Arithmetics of beta-expansions

by

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1. Beta-expansions. Let $\beta$ be a real number strictly greater than 1. A real number $x \geq 0$ can be represented using a sequence $(x_i)_{k \geq i > -\infty}$, $x_i \in \mathbb{Z}$, $0 \leq x_i < \beta$, such that

$$x = x_k \beta^k + x_{k-1} \beta^{k-1} + \ldots + x_1 \beta + x_0 + x_{-1} \beta^{-1} + x_{-2} \beta^{-2} + \ldots$$

for some $k \in \mathbb{Z}$. We then write

$$(x)_{\beta} = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots$$

A particular representation is the $\beta$-expansion of $x$ (see [7]). The digits $x_i$ of the $\beta$-expansion are computed by the “greedy” algorithm: Let $[y]$ denote the largest integer smaller than or equal to $y$. Find $k \in \mathbb{Z}$ for which $\beta^k \leq x < \beta^{k+1}$. Put $x_k = [x/\beta^k]$ and $r_k = x/\beta^k \mod 1$. For $i \in \mathbb{Z}$, $i < k$ put $x_i = [\beta r_{i+1}]$ and $r_i = \beta r_{i+1} \mod 1$. If $k < 0$, i.e. $0 < x < 1$, we put $x_0, x_1, \ldots, x_{k+1} = 0$ and write $(x)_{\beta} = 0 \bullet 00 \ldots 0 x_k x_{k-1} \ldots$ If an expansion ends in infinitely many zeros, it is said to be finite and the final zeros are omitted.

We denote by $\text{Fin}(\beta)$ the set of all $x$ for which $|x|$ has a finite $\beta$-expansion. The $\beta$-expansion of every $x \in \text{Fin}(\beta)$ has therefore the form

$$(x)_{\beta} = x_k x_{k-1} \ldots x_1 x_0 \bullet x_{-1} x_{-2} \ldots x_{-l},$$

where $x_k x_{k-1} \ldots x_1 x_0 \bullet$ is the $\beta$-integer part and $\bullet x_{-1} x_{-2} \ldots x_{-l}$ is the $\beta$-fractional part of $x$. We usually call it simply the integer and the fractional part of $x$. The length of the fractional part of $x$ is denoted by $\text{fp}_{\beta}(x)$. Elements of $\text{Fin}(\beta)$ with vanishing fractional part (i.e. $\text{fp}_{\beta}(x) = 0$) are called $\beta$-integers. The set of $\beta$-integers is denoted by $\mathbb{Z}_\beta$.

The sets $\mathbb{Z}_\beta$ and $\text{Fin}(\beta)$ are generally not closed under addition and multiplication. In spite of that it is sometimes useful in computer science to consider these operations in $\beta$-arithmetics. That is why it is important to

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study what fractional parts may appear as a result of addition and multiplication of $\beta$-integers.

**Definition 1.1.** Let $\beta > 1$. We define

$$L_\oplus(\beta) := \min\{L \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_\beta, x + y \in \text{Fin}(\beta) \Rightarrow \text{fp}_\beta(x + y) \leq L\},$$

$$L_\ominus(\beta) := \min\{L \in \mathbb{N}_0 \mid \forall x, y \in \mathbb{Z}_\beta, xy \in \text{Fin}(\beta) \Rightarrow \text{fp}_\beta(xy) \leq L\}.$$

The minimum of an empty set is defined to be $+\infty$.

The aim of this paper is to give some quantitative results for $L_\oplus(\beta)$ and $L_\ominus(\beta)$. Let us mention some of the known results. Frougny and Solomyak in [4] showed that $L_\oplus(\beta)$ is finite if $\beta$ is a Pisot number. A *Pisot number* $\beta$ is an algebraic integer such that $\beta > 1$ and all its algebraic conjugates are of modulus smaller than 1. Let us mention that to our knowledge no example is known of a $\beta$ such that $L_\oplus(\beta)$ or $L_\ominus(\beta)$ is infinite.

Results for the special case of quadratic Pisot units are found in [3]. The authors gave exact values for $L_\oplus(\beta)$ and $L_\ominus(\beta)$ when $\beta > 1$ is a solution either of the equation $x^2 = mx - 1$, $m \in \mathbb{N}$, $m \geq 3$ or of the equation $x^2 = mx + 1$, $m \in \mathbb{N}$. In the first case $L_\oplus(\beta) = L_\ominus(\beta) = 1$; in the second case $L_\oplus(\beta) = L_\ominus(\beta) = 2$.

In this article we provide estimates on $L_\oplus(\beta)$ and $L_\ominus(\beta)$ for those algebraic numbers $\beta > 1$ that have at least one conjugate of modulus smaller than 1. Other results are valid for Pisot numbers $\beta$. The last part of the paper is devoted to quadratic Pisot numbers. We recover the results of [3] as a special case.

2. **Beta-integers and cut-and-project sequences.** The Rényi development of unity plays an important role in the description of the properties of the sets $\mathbb{Z}_\beta$ and $\text{Fin}(\beta)$. For its definition we introduce the transformation $T_\beta(x) := \{\beta x\}$ for $x \in [0, 1]$. The *Rényi development of unity* is defined as

$$d(1, \beta) := t_1 t_2 \ldots t_i \ldots,$$

where $t_i := [\beta T_\beta^{i-1}(1)]$.

Parry [6] showed that $x = x_k x_{k-1} \ldots x_1 x_0 \cdot x_{-1} \ldots x_{-p}$ is a $\beta$-expansion if and only if $x_i x_{i-1} \ldots x_{-p}$ is lexicographically smaller than $t_1 t_2 \ldots t_i \ldots$ for every $-p \leq i \leq k$.

$\text{Fin}(\beta)$ and $\mathbb{Z}_\beta$ are centrally symmetric sets. While $\text{Fin}(\beta)$ is dense in $\mathbb{R}$, $\mathbb{Z}_\beta$ has no accumulation points. Distances between consecutive points in $\mathbb{Z}_\beta$ take values in $\{0 \cdot t_i t_{i+1} \ldots \mid i \in \mathbb{N}\}$. It is obvious that if $d(1, \beta)$ is eventually periodic, then $\mathbb{Z}_\beta$ has a finite number of distances between consecutive points. Numbers $\beta$ with this property are called *beta-numbers*. Some results and conjectures on beta-numbers are given in [2, 8]; a description of beta-numbers is provided in [9]. Note that every Pisot number $\beta$ is a beta-number.
The set $\mathbb{Z}_\beta$ of $\beta$-integers forms a ring only if $\beta$ is a rational integer, $\beta > 1$. If $\beta$ is an algebraic integer of order $q \geq 2$, then $\mathbb{Z}_\beta$ can be naturally embedded into the ring $\mathbb{Z}[\beta]$ defined as

$$
\mathbb{Z}[\beta] := \{n_0 + n_1 \beta + \ldots + n_{q-1} \beta^{q-1} \mid n_i \in \mathbb{Z}\}.
$$

Note that the ring $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$. In certain cases $\mathbb{Z}[\beta]$ coincides with $\text{Fin}(\beta)$, i.e. $\text{Fin}(\beta)$ is a ring (see [4]). Let us show that for $\beta$ an algebraic integer, the ring $\mathbb{Z}[\beta]$ is a projection of an integer lattice $\mathbb{Z}^q \subset \mathbb{R}^q$ on a one-dimensional subspace $V_1$ for a suitable decomposition $V_1 \oplus V_2$ of the space $\mathbb{R}^q$. A similar construction can be found in [1].

Denote by $\beta^{(1)} = \beta, \beta^{(2)}, \ldots, \beta^{(s)}$ the real roots of the minimal polynomial of $\beta$ and by $\beta^{(s+1)}, \beta^{(s+2)}, \ldots, \beta^{(q-1)}, \beta^{(q)}$ the non-real conjugates of $\beta$. We have ordered the complex roots in such a way that $\overline{\beta^{(s+1)}} = \beta^{(s+2)}, \ldots, \overline{\beta^{(q-1)}} = \beta^{(q)}$.

First we have to find (possibly) complex vectors

$$(\vec{x}^{(1)})^T = (x^{(1)}_0, x^{(1)}_1, \ldots, x^{(1)}_{q-1}), \ldots, (\vec{x}^{(q)})^T = (x^{(q)}_0, x^{(q)}_1, \ldots, x^{(q)}_{q-1}),$$

such that for any $\vec{x} = (n_0, n_1, \ldots, n_{q-1}) \in \mathbb{R}^q$ we have

$$(1) \quad \vec{x} = \left( \sum_{i=0}^{q-1} n_i (\beta^{(1)})^i \right) \vec{x}^{(1)} + \left( \sum_{i=0}^{q-1} n_i (\beta^{(2)})^i \right) \vec{x}^{(2)} + \ldots$$

$$+ \left( \sum_{i=0}^{q-1} n_i (\beta^{(q)})^i \right) \vec{x}^{(q)}.$$

Denote by $\mathbb{X}$ the $q \times q$ matrix with $(\mathbb{X})_{ij} = x^{(i)}_j$. Then (1) holds for each $\vec{x}$ if and only if

$$I_q = \mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)}) \cdot \mathbb{X},$$

where $\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)})$ is the Vandermonde matrix in variables $\beta^{(1)}, \ldots, \beta^{(q)}$,

$$
\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)}) := \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\beta^{(1)} & \beta^{(2)} & \ldots & \beta^{(q)} \\
(\beta^{(1)})^2 & (\beta^{(2)})^2 & \ldots & (\beta^{(q)})^2 \\
\vdots & \vdots & \ddots & \vdots \\
(\beta^{(1)})^{q-1} & (\beta^{(2)})^{q-1} & \ldots & (\beta^{(q)})^{q-1}
\end{pmatrix}.
$$

The determinant of $\mathbb{V}(\beta^{(1)}, \ldots, \beta^{(q)})$ is equal to $\prod_{q \geq i > j \geq 1} (\beta^{(i)} - \beta^{(j)})$. Since all conjugates are distinct, the determinant is non-zero.

Using the Cramer rule to compute $x^{(i)}_j$, we find that $\vec{x}^{(i)}$ is real if $\beta^{(i)}$ is real, and if $\beta^{(j)}$ and $\beta^{(j+1)}$ are complex conjugate roots then $\vec{x}^{(j)} = \overline{\vec{x}^{(j+1)}}$. 

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Thus we can define a real basis \( \tilde{y}^{(1)}, \ldots, \tilde{y}^{(q)} \) of \( \mathbb{R}^q \) in such a way that \( \tilde{y}^{(i)} = \tilde{x}^{(i)} \) if \( \tilde{x}^{(i)} \) is a real vector, and \( \tilde{y}^{(j)} = \tilde{x}^{(j)} + \overline{\tilde{x}^{(j)}} \), \( \tilde{y}^{(j+1)} = i(\tilde{x}^{(j)} - \overline{\tilde{x}^{(j)}}) \), if \( \tilde{x}^{(j)} \) and \( \tilde{x}^{(j+1)} = \overline{\tilde{x}^{(j)}} \) are complex conjugate vectors.

Note that the coordinates of a vector \( \tilde{x} = (n_0, n_1, \ldots, n_{q-1}) \in \mathbb{R}^q \) with respect to the basis \( \tilde{y}^{(1)}, \ldots, \tilde{y}^{(q)} \) are

\[
\sum_{p=0}^{q-1} n_p(\beta^{(i)})^p \quad \text{if} \quad \tilde{y}^{(i)} = \tilde{x}^{(i)},
\]

\[
\mathbb{R} \left[ \sum_{p=0}^{q-1} n_p(\beta^{(j)})^p \right] \quad \text{if} \quad \tilde{y}^{(j)} = \tilde{x}^{(j)} + \overline{\tilde{x}^{(j)}},
\]

\[
\mathbb{I} \left[ \sum_{p=0}^{q-1} n_p(\beta^{(j)})^p \right] \quad \text{if} \quad \tilde{y}^{(j)} = i(\tilde{x}^{(j)} - \overline{\tilde{x}^{(j)}}).
\]

If we put \( V_1 = \mathbb{R}\tilde{y}^{(1)} \) and \( V_2 = \mathbb{R}\tilde{y}^{(2)} + \mathbb{R}\tilde{y}^{(3)} + \ldots + \mathbb{R}\tilde{y}^{(q)} \), then the set \( \mathbb{Z}[^{\beta}] \) is the projection of \( \mathbb{Z}^q \) on \( V_1 \) along \( V_2 \).

Projections of crystallographic lattices and non-crystallographic lattices are studied by the theory of cut-and-project sets. Let us recall a special case of their definition, which will be used here.

**Definition 2.1.** Let \( U_1 \) and \( U_2 \) be linear subspaces of \( \mathbb{R}^d \) such that \( \dim U_1 = 1 \), \( \dim U_2 = d - 1 \) and \( U_1 \oplus U_2 = \mathbb{R}^d \). Denote by \( \pi_1 \) the projection on \( U_1 \) along \( U_2 \) and by \( \pi_2 \) the projection on \( U_2 \) along \( U_1 \). Let \( \Omega \subset U_2 \) be a bounded set with non-empty interior \( \Omega^o \), such that the closures of \( \Omega \) and \( \Omega^o \) coincide. If the mapping \( \pi_1 : \mathbb{Z}^q \rightarrow \pi_1(\mathbb{Z}^d) \) is one-to-one and \( \pi_2(\mathbb{Z}^d) \) is dense in \( V_2 \), then the set \( \Sigma(\Omega) = \{ \pi_1(x) \mid x \in \mathbb{Z}^d \}, \pi_2(x) \in \Omega \) is called a cut-and-project set with acceptance window \( \Omega \).

Basic properties of cut-and-project sets can be found in [5]. For us the most important property is that \( \Sigma(\Omega) \) is relatively dense and uniformly discrete, i.e. there exists a real increasing sequence \( (\alpha_n)_{n \in \mathbb{Z}} \) and constants \( r, R > 0 \) such that \( \Sigma(\Omega) = \{ \alpha_n \tilde{y} \mid n \in \mathbb{Z} \} \) and \( r \leq \alpha_{n+1} - \alpha_n \leq R \) for all \( n \in \mathbb{Z} \). In particular, the distances between consecutive points of \( \Sigma(\Omega) \) take only finitely many values, i.e. the set \( \{ \alpha_{n+1} - \alpha_n \mid n \in \mathbb{Z} \} \) is finite.

Let us consider again an algebraic integer \( \beta \) of order \( q \) and the decomposition \( \mathbb{R}^q = V_1 \oplus V_2 \) as described above. As shown by Akiyama [1], the projection \( \pi_1(\mathbb{Z}^2) = \mathbb{Z}[\beta] \) of \( \mathbb{Z}^2 \) on \( V_1 \) is one-to-one and the projection \( \pi_1(\mathbb{Z}^2) \) on \( V_2 \) is dense in \( V_2 \). For \( \alpha \in \mathbb{Q}[\beta] \) we denote by \( \alpha^{(k)} \) the image of \( \alpha \) under the \( k \)th Galois isomorphism \( \mathbb{Q}[\beta] \rightarrow \mathbb{Q}[\beta^{(k)}] \) induced by the assignment \( \beta \mapsto \beta^{(k)} \), i.e. if \( \alpha = \sum_{i=0}^{d-1} n_i \beta^i \) for \( n_i \in \mathbb{Q} \), then \( \alpha^{(k)} = \sum_{i=0}^{q-1} n_i (\beta^{(k)})^i \).

We shall focus on specific acceptance windows \( \Omega(h) \subset V_2 \) for \( h > 0 \). As the acceptance window \( \Omega(h) \subset V_2 \) we choose the cartesian product of one-dimensional line-segments \( \{ t\tilde{y}^{(i)} \mid |t| < h \} \) if \( \beta^{(i)} \) is real and two-dimensional
ellipses \{ty^{(j)} + sg^{(j+1)} \mid t^2 + s^2 < h^2\} if $\beta^{(j)}$ and $\beta^{(j+1)}$ are complex conjugate. Such an acceptance window $\Omega(h)$ satisfies the assumptions of Definition 2.1.

The point $\alpha y^{(1)}$ belongs to $\Sigma(\Omega(h))$ if and only if $\alpha \in \mathbb{Z}[\beta]$ and $|\alpha^{(k)}| < h$ for $k = 2, \ldots, q$. In other words, we have the following proposition.

**Proposition 2.2.** Let $\beta$ be an algebraic integer of order $q$. If $h > 0$, then the set

$$\Sigma(h) = \{ \alpha \in \mathbb{Z}[\beta] \mid |\alpha^{(k)}| < h, k = 2, \ldots, q\}$$

is relatively dense and uniformly discrete and the distances in $\Sigma(h)$ take only finitely many values.

In the following, the sets $\Sigma(h)$ are called cut-and-project sequences. In the case that $\beta$ is a Pisot number, we show the relation between cut-and-project sequences and $\beta$-integers $\mathbb{Z}_\beta$.

**Proposition 2.3.** Let $\beta$ be a Pisot number of order $q$. Set

$$l = [\beta] \max\{(1 - |\beta^{(i)}|)^{-1} \mid i = 2, \ldots, q\}.$$  

Then

$$\mathbb{Z}_\beta \subset \Sigma(l), \quad \mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \Sigma(2l), \quad \mathbb{Z}_\beta \mathbb{Z}_\beta \subset \Sigma(l^2).$$

**Proof.** Let $x \in \mathbb{Z}_\beta$, i.e. $x = \pm \sum_{i=0}^{n} x_i \beta^i$ for some $n$. Then

$$|x^{(j)}| \leq \sum_{i=0}^{n} [\beta] |\beta^{(j)}|^i < [\beta] \frac{1}{1 - |\beta^{(j)}|} \leq l \quad \text{for } j = 2, \ldots, q.$$  

The statement follows easily. ■

3. **Sufficient conditions for finiteness of $L_\oplus$ and $L_\odot$.** In this section we provide sufficient conditions on $\beta$ so that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite. First we demonstrate Theorem 3.1 stating that $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite for a Pisot $\beta$. The statement for $L_\oplus$ has been proven in [4], but we provide a different and simpler proof. We further show that this condition is not necessary. Theorem 3.3 provides a different sufficient condition together with bounds on $L_\oplus(\beta)$ and $L_\odot(\beta)$. In the next section we apply Theorem 3.3 to the case of quadratic Pisot numbers.

**Theorem 3.1.** Let $\beta$ be a Pisot number. Then $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite.

**Proof.** Let $x, y \in \mathbb{Z}_\beta$. To determine $L_\odot(\beta)$ it suffices to consider $x, y > 0$. Set $z_0 = \max\{z \in \mathbb{Z}_\beta \mid z \leq xy\}$ and $r := xy - z_0$. Since distances in $\mathbb{Z}_\beta$ are bounded by 1, we have $0 \leq r < 1$. Therefore obviously the remainder $r$ is the fractional part of the $\beta$-expansion of $xy$, i.e. $xy \in \text{Fin}(\beta)$ if and only if $r \in \text{Fin}(\beta)$. Since $l > 1$, we have $\Sigma(l) \subset \Sigma(l^2)$ and according to Proposition 2.3 both $xy$ and $z_0$ belong to $\Sigma(l^2)$.  


According to Proposition 2.2 distances in $\Sigma(l^2)$ take only finitely many values, say $f_1, \ldots, f_T$. The gap $r$ between $z_0$ and $xy$ must be composed of these distances. Therefore $1 > r = xy - z_0 = \sum h_if_i$, where $h_i \in \mathbb{N}_0$. Fractional parts of all results of multiplication $xy$ belong to the set 

$$F := \left\{ \sum_i h_if_i < 1 \mid h_i \in \mathbb{N}_0 \right\},$$

which is finite and therefore 

$$L_\circ(\beta) \leq \max\{fp_\beta(r) \mid r \in F \cap \text{Fin}(\beta)\}.$$ 

To derive the finiteness of $L_\oplus(\beta)$ one uses an analogous argument. ■

A simple consequence of the above proof is that $\mathbb{Z}_\beta$ is a Meyer set.

**Corollary 3.2.** Let $\beta$ be a Pisot number. Then there exists a finite set $F$ such that 

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F, \quad \mathbb{Z}_\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F.$$ 

Theorem 3.1 gives a sufficient condition for finiteness of $L_\oplus(\beta)$ and $L_\circ(\beta)$. An upper bound on $L_\oplus(\beta)$ is determined in [10] using some complicated techniques. However, their result applies only to a class of Pisot numbers. The condition that $\beta$ is Pisot is however not necessary. In the following theorem we provide a similar estimate on $L_\oplus(\beta)$ with less restrictive criteria for $\beta$. Moreover, we determine an upper bound for $L_\circ(\beta)$.

**Theorem 3.3.** Let $\beta > 1$ be an irrational algebraic number such that at least one of its conjugates is of modulus smaller than $1$. Define 

$$H = \sup\{|z'| \mid z \in \mathbb{Z}_\beta\}, \quad K = \inf\{|z'| \mid z \in \mathbb{Z}_\beta, z \notin \beta\mathbb{Z}_\beta\}.$$ 

If $K > 0$, then $L_\oplus(\beta)$ and $L_\circ(\beta)$ are finite and 

\begin{align*}
(2) \quad & \left( \frac{1}{|\beta'|} \right)^{L_\oplus(\beta)} < \frac{2H}{K}, \\
(3) \quad & \left( \frac{1}{|\beta'|} \right)^{L_\circ(\beta)} < \frac{H^2}{K}.
\end{align*}

**Proof.** Let $x, y \in \mathbb{Z}_\beta$ and $x + y \in \text{Fin}(\beta)$, $x + y = \sum_{i=-L}^{k} a_i\beta^i$, $a_{-L} \geq 1$. Then $\beta^L(x + y) \in \mathbb{Z}_\beta$ and $\beta^L(x + y) \notin \beta\mathbb{Z}_\beta$. Thus 

$$K \leq |\beta'|^L|x'| + |y'| \leq |\beta'|^L(|x'| + |y'|) < 2H|\beta'|^L,$$

which implies (2). Note that the supremum $H$ is never attained, i.e. $|z'| < H$ for all $z \in \mathbb{Z}_\beta$. The proof for multiplication is similar. ■

**Remark 3.4.** 1. Using the same inequalities as in the proof of Proposition 2.3 we obtain 

$$H \leq |\beta| \frac{1}{1 - |\beta'|}.$$
2. If $\beta' \in (0, 1)$, then $K = 1$. Indeed, for $z = \sum_{i=0}^{n} z_i \beta^i$, $z_0 \neq 0$, one has

$$z' = \sum_{i=0}^{n} z_i (\beta')^i \geq z_0 \geq 1.$$

**Corollary 3.5.** Let $\beta > 1$ be an algebraic integer such that at least one of its conjugates, say $\beta'$, belongs to $(0, 1)$. Then

$$\left( \frac{1}{|\beta'|} \right)^{L_\oplus(\beta)} < \frac{2|\beta|}{1 - \beta'}, \quad \left( \frac{1}{|\beta'|} \right)^{L_\odot(\beta)} < \frac{|\beta|^2}{(1 - \beta')^2}.$$

4. **Theorem 3.3 for quadratic Pisot numbers.** So far we have been interested in $L_\oplus(\beta)$ and $L_\odot(\beta)$ for general algebraic integers $\beta$. From now on we shall focus on quadratic Pisot numbers. In the quadratic case the Pisot condition implies that $\beta$ is a solution of an equation

$$x^2 = mx - n, \quad m, n \in \mathbb{N}, \ m \geq n + 2,$$

$$x^2 = mx + n, \quad m, n \in \mathbb{N}, \ m \geq n.$$

We shall try to apply Theorem 3.3 for such $\beta$ and derive the corresponding bounds on $L_\oplus(\beta)$ and $L_\odot(\beta)$. It will be seen that the situation drastically differs for the two types of quadratic equations.

Note that for $n = 1$, the root $\beta$ is a quadratic Pisot unit. For such $\beta$ the values of $L_\oplus(\beta)$ and $L_\odot(\beta)$ have been determined in [3].

Let us now study the case of $\beta > 1$ solving the equation $x^2 = mx - n$, $m, n \in \mathbb{N}, m \geq n + 2$. Note that $|\beta| = m - 1$, thus the digits in $\beta$-expansions are $0, 1, \ldots, m-1$. The conjugate $\beta'$ of $\beta$ satisfies $\beta' \in (0, 1)$, and the $\beta$-development of unity is $d(1, \beta) = (m-1)(m-n-1)^{\omega}$. For $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^{n} z_i \beta^i$ we have

$$z' = \sum_{i=0}^{n} z_i (\beta')^i < (m-1) + (m-2)\beta' + (m-2)\beta'^2 + \ldots$$

$$= 1 + (m-2) \frac{1}{1 - \beta'} = \frac{\beta(\beta - 1)}{\beta - n} = H.$$

Clearly, $\beta(\beta - 1)/(\beta - n)$ above is the desired supremum $H$ of Theorem 3.3, since we can construct a sequence of numbers

$$z_n = (m - 1)\beta^0 + \sum_{i=1}^{n} (m - 2)\beta^i \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$$

such that $\lim_{n \to \infty} |z'_n| = H$. For the relation (4) we have considered the admissibility of sequences of digits in $\beta$-expansions. According to Remark 3.4 we have $K = 1$, and hence we can use Theorem 3.3 to derive results for $L_\oplus(\beta)$ and $L_\odot(\beta)$. 
Proposition 4.1. Let $\beta^2 = m\beta - n$, $m \geq n + 2$. Then

$$L_\oplus(\beta) \leq 3m \ln m, \quad L_\ominus(\beta) \leq 4m \ln m.$$ 

In particular, if $n = 1$, then $L_\oplus(\beta) = L_\ominus(\beta) = 1$.

Proof. Since $K = 1$ and $H = \beta(\beta - 1)/(\beta - n) = (\beta - 1)^2/(m - n - 1)$ we can estimate

$$\left(\frac{m - 1}{n}\right)^{L_\oplus} < \left(\frac{\beta}{n}\right)^{L_\oplus} = \left(\frac{1}{\beta'}\right)^{L_\oplus} < 2 - \frac{(\beta - 1)^2}{m - n - 1} < 2 - \frac{(m - 1)^2}{m - n - 1}.$$ 

For $n = 1$ we obtain directly $L_\oplus \leq 1$. For general $n \leq m - 2$ we estimate the left hand side of the inequality by

$$\left(\frac{m - 1}{n}\right)^{L_\ominus} \geq \left(\frac{m - 1}{m - 2}\right)^{L_\ominus} > e^{L_\ominus/m},$$ 

where we have used $(1 + 1/k)^{k+1} > e$ for $k \in \mathbb{N}$. The right hand side of the inequality is estimated by $m^3$. Altogether we get $L_\oplus(\beta) \leq 3m \ln m$. The estimate for $L_\ominus(\beta)$ is derived analogously, the first step for $n = 1$ being

$$\beta^{L_\ominus} = \left(\frac{1}{\beta'}\right)^{L_\ominus} < \left(\frac{\beta(\beta - 1)}{\beta - 1}\right)^2 = \beta^2 \Rightarrow L_\ominus \leq 1.$$ 

In order to show that for $n = 1$ we have $L_\oplus(\beta) = L_\ominus(\beta) = 1$ it suffices to observe that

$$((m - 1) + (m - 1))_\beta = (2 \cdot (m - 1))_\beta = \left(\beta + (m - 2) + \frac{1}{\beta}\right)_\beta = 1(m - 2) \bullet 1.$$ 

Let us now study the case of $\beta > 1$ solving the equation $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. Note that $[\beta] = m$. Therefore the digits in $\beta$-expansions are $0, 1, \ldots, m$. The $\beta$-development of unity is $d(1, \beta) = mn$. Now the conjugate $\beta'$ of $\beta$ satisfies $\beta' \in (-1, 0)$. If $w \in \mathbb{Z}_\beta$, $w = \sum_{i=0}^{n} w_i \beta^i$, we have

$$\ldots + m\beta'^3 + m\beta' < w' < m + m\beta'^2 + m\beta'^4 + \ldots,$$

$$-1 < w' < \frac{m}{1 - \beta'^2} = \frac{\beta'^2m}{m\beta + n - n^2} = H.$$ 

Unfortunately, in this case $K = 0$ for all $n \in \mathbb{N}$ except $n = 1$. Therefore only for $n = 1$ can we use Theorem 3.3 to find $L_\oplus(\beta)$ and $L_\ominus(\beta)$. In this case for $z \in \mathbb{Z}_\beta$, $z = \sum_{i=0}^{n} z_i \beta^i$ with $z_0 \neq 0$, we have

$$z' \geq z_0 + z_1 \beta' + z_3 \beta'^3 + z_5 \beta'^5 + \ldots$$

$$\geq 1 + (m - 1)\beta' + m\beta'^3 + m\beta'^5 + \ldots$$

$$= 1 - \beta' + \frac{m\beta'}{1 - \beta'} = -\beta' = \frac{1}{\beta} = K.$$
Note that $H$ is equal to $\beta$ for $n = 1$. Using (2) and (3), we obtain for $m \geq 2$
\[
\begin{align*}
\beta^{L_\oplus} &< 2\beta^2 < \beta^3 \\
\beta^{L_\odot} &< \beta^3
\end{align*}
\]

To prove that $L_\oplus(\beta) = L_\odot(\beta) = 2$ we calculate
\[
(m+m)\beta = (2\cdot m)\beta = \left(\beta + (m-1) + \frac{m-1}{\beta} + \frac{1}{\beta^2}\right) \beta = 1(m-1) \cdot (m-1)1.
\]

For $m = 1$, i.e. $\beta$ the golden ratio, it is not true that $2\beta^2 < \beta^3$. A slightly finer discussion is necessary to obtain the exact bound on the number of fractional digits of the sum $x + y$.

In the above considerations we are not able to derive any estimates on $L_\oplus(\beta)$ and $L_\odot(\beta)$ if $\beta$ is a solution of $x^2 = mx + n, m,n \in \mathbb{N}, m \geq n \geq 2$. Therefore in the rest of the paper we focus on such quadratic Pisot numbers.

First we give an estimate on $L_\odot(\beta)$ using $L_\oplus(\beta)$ and then we determine the value of $L_\oplus(\beta)$.

5. Relation of $L_\oplus$ and $L_\odot$ for quadratic Pisot numbers. In Section 2 we have shown that $\mathbb{Z}_\beta$ can be embedded into a cut-and-project sequence with a suitably chosen window. In our case $\beta$ is a solution of $x^2 = mx + n, m,n \in \mathbb{N}, m \geq n \geq 2$. Therefore we choose $\Sigma(H)$, where $H = m/(1-\beta^2)$. We show that a cut-and-project set with arbitrary window can be embedded into a finite union of shifted copies of $\mathbb{Z}_\beta$, where the shifts belong to $\mathbb{Z}[\beta]$. In fact, a product $xy$ of $x,y \in \mathbb{Z}_\beta$ can be expressed as a sum of a $\beta$-integer and a small rational integer and therefore we can find an upper estimate of $L_\odot(\beta)$ using $L_\oplus(\beta)$. A similar result can also be proven for non-quadratic Pisot $\beta$. The demonstration is however rather technical.

**Theorem 5.1.** Let $\beta > 1$ be a solution of $x^2 = mx + n, m,n \in \mathbb{N}, m \geq n$, and let $h > 0$. Then there exists $p \in \mathbb{N}$ such that
\[
\Sigma(h) \subset \mathbb{Z}_\beta + \{-p, -p+1, \ldots, -1, 0, 1, \ldots, p-1, p\},
\]
where
\[
p \leq h - \beta'H = h - \beta' \left(\frac{m}{1-\beta^2}\right).
\]

**Proof.** Since $\beta$ is a quadratic integer, we can write every power $\beta^k$ as an integer combination of $1$ and $\beta$. Define $F_k, G_k$ by
\[
\beta^k = F_k\beta + G_k.
\]
Since $\beta^{k+1} = \beta(F_k\beta + G_k) = F_km\beta + F_kn + G_k\beta$, the sequences $(F_k)_{k \in \mathbb{N}_0}$, $(G_k)_{k \in \mathbb{N}_0}$ satisfy $F_{k+1} = mF_k + G_k, G_{k+1} = nF_k$, which gives a recurrence
relation
\[ F_{k+2} = mF_{k+1} + nF_k, \quad \text{where} \quad F_0 = 0, \quad F_1 = 1. \]

It is easy to see that every \( x \in \mathbb{N} \) can be written in the form \( x = \sum_{i=1}^{j} c_i F_i \), where \( c_i \in \{0, 1, \ldots, m\} \) and \( c_i c_{i-1} \) is lexicographically smaller than \( mn \). The coefficients \( c_j, c_{j-1}, \ldots, c_1 \) can be found by the so-called greedy algorithm. Thus \( j \) is a number for which \( F_j \leq x < F_{j+1} \) and \( c_j := \lfloor x F_j^{-1} \rfloor \). We obtain the coefficients \( c_i, \ i < j \), by applying the same steps to the integer \( \bar{x} = x - c_j F_j \).

Let \( z \in \Sigma(h) \), i.e. \( z = a + b\beta \) and \( |z'| < h \). Since both \( \Sigma(h) \) and \( \mathbb{Z}_\beta \) are symmetric with respect to the origin, it suffices to show the statement for \( b \geq 0 \). Let \( b = \sum_{i=1}^{j} c_i F_i \). Then
\[
(5) \quad z = \sum_{i=1}^{j} c_i (F_i \beta + G_i) - \sum_{i=1}^{j} c_i G_i + a = z_1 + z_2,
\]
where \( z_2 := a - \sum_{i=1}^{j} c_i G_i \in \mathbb{Z} \) and \( z_1 := \sum_{i=1}^{j} c_i \beta^i \in \beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta \). Applying the Galois automorphism to the equality \( z = z_1 + z_2 \) gives \( z_2 = z' - z'_1 \). Since \( |z'| < h \) and \( |z'| < -\beta'H \), the integer \( z_2 \) belongs to the interval \((-h + \beta'H, h - \beta'H)\).

**Corollary 5.2.**
\[
\mathbb{Z}_\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{-p, \ldots, p\}, \quad \text{where} \quad p \leq (m + 2)^4/4.
\]

**Proof.** Since \( \mathbb{Z}_\beta \subset \Sigma(H) \), we have \( \mathbb{Z}_\beta \mathbb{Z}_\beta \subset \Sigma(H^2) \). The proof will be completed if we verify that \( H^2 - \beta'H \leq \frac{1}{4}(m + 2)^4 \). Let us first show that
\[
(6) \quad \frac{1}{1 - \beta'^2} < \frac{m + 3}{2}.
\]
We have \( -\beta' = n/\beta \), thus for \( n \leq m - 1 \),
\[
1 - \beta'^2 = 1 - \frac{n^2}{\beta^2} > 1 - \frac{n^2}{m^2} \geq 1 - \frac{(m - 1)^2}{m^2} = \frac{2m - 1}{m^2} \geq \frac{2}{m + 3}.
\]
For \( n = m \) the inequality (6) is verified directly using \( \beta' = \frac{1}{2}(m - \sqrt{m^2 + 4m}) \). Therefore
\[
H^2 - \beta'H \leq H^2 + H = \frac{m^2}{(1 - \beta'^2)^2} + \frac{m}{1 - \beta'^2} < \frac{m^2(m + 3)^2}{4} + \frac{m(m + 3)}{2}
\]
\[
= \frac{1}{4} m(m + 1)(m + 2)(m + 3) \leq \frac{1}{4} (m + 2)^4. \]

The above corollary states that a product of two \( \beta \)-integers can be written as a sum of a \( \beta \)-integer and a rational integer. Let us derive the number of fractional digits of the \( \beta \)-expansion of a rational integer \( p \).

**Lemma 5.3.** Let \( p \in \mathbb{N} \). Then
\[
fp_\beta(p) \leq (1 + \log_2 p)L_{\oplus}(\beta).
\]
Proof. The proof is based on the simple observation that
\[
\text{fp}_\beta(x + y) \leq \max\{\text{fp}_\beta(x), \text{fp}_\beta(y)\} + L_\oplus(\beta),
\]
which in particular gives \(\text{fp}_\beta(2x) \leq \text{fp}_\beta(x) + L_\oplus(\beta)\). Applying the latter \(k\) times we obtain \(\text{fp}_\beta(2^k x) \leq kL_\oplus(\beta)\). We use induction on \(j\) to prove that if \(p\) has a binary expansion \(p = \sum_{i=0}^{j} a_i 2^i\) then \(\text{fp}_\beta(p) \leq (j + 1) L_\oplus(\beta)\). Using the hypothesis for \(p = \sum_{i=0}^{j} a_i 2^i = 2^j + \sum_{i=0}^{j-1} a_i 2^i\) we obtain
\[
\text{fp}_\beta(p) \leq \max\{\text{fp}_\beta(2^j), \text{fp}_\beta\left(\sum_{i=0}^{j-1} a_i 2^i\right)\} + L_\oplus(\beta)
\leq \max\{jL_\oplus(\beta), jL_\oplus(\beta)\} + L_\oplus(\beta) = (j + 1)L_\oplus(\beta).
\]
The statement of the lemma follows easily from the fact that \(j \leq \log_2 p\).

The following theorem is a simple consequence of Corollary 5.2 and Lemma 5.3.

THEOREM 5.4. Let \(\beta > 1\) be a solution of \(x^2 = mx + n\), \(m, n \in \mathbb{N}\), \(m \geq n\). Then
\[
L_\oplus(\beta) \leq 4L_\oplus(\beta) \log_2(m + 2).
\]

Proof. Let \(x, y \in \mathbb{Z}_\beta\). Using Corollary 5.2, we have \(\text{fp}_\beta(xy) = \text{fp}_\beta(z + p)\) for some \(z \in \mathbb{Z}_\beta\) and \(p \in \mathbb{N}\), \(p \leq \frac{1}{4}(m + 2)^4\). Now, due to (7),
\[
\text{fp}_\beta(z + p) \leq \text{fp}_\beta(p) + L_\oplus(\beta) \leq (2 + \log_2 p)L_\oplus(\beta)
\leq \left(2 + \log_2 \left(\frac{m + 2}{4}^4\right)\right)L_\oplus(\beta).
\]
The statement of the theorem follows easily.

6. \(L_\oplus\) for quadratic \(\beta\). In this section we obtain an upper bound on \(L_\oplus(\beta)\). This is done in two steps: first we find an upper bound on \(\text{fp}(x + y)\) where \(x\) is an arbitrary \(\beta\)-integer and \(y\) is a \(\beta\)-integer of a specific form. Then we show that any \(\beta\)-integer can be written as a finite sum of numbers of this specific form. An upper bound on \(L_\oplus(\beta)\) is obtained by combining both results.

Let \(\beta > 1\) be a solution of \(x^2 = mx + n\), \(m, n \in \mathbb{N}\), \(m \geq n\). Let \((x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_1 x_2 \ldots x_p\) be a \(\beta\)-representation of \(x\), i.e. \(0 \leq x_i \leq m\). The \(\beta\)-representation \((x)_\beta\) is a \(\beta\)-expansion of \(x\) if and only if \(x_i x_{i-1}\) is lexicographically smaller than \(mn = d(1, \beta)\) for every \(i\).

The following lemma is an easy consequence of the result of Frougny and Solomyak in [4]. It is mentioned here in order to make the article self-contained.

LEMMA 6.1. Let \((x)_\beta = x_k x_{k-1} \ldots x_1 x_0 \bullet x_1 x_2 \ldots x_p\) be a \(\beta\)-representation of \(x\). Then \(\text{fp}_\beta(x) \leq p\).
Proof. If the representation is already in the form of a $\beta$-expansion, then $fp_\beta(x) = p$. Otherwise we can find the largest $j$ such that $x_jx_{j-1}$ is lexicographically greater than or equal to $mn$. Since $x_i \leq m$ for all $i$, necessarily $x_j = m$ and $x_{j-1} \geq n$. Since $j$ was the largest index with this property, $x_{j+1} < m$. Therefore we can define a new representation of $x$ as

$$(x)_\beta = \tilde{x}_k\tilde{x}_{k-1} \ldots \tilde{x}_1\tilde{x}_0 \cdot \tilde{x}_{-1}\tilde{x}_{-2} \ldots \tilde{x}_{-p},$$

where $\tilde{x}_j := x_j - m$, $\tilde{x}_{j-1} := x_{j-1} - n$, $\tilde{x}_{j+1} := x_{j+1} + 1$, and $\tilde{x}_i = x_i$ otherwise. In the new representation the sum of digits is strictly smaller than in the previous one. This procedure can be repeated and in finitely many steps we obtain the $\beta$-expansion of $x$. The result follows easily, since in each step the number of digits in the fractional part of the representation does not increase.

Let us first determine a lower bound on $L_\beta(^\oplus)$. It suffices to find a single example of addition with specified fractional part length. We use the following example.

**Example 6.2.** Consider $x = m \sum_{i=0}^{k-1} \beta^{2i}$. Then it can be shown by induction on $k$ that

$$x + x = \sum_{i=0}^{k-1} (A_{k-i}\beta + B_{k-i})\beta^{2i} + \sum_{i=0}^{k-1} \left( \frac{a_{k-i}}{\beta} + \frac{b_{k-i}}{\beta^2} \right)\beta^{-2i},$$

where the coefficients $A_i$, $B_i$, $a_i$ and $b_i$, $i \in \mathbb{N}$, are defined by

$$A_i = i(m - n + 1) - m + n,$$

$$B_i = 2m - n - i(m - n + 1),$$

$$a_i = i(m - n + 1) - 1,$$

$$b_i = m + 1 - i(m - n + 1).$$

Formally, we have

$$x + x = A_1B_1A_2B_2 \ldots A_kB_k \cdot a_kb_k \ldots a_2b_2a_1b_1.$$
Corollary 6.3. Let $\beta$ be the larger solution of $x^2 = mx + n$, $m, n \in \mathbb{Z}$, $m \geq n > 0$. Then

$$L_\oplus(\beta) \geq \begin{cases} 2m & \text{if } m = n, \\ 2k_0 & \text{if } m > n. \end{cases}$$

From now on we focus on determining the upper bound for $L_\oplus(\beta)$.

Lemma 6.4. Let $x, y \in \mathbb{Z}_\beta$, $x, y \geq 0$, with $\beta$-expansions

$$(x)_\beta = x_l x_{l-1} \ldots x_1 x_0 \ast,$$

$$(y)_\beta = y_k y_{k-1} \ldots y_1 y_0 \ast,$$

where $y_i \leq m - n + 1$ for $i = 0, 1, \ldots, k - 2, k - 1$. Then the $\beta$-expansion of $x + y$ is

$$(x + y)_\beta = z_r z_{r-1} \ldots z_1 z_0 \ast z_{-1} z_{-2},$$

where

$$\frac{z_{-1}}{\beta} + \frac{z_{-2}}{\beta^2} \in \left\{ 0, \frac{n}{\beta}, \frac{m - n}{\beta} + \frac{n}{\beta^2} \left( = 1 - \frac{n}{\beta} \right) \right\}.$$ 

Proof. We make use of the relation $m + p = \beta + p - 1 + (m - n) \beta^{-1} + n \beta^{-2}$ for $p \leq m$, i.e. $(m + p)_\beta = 1(p-1) \ast (m-n)n$. Symbolically it may be rewritten as

$$
\begin{array}{c|cc|c}
\hline
& m & p & 1 \\
\hline
\beta & (p-1) & (m-n)n \\
\hline
\end{array}
$$

We proceed by induction on the values of $y$. Let $y = y_0 \leq m - n + 1$. Then, according to (8), the $\beta$-representation of $x + y$ is

$$(x + y)_\beta = \begin{cases} x_l \ldots x_1(x_0 + y_0) \ast & \text{if } x_0 + y_0 \leq m, \\ x_l \ldots (x_1 + 1)(x_0 + y_0 - m - 1) \ast (m-n)n & \text{if } x_0 + y_0 > m. \end{cases}$$

Note that $x_1 + 1 \leq m$ in the second case, since $x_1 = m$ implies $x_0 \leq n - 1$, and thus $x_0 + y_0 \leq n - 1 + m - n + 1 = m$, which is a contradiction.

Now assume that the statement holds for all $\tilde{y} < y$ satisfying the conditions of the lemma. Suppose that there exists an index $i$ such that $y_i > 0$ and $x_i < m$. Then $x + y = \tilde{x} + \tilde{y}$, where according to Lemma 6.1, $\tilde{x} = x + \beta^i \in \mathbb{Z}_\beta$ and $\tilde{y} = y - \beta^i$ satisfies the conditions of the lemma. We may thus use the induction hypothesis.

Suppose that $y_i > 0$ implies $x_i = m$ for all $i \leq k$. Since $x_l x_{l-1} \ldots x_1 x_0$ is an expansion, $x_i = m$ implies $x_{i-1} \leq n - 1 < m$. Thus $y_i > 0$ implies $y_{i-1} = 0$. Since $y_k > 0$, we have $x_k = m$ and $x_{k+1} < m$. Without loss of generality we can consider only the case when $l \leq k + 1$. Therefore we have
the following situation:

\[
\begin{array}{cccccccc}
  x_{k+1} & m & x_{k-1} & x_{k-2} & \ldots & x_1 & x_0 \\
  y_k & m & y_{k-2} & \ldots & y_1 & y_0
\end{array}
\]

Let \( j \) be the smallest integer among \( \{1, 2, \ldots, [k/2]\} \) such that \( y_{k-2j} < m - n + 1 \). Then

\[
\begin{array}{cccccccc}
  x_{k+1} & m & x_{k-1} & m & \ldots & x_{k-2j+3} & m & x_{k-2j+1} & x_{k-2j} & x_{k-2j-1} & \ldots & x_0 \\
  y_k & 0 & m - n + 1 & \ldots & 0 & m - n + 1 & 0 & y_{k-2j} & y_{k-2j-1} & \ldots & y_0
\end{array}
\]

We may check by elementary algebra using the relation \( \beta^2 = m\beta + n \) that

\[
(9) \quad m\beta^k + (m - n + 1) \sum_{i=1}^{j-1} \beta^{k-2i} = \beta^{k+1} - \beta^k + (m - n + 1)\beta \sum_{i=1}^{j-1} \beta^{k-2i} + (m - n)\beta^{k-2j+1} + n\beta^{k-2j}.
\]

Using this relation, we may write the sum \( x + y = \tilde{x} + \tilde{y} \) as

\[
(\tilde{x}_{k+1} + 1) (\tilde{y}_{k-1}) \quad m \quad \ldots \quad \tilde{x}_{k-2j+3} \quad m \quad \tilde{x}_{k-2j+1} \quad x_{k-2j} \quad x_{k-2j-1} \quad \ldots \quad x_0 \\
(\tilde{y}_{k-2j} + n) \quad y_{k-2j-1} \quad \ldots \quad y_0
\]

where \( \tilde{x}_{k-2i+1} = x_{k-2i+1} + m - n + 1 \) for \( i = 1, \ldots, j - 1 \) and \( \tilde{x}_{k-2j+1} = x_{k-2j+1} + m - n \). The first row represents the summand \( \tilde{x} \), the second row the summand \( \tilde{y} \). Due to (9) we have \( x + y = \tilde{x} + \tilde{y} \). Obviously \( \tilde{x}, \tilde{y} \in \mathbb{Z}_\beta \), the digits of \( \tilde{y} \) are \( \leq m - n + 1 \), except its first non-zero digit from the left. We have \( \tilde{y} < y \) and thus we may use the induction hypothesis.

It remains to solve the case where \( y_{k-2i} = m - n + 1 \) for all \( i \in \{1, 2, \ldots, [k/2]\} \). Then either

\[
y = y_k 0 (m - n + 1) 0 (m - n + 1) \ldots 0 (m - n + 1),
\]

or

\[
y = y_k 0 (m - n + 1) 0 (m - n + 1) \ldots 0 (m - n + 1) 0,
\]

i.e.

\[
y = y_k \beta^k + (m - n + 1) \sum_{i=1}^{[k/2]} \beta^{k-2i}
\]

for \( k \) even or odd. We may deduce from (9) that the results of the addition \( x + y \) have fractional parts \( 1 - n/\beta \) and \( n/\beta \) respectively. This completes the proof. \( \blacksquare \)
Lemma 6.5. Let $x, y \in \mathbb{Z}_\beta$, $x > y \geq 0$. Then

$$x - y = \begin{cases} 
z + (n - 1) \sum_{i=0}^{k} \beta^{2i} + 1 & \text{with } z \in \mathbb{Z}_\beta, z \geq 0, k \geq 0. \\
z + (n - 1) \sum_{i=1}^{k} \beta^{2i-1} + \frac{n}{\beta} & 
\end{cases}$$

Proof. First note that for every $x \in \mathbb{Z}_\beta$ there exists a $\beta$-representation $(x)_\beta = x_l \ldots x_1 x_0 \cdot$ such that $x_i + x_{i-1} > 0$ for all $0 < i \leq l$, i.e. the $\beta$-representation is “dense”. The dense form can be found by the following procedure: Find the first pair of zeros from the left, say $x_i = x_{i-1} = 0$, $x_{i+1} > 0$. Put $\tilde{x}_{i+1} = x_{i+1} - 1$, $\tilde{x}_i = m$, $\tilde{x}_{i-1} = n$, and $\tilde{x}_j = x_j$ for all other $0 \leq j \leq l$. The new $\beta$-representation $(x)_\beta = \tilde{x}_l \ldots \tilde{x}_1 \tilde{x}_0 \cdot$ has strictly lower number of vanishing coefficients. Thus the procedure is finite.

The proof of the lemma is by induction on the value of $y$. Without loss of generality we may assume that both $(x)_\beta = x_l \ldots x_1 x_0 \cdot$ and $(y)_\beta = y_k \ldots y_1 y_0 \cdot$ are written in their dense form.

Assume that there is an index $i$ such that both $x_i$ and $y_i$ are non-zero. Then $x - y = \tilde{x} - \tilde{y}$, where $\tilde{x} = x - \beta^i$ and $\tilde{y} = y - \beta^i$. Clearly, $\tilde{x}, \tilde{y} \in \mathbb{Z}_\beta$ and $\tilde{y} < y$, thus we may use the induction hypothesis.

Assume that $y_i > 0$ implies $x_i = 0$ for all indices $i$. Since $x_i + x_{i-1} > 0$, we have $y_{i-1} = 0$. Since $y_k > 0$, we have $x_k = 0$ and $x_{k+1} > 0$. Without loss of generality we consider $l = k + 1$ and $x_{k+1} = 1$. Since both $x$ and $y$ are in their dense form, the remaining cases are as follows. First assume that the maximal index $k$ such that $y_k$ is non-zero, is even. We have $x - y$ equal to

| 1 0 $x_{k-1}$ 0 $x_{k-3}$ \ldots $x_1$ 0 |
|---|---|---|---|---|---|---|
| $-y_k$ 0 0 0 0 0 $y_0$ |
| 1 0 $x_{k-1}$ 0 $x_{k-3}$ \ldots $x_1$ 0 |
| $-1$ 0 0 0 0 0 0 |
| $+m (n-1)$ $m (n-1)$ \ldots $(n-1)$ $m n$ |
| $-y_k$ 0 0 0 0 $y_0$ |
| $(m-y_k)$ $x_{k-1}$ $(m-y_{k-2})$ $x_{k-3}$ \ldots $x_1$ $(m-y_0)$ |
| $+(n-1)$ 0 $(n-1)$ \ldots $(n-1)$ 0 $n$ |

which corresponds to the statement of the lemma. For $k$ odd we may write
similarly that $x - y$ equals

\[
\begin{array}{cccccccc}
1 & 0 & x_{k-1} & 0 & x_{k-3} & \ldots & x_2 & 0 & x_0 \\
- & y_k & 0 & y_{k-2} & 0 & \ldots & 0 & y_1 & 0 \\
(m - y_k) & x_{k-1} & (m - y_{k-2}) & x_{k-3} & \ldots & x_2 & (m - y_1) & x_0 \\
+ & (n - 1) & 0 & (n - 1) & \ldots & (n - 1) & 0 & n \\
\end{array}
\]

which is of the desired form. ■

**Theorem 6.6.** Let $\beta$ be the larger solution of $x^2 = mx + n$, $m, n \in \mathbb{Z}$, $m \geq n > 0$. Then

\[ L_\oplus(\beta) = 2m \text{ if } m = n \]

and

\[ 2 \left\lfloor \frac{m + 1}{m - n + 1} \right\rfloor \leq L_\oplus(\beta) \leq 2 \left\lfloor \frac{m}{m - n + 1} \right\rfloor \text{ if } m > n. \]

**Proof.** Let $x, y \in \mathbb{Z}_\beta$, $xy > 0$. Every $y$ splits as $y = y_{(1)} + \ldots + y_{(s)}$, for some $s$, where the summands $y_{(i)}$ have digits $\leq m - n + 1$, and thus satisfy the assumptions of Lemma 6.4. We can always choose $y_{(i)}$ in such a way that the sum has at most

\[ s_0 := \left\lceil \frac{m}{m - n + 1} \right\rceil \]

non-vanishing summands. Lemma 6.4 then implies that

\[ \text{fp}_\beta(x + y) \leq 2s_0. \]

Now let $xy < 0$, without loss of generality $x > -y$. Then, according to Lemma 6.5, $x + y$ can be written either as $z + w$ for some $0 \leq z, w \in \mathbb{Z}_\beta$, or

\[ x + y = z + (n - 1) \sum_{i=1}^{k} \beta^{2i-1} + n/\beta \quad \text{for } 0 \leq z \in \mathbb{Z}_\beta. \]

The sum $(n - 1) \sum_{i=1}^{k} \beta^{2i-1}$ can be written as addition of $\left\lfloor \frac{n-1}{m-n+1} \right\rfloor = s_0 - 1$ summands with digits $\leq m - n + 1$. Therefore

\[ \text{fp}_\beta \left( z + (n - 1) \sum_{i=1}^{k} \beta^{2i-1} \right) \leq 2(s_0 - 1). \]

Adding $n/\beta$ to the result may yield only two more fractional digits (cf. Lemma 6.4).

Thus the proof for the upper bound on $L_\oplus(\beta)$ is finished. The lower bound of $L_\oplus(\beta)$ is given by Corollary 6.3. ■
The last two sections were devoted to the study of arithmetic of $\beta$-expansions for $\beta > 1$ a solution of $x^2 = mx + n$, $m, n \in \mathbb{N}$, $m \geq n$. This is the case where Theorem 3.3 does not provide us with any results, since $K = 0$. Let us comment on the results obtained in Sections 5 and 6:

1. The lower and upper bounds for $L_{\beta}(\beta)$ found in Theorem 6.6 differ at most by 2. They coincide if and only if

$$m - n + 1 \mid m \text{ or } m + 1.$$ Based on observation, we conjecture that for $m > n$ we actually have $L_{\beta}(\beta) = 2k_0$. We also note that for $m > n$ the results of subtraction $x - y$, where $x, y > 0$, have lower numbers of fractional digits than addition, more precisely, $\text{fp}_\beta(x - y) \leq 2k_0 - 1$.

2. According to Theorem 5.4 we may use the bound on $L_{\beta}(\beta)$ to derive an upper estimate on $L_{\circ}(\beta)$. For example for $m = n$ this gives

$$L_{\circ}(\beta) \leq 8m(\log_2(m + 2)).$$

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