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Sums of twelve squares

by

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1. Introduction. Let $\mathbb{N}$ denote the set of all positive integers, $\mathbb{Z}$ the set of all integers, and $\mathbb{Q}$ the set of all rational numbers. For $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ we let $r_k(n)$ denote the number of representations of n as the sum of k squares, that is,

$$r_k(n) := \sum_{\substack{(x_1,\ldots,x_k)\in\mathbb{Z}^k \\ x_1^2+\ldots+x_k^2=n}} 1.$$

In the first decade of the twentieth century Glaisher obtained formulae for $r_k(n)$ for $k = 2, 4, 6, 8, 10, 12, 14, 16$ and 18 in a systematic manner (see [2]). All of Glaisher's results were obtained from formulae derived from the theory of elliptic functions and so cannot be considered elementary. Nathanson's book [7] contains elementary proofs of formulae for $r_k(n)$ for $k = 2, 4, 6, 8, 10$.

In this paper we consider the case $k = 12$. In 1864 Liouville [5] stated the formula

$$(1.1)\qquad r_{12}(n) = \frac{24}{31}\,(21 + 2^{5\alpha+1}5)\sigma_5(n/2^\alpha) \quad \text{if } n \equiv 0 \pmod 2,$$

where $\alpha \in \mathbb{N}$ is such that $2^\alpha \parallel n$, and, for $k \in \mathbb{N}$ and $x \in \mathbb{Q}$,

$$\sigma_k(x) := \begin{cases} \sum_{d\mid x} d^k & \text{if } x \in \mathbb{N}, \\ 0 & \text{if } x \in \mathbb{Q},\ x \notin \mathbb{N}. \end{cases}$$

Petr [8] proved Liouville's formula (1.1) in 1905 using theta functions and Humbert [4] proved it in 1907 using elliptic functions. In 1907 Glaisher in his paper [2, pp. 480–481] gave the formulae

$$(1.2)\qquad r_{12}(n) = -8\sum_{d\mid n}(-1)^{d+(n/d)}d^5 \quad \text{if } n \equiv 0 \pmod 2,$$

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$$(1.3)\qquad r_{12}(n) = 8\sigma_5(n) + 2 \sum_{\substack{a,b,c,d\in\mathbb{Z}\\ a^2+b^2+c^2+d^2=n}} F(a,b,c,d) \quad \text{if } n \equiv 1 \pmod 2,$$

where

$$F(a,b,c,d) := (a^4+b^4+c^4+d^4) - 2(a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2),$$

and he pointed out that his formula (1.2) is equivalent to Liouville's formula (1.1). In 1987 Ewell [1, p. 298] gave the formulae

$$(1.4)\qquad r_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) \quad \text{if } n \equiv 0 \pmod 2,$$

$$(1.5)\qquad r_{12}(n) = 8\sigma_5(n) + 16\sigma(n) + 256 \sum_{d<n/2} (-1)^d d^3 \sum_{k<n/2d} \sigma(n-2kd) \quad \text{if } n \equiv 1 \pmod 2,$$

where $\sigma(x) := \sigma_1(x)$ for all $x \in \mathbb{Q}$. An easy calculation shows that (1.4) is equivalent to Liouville's formula (1.1) and thus to Glaisher's formula (1.2). Although Ewell did not indicate how he proved his formulae (1.4) and (1.5), he presumably used infinite product identities as in his proof of his formula for $r_{16}(n)$.

In Section 2 of this paper we prove in an elementary manner two new convolution identities (see Theorem 2). In Section 3 we give the first elementary proof of Liouville's formula (1.1) in the form (1.4). In Section 4, we prove in an elementary fashion a new formula for $r_{12}(n)$ when n is odd which is simpler than both (1.3) and (1.5). We prove

THEOREM 1. *Let n be a positive odd integer. Then*

$$(1.6)\qquad r_{12}(n) = 16\sigma_3(n) + 8\sigma(n) + 128 \sum_{ax+by=n} (-1)^{a+b+x} ab^3,$$

where a, b, x, y run through all ordered quadruples of positive integers such that $ax+by=n$.

This formula is similar to one for $r_{16}(n)$, which can be deduced from Milne [6, Theorem 1.4]. Finally in Section 5 we deduce Ewell's formula (1.5) from Theorems 1 and 2. Our main tool is the following recent identity due to Huard, Ou, Spearman and Williams [3, Theorem 1], whose proof involves nothing more than the manipulation of finite sums.

PROPOSITION. *Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$(1.7)\qquad f(a,b,x,y) - f(x,y,a,b) = f(-a,-b,x,y) - f(x,y,-a,-b)$$

for all integers a, b, x and y. Then

$$\sum_{ax+by=n}(f(a,b,x,-y)-f(a,-b,x,y)+f(a,a-b,x+y,y) \tag{1.8}$$
$$-f(a,a+b,y-x,y)+f(b-a,b,x,x+y)-f(a+b,b,x,x-y))$$
$$=\sum_{d\mid n}\sum_{x<d}(f(0,n/d,x,d)+f(n/d,0,d,x)+f(n/d,n/d,d-x,-x)$$
$$-f(x,x-d,n/d,n/d)-f(x,d,0,n/d)-f(d,x,n/d,0)),$$

where the sum on the left hand side of (1.8) *is over all ordered quadruples of positive integers a, b, x, y satisfying $ax+by=n$, the inner sum on the right hand side is over all positive integers x satisfying $x<d$, and the outer sum on the right hand side is over all positive integers d dividing n.*

We also make use of the classical formulae for $r_4(n)$ and $r_8(n)$, which can be deduced in an elementary way from the Proposition (see [3], [9] and [10]):

$$r_4(n)=8\sigma(n)-32\sigma(n/4), \tag{1.9}$$
$$r_8(n)=16(-1)^{n-1}(\sigma_3(n)-16\sigma_3(n/2)). \tag{1.10}$$

Clearly

$$r_{12}(n)=r_4(n)+r_8(n)+\sum_{k=1}^{n-1}r_4(n-k)r_8(k). \tag{1.11}$$

We note for later use the following elementary identity:

$$\sigma_e(2n)-(2^e+1)\sigma_e(n)+2^e\sigma_e(n/2)=0, \quad e,n\in\mathbb{N}. \tag{1.12}$$

2. Preliminary results. For $e,f,n\in\mathbb{N}$ we define

$$S_{e,f}(n):=\sum_{m=1}^{n-1}\sigma_e(m)\sigma_f(n-m). \tag{2.1}$$

Clearly

$$S_{e,f}(n)=\sum_{ax+by=n}a^eb^f=S_{f,e}(n). \tag{2.2}$$

The sums $S_{e,f}(n)$ can be evaluated in an elementary manner for $e,f\in\mathbb{N}$ satisfying

$$e\equiv f\equiv 1 \pmod 2, \quad e+f=2,4,6,8,12, \tag{2.3}$$

by taking particular choices of $f(a,b,x,y)$ in the Proposition (see [3]). We just need the value of $S_{1,3}(n)$, namely,

$$S_{1,3}(n)=\frac{1}{240}\,(21\sigma_5(n)+(10-30n)\sigma_3(n)-\sigma(n)). \tag{2.4}$$

Secondly, for $e, f, n \in \mathbb{N}$, we define

$$(2.5)\qquad A_{e,f}(n) := \sum_{m<n/2} \sigma_e(m)\sigma_f(n-2m) = \sum_{2ax+by=n} a^e b^f,$$

where m runs through the positive integers satisfying $m < n/2$. We note that

$$(2.6)\qquad A_{e,f}(n) = \sum_{a<n/2} a^e \sum_{m<n/2a} \sigma_f(n-2am).$$

The values of $A_{1,1}(n)$, $A_{1,3}(n)$ and $A_{3,1}(n)$ were derived in an elementary manner in [3] from the Proposition. We just need

$$(2.7)\qquad A_{1,3}(n) = \frac{1}{240}\,(5\sigma_5(n) + (10-15n)\sigma_3(n) + 16\sigma_5(n/2) - \sigma(n/2)),$$

$$(2.8)\qquad A_{3,1}(n) = \frac{1}{240}\,(\sigma_5(n) - \sigma(n) + 20\sigma_5(n/2) + (10-30n)\sigma_3(n/2)).$$

Thirdly, for $e, f, n \in \mathbb{N}$, we define

$$(2.9)\qquad B_{e,f}(n) := \sum_{m<n/4} \sigma_e(m)\sigma_f(n-4m) = \sum_{4ax+by=n} a^e b^f,$$

where m runs through the positive integers satisfying $m < n/4$. We note that

$$(2.10)\qquad B_{e,f}(n) = \sum_{a<n/4} a^e \sum_{m<n/4a} \sigma_f(n-4am).$$

The value of $B_{1,1}(n)$ was derived in [3] from the Proposition. We need the following new evaluations of $B_{1,3}(n)$ and $B_{3,1}(n)$ when n is odd.

Theorem 2. *Let n be a positive odd integer. Then*

$$384B_{1,3}(n) = 5\sigma_5(n) + (10-12n)\sigma_3(n) - 3\sigma(n) - 48\sum_{ax+by=n} (-1)^{a+b+x} ab^3,$$

$$7680B_{3,1}(n) = -13\sigma_5(n) + 30\sigma_3(n) - 17\sigma(n) + 240\sum_{ax+by=n} (-1)^{a+b+x} ab^3.$$

Proof. We first use the Proposition to prove that

$$(2.11)\qquad B_{1,3}(n) + 4B_{3,1}(n) = \frac{1}{480}\,(3\sigma_5(n) + (20-15n)\sigma_3(n) - 8\sigma(n)).$$

For $m \in \mathbb{Z}$ we set

$$F_2(m) = \begin{cases} 1 & \text{if } 2 \mid m, \\ 0 & \text{if } 2 \nmid m. \end{cases}$$

We choose

$$f(a,b,x,y) = ab^3 F_2(a) F_2(x).$$

It is easy to see that $f(a,b,x,y)$ satisfies condition (1.7) of the Proposition. With this choice we now examine the various terms occurring in the identity (1.8).

The first two terms on the left hand side give

$$2\sum_{ax+by=n} ab^3 F_2(a)F_2(x) = 4\sum_{4ax+by=n} ab^3 = 4B_{1,3}(n).$$

The third and fourth terms on the left hand side give (since $F_2(x+y) = F_2(y-x) = F_2(x-y)$ and n is odd)

$$\begin{aligned}
&\sum_{ax+by=n} (a(a-b)^3 - a(a+b)^3)F_2(a)F_2(x-y) \\
&\qquad = \sum_{\substack{2ax+by=n \\ x\equiv y \pmod 2}} (-48a^3b - 4ab^3) = \sum_{\substack{2ax+by=n \\ x\equiv 1 \pmod 2}} (-48a^3b - 4ab^3) \\
&\qquad = \sum_{2ax+by=n} (-48a^3b - 4ab^3) - \sum_{\substack{2ax+by=n \\ 2|x}} (-48a^3b - 4ab^3) \\
&\qquad = -48A_{3,1}(n) - 4A_{1,3}(n) + \sum_{4ax+by=n} (48a^3b + 4ab^3) \\
&\qquad = 4B_{1,3}(n) + 48B_{3,1}(n) - 4A_{1,3}(n) - 48A_{3,1}(n).
\end{aligned}$$

The fifth and sixth terms on the left hand side give (as $F_2(a+b) = F_2(b-a) = F_2(a-b)$ and n is odd)

$$\begin{aligned}
&\sum_{ax+by=n} ((b-a)b^3 - (a+b)b^3)F_2(a-b)F_2(x) \\
&\qquad = -2\sum_{\substack{2ax+by=n \\ a\equiv b \pmod 2}} ab^3 = -2\sum_{\substack{2ax+by=n \\ a\equiv 1 \pmod 2}} ab^3 \\
&\qquad = -2\Big(\sum_{2ax+by=n} ab^3 - \sum_{\substack{2ax+by=n \\ 2|a}} ab^3\Big) \\
&\qquad = -2\Big(A_{1,3}(n) - 2\sum_{4ax+by=n} ab^3\Big) = -2A_{1,3}(n) + 4B_{1,3}(n).
\end{aligned}$$

Thus the left hand side is equal to

$$12B_{1,3}(n) + 48B_{3,1}(n) - 6A_{1,3}(n) - 48A_{3,1}(n).$$

As n is odd, all but the fifth term on the right hand side of (1.8) vanish.

The fifth term is

$$\sum_{d\mid n}\sum_{x<d} xd^3F_2(x) = 2\sum_{d\mid n} d^3 \sum_{1\le x\le (d-1)/2} x = \frac{1}{4}\sum_{d\mid n} d^3(d^2-1)$$
$$= \frac{1}{4}\left(\sigma_5(n) - \sigma_3(n)\right).$$

Thus, by the Proposition, we obtain

$$12B_{1,3}(n) + 48B_{3,1}(n) - 6A_{1,3}(n) - 48A_{3,1}(n) = -\frac{1}{4}\sigma_5(n) + \frac{1}{4}\sigma_3(n).$$

Appealing to (2.7) and (2.8) for the values of $A_{1,3}(n)$ and $A_{3,1}(n)$ respectively, we obtain (2.11).

Next we show that

$$\text{(2.12)}\quad \sum_{ax+by=n} (-1)^{a+b+x}ab^3 = -S_{1,3}(n) + 6A_{1,3}(n) + 16A_{3,1}(n) - 8B_{1,3}(n),$$

by using the identity

$$\begin{aligned}(-1)^{a+b+x} &= (1+(-1)^a)(1+(-1)^b)(1+(-1)^x) - (1+(-1)^a)(1+(-1)^b)\\ &\quad - (1+(-1)^a)(1+(-1)^x) - (1+(-1)^b)(1+(-1)^x)\\ &\quad + (1+(-1)^a) + (1+(-1)^b) + (1+(-1)^x) - 1\end{aligned}$$

on the left hand side of (2.12). We obtain eight sums and evaluate two of them. The rest can be evaluated in a similar fashion. As n is odd, we have

$$\sum_{ax+by=n} (1+(-1)^a)(1+(-1)^b)(1+(-1)^x)ab^3 = 8 \sum_{\substack{ax+by=n\\ 2\mid a,\, 2\mid b,\, 2\mid x}} ab^3 = 0$$

and

$$\sum_{ax+by=n} (1+(-1)^a)ab^3 = 2\sum_{\substack{ax+by=n\\ 2\mid a}} ab^3 = 4\sum_{2ax+by=n} ab^3 = 4A_{1,3}(n).$$

From (2.4), (2.7), (2.8) and (2.12), we deduce the first formula of Theorem 2. The second formula of Theorem 2 then follows from (2.11).

3. Elementary proof of Liouville's formula for $r_{12}(n)$ when n is even. Let n be a positive even integer. Set $n = 2N$, where $N \in \mathbb{N}$. By (1.9)–(1.12), we have

$$\text{(3.1)}\quad r_4(2N) = 8\sigma(2N) - 32\sigma(N/2) = 24\sigma(N) - 48\sigma(N/2),$$

$$\text{(3.2)}\quad r_8(2N) = -16\sigma_3(2N) + 256\sigma_3(N) = 112\sigma_3(N) + 128\sigma_3(N/2),$$

$$\text{(3.3)}\quad r_{12}(2N) = 112\sigma_3(N) + 128\sigma_3(N/2) + 24\sigma(N) - 48\sigma(N/2) + T_0 + T_1,$$

where

$$T_i := \sum_{\substack{k=1 \\ k\equiv i \pmod 2}}^{2N-1} r_4(2N-k)r_8(k), \qquad i=0,1. \tag{3.4}$$

We first evaluate T_0. Appealing to (3.1), (3.2) and (3.4), we obtain

$$\begin{aligned} T_0 &= \sum_{\substack{k=1 \\ 2\mid k}}^{2N-1} r_4(2N-k)r_8(k) = \sum_{k=1}^{N-1} r_4(2N-2k)r_8(2k) \\ &= \sum_{k=1}^{N-1}(24\sigma(N-k)-48\sigma((N-k)/2))(112\sigma_3(k)+128\sigma_3(k/2)), \end{aligned}$$

that is,

$$T_0 = 2688U_1 - 5376U_2 + 3072U_3 - 6144U_4, \tag{3.5}$$

where

$$\begin{aligned} U_1 &:= \sum_{k=1}^{N-1} \sigma(N-k)\sigma_3(k) = S_{1,3}(N), \\ U_2 &:= \sum_{k=1}^{N-1} \sigma((N-k)/2)\sigma_3(k) = \sum_{k<N/2} \sigma(k)\sigma_3(N-2k) = A_{1,3}(N), \\ U_3 &:= \sum_{k=1}^{N-1} \sigma(N-k)\sigma_3(k/2) = \sum_{k<N/2} \sigma(N-2k)\sigma_3(k) = A_{3,1}(N), \\ U_4 &:= \sum_{k=1}^{N-1} \sigma((N-k)/2)\sigma_3(k/2) = \sum_{k<N/2} \sigma(N/2-k)\sigma_3(k) = S_{1,3}(N/2), \end{aligned}$$

so that from (2.4), (2.7), (2.8) and (3.5), we obtain

$$\begin{aligned} T_0 = {} & 136\sigma_5(N) - 112\sigma_3(N) - 24\sigma(N) \\ & - 640\sigma_5(N/2) - 128\sigma_3(N/2) + 48\sigma(N/2). \end{aligned} \tag{3.6}$$

Now we turn to the evaluation of T_1. By (1.9) and (1.10) we have

$$\begin{aligned} T_1 &= \sum_{\substack{k=1 \\ 2\nmid k}}^{2N-1} r_4(2N-k)r_8(k) = \sum_{k=1}^{N} r_4(2N-(2k-1))r_8(2k-1) \\ &= 128\sum_{k=1}^{N} \sigma(2N-(2k-1))\sigma_3(2k-1) = 128V_1 - 128V_2, \end{aligned}$$

where

$$V_1 := \sum_{k=1}^{2N-1} \sigma(2N-k)\sigma_3(k), \qquad V_2 := \sum_{k=1}^{N-1} \sigma(2N-2k)\sigma_3(2k).$$

First we evaluate V_1. We have by (2.4) and (1.12)

$$\begin{aligned} 240V_1 &= 240S_{1,3}(2N) = 21\sigma_5(2N) + (10-60N)\sigma_3(2N) - \sigma(2N) \\ &= 693\sigma_5(N) + (90-540N)\sigma_3(N) - 3\sigma(N) \\ &\quad - 672\sigma_5(N/2) - (80-480N)\sigma_3(N/2) + 2\sigma(N/2). \end{aligned}$$

Next we evaluate V_2. By (1.12) we have

$$\begin{aligned} V_2 &= \sum_{k=1}^{N-1} (3\sigma(N-k) - 2\sigma((N-k)/2))(9\sigma_3(k) - 8\sigma_3(k/2)) \\ &= 27S_{1,3}(N) - 18A_{1,3}(N) - 24A_{3,1}(N) + 16S_{1,3}(N/2). \end{aligned}$$

Appealing to (2.4), (2.7) and (2.8), we obtain

$$\begin{aligned} 240V_2 = 453\sigma_5(N) &+ (90-540N)\sigma_3(N) - 3\sigma(N) \\ &- 432\sigma_5(N/2) + (-80+480N)\sigma_3(N/2) + 2\sigma(N/2). \end{aligned}$$

Hence

$$T_1 = 128\sigma_5(N) - 128\sigma_5(N/2). \tag{3.7}$$

Thus, by (3.3), (3.6) and (3.7), we obtain

$$r_{12}(2N) = 264\sigma_5(N) - 768\sigma_5(N/2).$$

Therefore for $n \equiv 0 \pmod 2$ we have, by (1.12),

$$r_{12}(n) = 264\sigma_5(n/2) - 768\sigma_5(n/4) = 8\sigma_5(n) - 512\sigma_5(n/4),$$

which is Ewell's formula (1.4) and so is equivalent to Liouville's formula (1.1).

4. Elementary proof of Theorem 1. Let n be a positive odd integer. By (1.9)–(1.11), we obtain

$$\begin{aligned} &r_{12}(n) - 16\sigma_3(n) - 8\sigma(n) \\ &\qquad = \sum_{k=1}^{n-1} r_4(n-k)r_8(k) \\ &\qquad = 128\sum_{k=1}^{n-1} (-1)^{k-1}(\sigma(n-k) - 4\sigma((n-k)/4))(\sigma_3(k) - 16\sigma_3(k/2)) \\ &\qquad = 128X_1 - 512X_2 - 2048X_3 + 8192X_4, \end{aligned}$$

where

$$\begin{aligned}
X_1 &:= \sum_{k=1}^{n-1} (-1)^{k-1} \sigma(n-k)\sigma_3(k) \\
&= \sum_{k=1}^{n-1} \sigma(n-k)\sigma_3(k) - 2 \sum_{k<n/2} \sigma(n-2k)\sigma_3(2k) \\
&= S_{1,3}(n) - 2 \sum_{k<n/2} \sigma(n-2k)(9\sigma_3(k) - 8\sigma_3(k/2)) \quad \text{(by (1.12))} \\
&= S_{1,3}(n) - 18 \sum_{k<n/2} \sigma(n-2k)\sigma_3(k) + 16 \sum_{k<n/4} \sigma(n-4k)\sigma_3(k) \\
&= S_{1,3}(n) - 18A_{3,1}(n) + 16B_{3,1}(n), \\
X_2 &:= \sum_{k=1}^{n-1} (-1)^{k-1} \sigma((n-k)/4)\sigma_3(k) = \sum_{k<n/4} \sigma(k)\sigma_3(n-4k) = B_{1,3}(n), \\
X_3 &:= \sum_{k=1}^{n-1} (-1)^{k-1} \sigma(n-k)\sigma_3(k/2) = -\sum_{k<n/2} \sigma(n-2k)\sigma_3(k) = -A_{3,1}(n), \\
X_4 &:= \sum_{k=1}^{n-1} (-1)^{k-1} \sigma((n-k)/4)\sigma_3(k/2) = 0,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&r_{12}(n) - 16\sigma_3(n) - 8\sigma(n) \\
&\qquad = 128S_{1,3}(n) - 256A_{3,1}(n) + 2048B_{3,1}(n) - 512B_{1,3}(n).
\end{aligned}$$

Hence Theorem 1 follows by appealing to (2.4), (2.8) and Theorem 2.

5. Elementary proof of Ewell's formula when n is odd. Let n be a positive odd integer. By (2.6) and (2.10) we have

$$\begin{aligned}
&\sum_{d<n/2} (-1)^d d^3 \sum_{k<n/2d} \sigma(n-2kd) \\
&\qquad = 2 \sum_{\substack{d<n/2 \\ 2|d}} d^3 \sum_{k<n/2d} \sigma(n-2kd) - \sum_{d<n/2} d^3 \sum_{k<n/2d} \sigma(n-2kd) \\
&\qquad = 16 \sum_{d<n/4} d^3 \sum_{k<n/4d} \sigma(n-4kd) - \sum_{d<n/2} d^3 \sum_{k<n/2d} \sigma(n-2kd) \\
&\qquad = 16B_{3,1}(n) - A_{3,1}(n).
\end{aligned}$$

Finally, by (2.8), Theorem 2 and Theorem 1, we obtain

$$8\sigma_5(n)+16\sigma(n)+256\sum_{d<n/2}(-1)^d d^3\sum_{k<n/2d}\sigma(n-2kd)$$
$$=16\sigma_3(n)+8\sigma(n)+128\sum_{ax+by=n}ab^3=r_{12}(n),$$

which is Ewell's formula (1.5).

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