (6), or equivalently that \( y \) belongs to infinitely many balls of the form \( B(\gamma x, |\gamma|^{-\mu}) \). We now restrict our attention to points \( y \) with \( \mu(x, y) > \mu \) lying in an annulus

\[ \Omega' = \{ z \in \mathbb{R}^2; a' \leq |z| \leq b' \} \]

where \( b' > a' > 0 \) are arbitrarily fixed. Since \( y \in \Omega' \cap B(\gamma x, |\gamma|^{-\mu}) \), the triangle inequality yields

\[ a' - |\gamma|^{-\mu} \leq |\gamma x| \leq b' + |\gamma|^{-\mu} \]

If \( a < a' \) and \( b > b' \), then the center \( \gamma x \) lies in the larger annulus \( \Omega = \{ z \in \mathbb{R}^2; a \leq |z| \leq b \} \), provided that \( |\gamma| \) is large enough. It follows that \( y \) falls inside the union of balls

\[ \bigcup_{|\gamma| \geq N, \gamma x \in \Omega} B(\gamma x, |\gamma|^{-\mu}) \]

considered in Lemma 6 for every integer \( N \) large enough, and thus \( y \in B \). However, Lemma 6 asserts that \( B \) is a null set, which is a contradiction.

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References


Inhomogeneous approximation with coprime integers


Michel Laurent, Arnaldo Nogueira
Institut de Mathématiques de Luminy
Case 907
163 avenue de Luminy
13288 Marseille Cédex 9, France
E-mail: michel-julien.laurent@univ-amu.fr
arnaldo.nogueira@univ-amu.fr

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