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An analog of crank for a certain kind of partition function arising from the cubic continued fraction

by

Byungchan Kim (Seoul)

1. Introduction and statement of results. In a series of papers ([4]–[6]) H.-C. Chan studied congruence properties of a certain kind of partition function $a(n)$, which arises from Ramanujan's cubic continued fraction and is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}. \tag{1.1}$$

Here and below, we use the following standard q-series notation:

$$\begin{aligned}(a;q)_0 &:= 1,\\ (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n \geq 1,\\ (a;q)_\infty &:= \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.\end{aligned}$$

We can interpret $a(n)$ as the number of 2-color partitions of n with colors r and b subject to the restriction that color b appears only in even parts. For example, there are three such partitions of 2:

$$2_r, \ \ 2_b, \ \ 1_r + 1_r.$$

Since $a(n)$ is closely related to Ramanujan's cubic continued fraction (see [4]), we will say that $a(n)$ is the *number of cubic partitions* of n.

In particular, by using identities for the cubic continued fraction, Chan found a result analogous to "Ramanujan's most beautiful identity" (in the words of G. H. Hardy [17, p. xxxv]), namely,

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},$$

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where $p(n)$ is the number of ordinary partitions of n. Chan's identity is

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\frac{(q^3;q^3)_\infty (q^6;q^6)_\infty}{(q;q)_\infty^4 (q^2;q^2)_\infty}.$$

This implies immediately that

$$a(3n+2) \equiv 0 \pmod 3. \tag{1.2}$$

To give a combinatorial explanation of the famous Ramanujan partition congruences

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5,\\ p(7n+5) &\equiv 0 \pmod 7,\\ p(11n+6) &\equiv 0 \pmod{11}, \end{aligned}$$

G. E. Andrews and F. G. Garvan [2] introduced the crank of a partition. For a given partition λ, the *crank* $c(\lambda)$ of a partition is defined as

$$c(\lambda) := \begin{cases} \ell(\lambda) & \text{if } r = 0,\\ \omega(\lambda) - r & \text{if } r \geq 1, \end{cases}$$

where r is the number of 1's in λ, $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r, and $\ell(\lambda)$ is the largest part in λ.

Let $M(m,n)$ be the number of ordinary partitions of n with crank m. Andrews and Garvan showed that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)x^m q^n = (1-x)q + \frac{(q;q)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty}. \tag{1.3}$$

This equation is equivalent to

$$\frac{(q;q)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty} = 1 + (-1+x+x^{-1})q + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)x^m q^n.$$

Let $M(k,N,n)$ be the number of ordinary partitions of n with crank $\equiv k \pmod N$. In [2] and [9], Andrews and Garvan showed that for all $n \geq 0$,

$$\begin{aligned} M(i,5,5n+4) &= M(j,5,5n+4) \qquad \text{for all } 0 \leq i \leq j \leq 4,\\ M(i,7,7n+5) &= M(j,7,7n+5) \qquad \text{for all } 0 \leq i \leq j \leq 6,\\ M(i,11,11n+6) &= M(j,11,11n+6) \qquad \text{for all } 0 \leq i \leq j \leq 10. \end{aligned}$$

These identities clearly imply Ramanujan's congruences.

As Chan mentioned in his paper [6], it is natural to seek an analog of the crank of the ordinary partition to give a combinatorial explanation of (1.2). In light of (1.3), it is natural to conjecture that

$$F(x,q) = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty (xq^2;q^2)_\infty (x^{-1}q^2;q^2)_\infty} \tag{1.4}$$

gives an analogous crank for cubic partitions. In Section 2, we will review the crank of Andrews and Garvan of the ordinary partition function, and by giving a combinatorial interpretation of (1.4), we will define an analog c_a of crank. To this end, we need to extend the set of cubic partitions to a new set which we will call extended cubic partitions. Then we define a cubic crank c_a as a weighted count of extended cubic partitions according to a weight wt_a. (For the exact definition, see Section 2.) By using q-series identities, we will prove our first theorem.

THEOREM 1.1. *Let $M'(m, N, n)$ be the number of extended cubic partitions of n with cubic crank $\equiv m \pmod{N}$ counted according to the weight wt_a. Then*

$$M'(0, 3, 3n + 2) \equiv M'(1, 3, 3n + 2) \equiv M'(2, 3, 3n + 2) \pmod{3}$$

for all nonnegative integers n.

Since

$$a(n) = \sum_{k=0}^{N-1} M'(k, N, n),$$

this immediately implies the following corollary.

COROLLARY 1.2. *For all nonnegative integers n,*

$$a(3n + 2) \equiv 0 \pmod{3}.$$

Let us define

$$c_k := \begin{cases} \dfrac{7 \cdot 3^n + 1}{8} & \text{if } k \text{ is even,} \\ \dfrac{5 \cdot 3^n + 1}{8} & \text{if } k \text{ is odd.} \end{cases} \tag{1.5}$$

In [5], Chan proved the following congruences for cubic partitions, which are originally due to P. Eggan [7].

THEOREM 1.3 (Theorem 1 in [5]). *For all nonnegative n, $a(3^k n+c_k) \equiv 0 \pmod{3^{2\lfloor k/2\rfloor+1}}$.*

Surprisingly, our cubic crank can explain these congruences partially. To see this, we will prove the following theorem.

THEOREM 1.4. *For all nonnegative n,*

$$M'(0, 3, 3^k n + c_k) - M'(1, 3, 3^k n + c_k) \equiv 0 \pmod{3^{\lfloor k/2\rfloor+1}}.$$

By (2.7), Theorem 1.4 implies that

$$M'(0, 3, 3^k n + c_k) \equiv M'(1, 3, 3^k n + c_k) \equiv M'(2, 3, 3^k n + c_k) \pmod{3^{\lfloor k/2\rfloor+1}}.$$

Moreover, from Theorem 1.3, we find that

$$M'(0, 3, 3^k n+c_k) \equiv M'(1, 3, 3^k n+c_k) \equiv M'(2, 3, 3^k n+c_k) \equiv 0 \pmod{3^{\lfloor k/2\rfloor}}.$$

Therefore, we can see that the cubic crank gives a combinatorial explanation for the congruences

$$a(3^k n + c_k) \equiv 0 \pmod{3^{\lfloor k/2 \rfloor + 1}}$$

for all nonnegative integers n. Though this cubic crank does not give a full explanation for Theorem 1.3, as far as the author knows, this is the first crank which explains infinitely many congruences for a fixed arithmetic progression. In Section 3, we will review basic properties of modular forms. With this equipment, we will prove Theorem 1.4 in Section 4.

In [14], K. Mahlburg proved that there are infinitely many arithmetic progressions $An + B$ such that

$$M(m, \ell^j, An + B) \equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0 \le m \le \ell^j - 1$, where $\ell \ge 5$ is a prime and τ, j are positive integers. This implies that $p(An + B) \equiv 0 \pmod{\ell^\tau}$.

By using the theory of modular forms, in Section 4, we will prove our third theorem, which is analogous to Mahlburg's result.

THEOREM 1.5. *There are infinitely many arithmetic progressions $An+B$ such that*

$$M'(m, \ell^j, An + B) \equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0 \le m \le \ell^j - 1$, where $\ell \ge 5$ is a prime and τ, j are positive integers.

2. A cubic crank for $a(n)$. Before defining a cubic crank, we need to introduce some notation and review the definition of the crank of ordinary partitions. After Andrews and Garvan [2], for a partition λ, we denote by $\#(\lambda)$ the number of parts in λ, and by $\sigma(\lambda)$ the sum of the parts of λ, with the convention $\#(\lambda) = \sigma(\lambda) = 0$ for the empty partition λ. Let $\mathcal{P}$ be the set of all ordinary partitions, and $\mathcal{D}$ be the set of all partitions into distinct parts. We define

$$V = \{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1 \in \mathcal{D} \text{ and } \lambda_2, \lambda_3 \in \mathcal{P}\}.$$

For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, we define the sum of parts s, a weight w, and a crank t, by

$$\begin{aligned} s(\lambda) &= \sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3), \\ w(\lambda) &= (-1)^{\#(\lambda_1)}, \\ t(\lambda) &= \#(\lambda_2) - \#(\lambda_3). \end{aligned}$$

We say λ is a *vector partition* of n if $s(\lambda) = n$. Let $N_V(m, n)$ denote the number of vector partitions of n with crank m counted according to the weight w, so that

$$N_V(m,n) = \sum_{\substack{\lambda\in V\\ s(\lambda)=n\\ t(\lambda)=m}} w(\lambda).$$

Then

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V(m,n)x^m q^n = \frac{(q;q)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty}. \tag{2.1}$$

By putting $x=1$ in (2.1) we find

$$\sum_{m=-\infty}^{\infty} N_V(m,n) = p(n).$$

Andrews and Garvan showed that this vector crank actually gives the crank for ordinary partitions.

THEOREM 2.1 (Theorem 1 in [2]). *For all $n>1$, $M(m,n)=N_V(m,n)$.*

Now, we are ready to define a cubic crank for cubic partitions. For a given cubic partition λ, we define λ_r to be the partition that consists of all parts with color r, and λ_b to be the partition that is formed by dividing each of the parts with color b by 2. The generating function (1.4) suggests that it is natural to define an analog $N_V^a(m,n)$ of vector crank as

$$N_V^a(m,n) = \sum_{\substack{\lambda_r,\lambda_b\in V\\ s(\lambda_r)+2s(\lambda_b)=n\\ t(\lambda_r)+t(\lambda_b)=m}} w(\lambda_r)w(\lambda_b).$$

Then

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N_V^a(m,n)x^m q^n = \frac{(q;q)_\infty (q^2;q^2)_\infty}{(xq;q)_\infty (x^{-1}q;q)_\infty (xq^2;q^2)_\infty (x^{-1}q^2;q^2)_\infty}. \tag{2.2}$$

By putting $x=1$ in (2.2), we find

$$\sum_{m=-\infty}^{\infty} N_V^a(m,n) = a(n).$$

Since Theorem 2.1 does not hold when $n=1$, we need to extend the set of partitions $\mathcal{P}$ to a new set $\mathcal{P}^*$ by adding two additional copies of the partition 1, say 1^* and 1^{**}. We identify these three partitions of 1 with the three vector partitions of 1:

$$1 = ((1),\emptyset,\emptyset), \qquad 1^* = (\emptyset,(1),\emptyset), \qquad 1^{**} = (\emptyset,\emptyset,(1)),$$

so that

$$\mathcal{P}^* = \{(\emptyset),\ (1),\ (1^*),\ (1^{**}),\ (1,1),\ (2),\ (1,1,1),\ (1,2),\ (3),\ \dots\}.$$

We define the weight $\mathrm{wt}(\lambda)$ for $\lambda \in \mathcal{P}^*$ by

$$\mathrm{wt}(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mathcal{P}, \\ w(\lambda) & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned} \mathrm{wt}(1) &= \mathrm{wt}((1), \emptyset, \emptyset) = -1, \\ \mathrm{wt}(1^*) &= \mathrm{wt}(\emptyset, (1), \emptyset) = 1, \\ \mathrm{wt}(1^{**}) &= \mathrm{wt}(\emptyset, \emptyset, (1)) = 1. \end{aligned}$$

We extend the definition of the crank function $c(\lambda)$ to $\mathcal{P}^*$ by

$$c^*(\lambda) = \begin{cases} c(\lambda) & \text{if } \lambda \in \mathcal{P}, \\ t(\lambda) & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned} c^*(1) &= t((1), \emptyset, \emptyset) = 0, \\ c^*(1^*) &= t(\emptyset, (1), \emptyset) = 1, \\ c^*(1^{**}) &= t(\emptyset, \emptyset, (1)) = -1. \end{aligned}$$

The sum-of-parts function $\sigma(\lambda)$ is extended in the natural way:

$$\sigma^*(\lambda) = \begin{cases} \sigma(\lambda) & \text{if } \lambda \in \mathcal{P}, \\ s(\lambda) & \text{otherwise.} \end{cases}$$

In this way we see that

$$\begin{aligned} \frac{(q;q)_\infty}{(xq;q)_\infty\,(x^{-1}q;q)_\infty} &= 1 + (-1 + x + x^{-1})q + (x^2 + x^{-2})q^2 \\ &\quad + (x^3 + 1 + x^{-3})q^3 + \cdots \\ &= \sum_{\lambda \in \mathcal{P}^*} \mathrm{wt}(\lambda) x^{c^*(\lambda)} q^{\sigma^*(\lambda)}. \end{aligned}$$

Now we need to extend the definition of cubic partition. Note that we may identify a cubic partition of n (i.e. a partition into two colors r and b where color b is only available for even parts) with an element of (λ_r, λ_b) in $\mathcal{P} \times \mathcal{P}$ such that $\sigma(\lambda_r) + 2\sigma(\lambda_b) = n$. We extend the definition of cubic partitions in the natural way by defining them to be elements of $\mathcal{P}^* \times \mathcal{P}^*$. For the set of extended cubic partitions we define the sum-of-parts function σ_a, weight function wt_a, and crank function c_a in the natural way. For $\lambda = (\lambda_r, \lambda_b) \in \mathcal{P}^* \times \mathcal{P}^*$ we set

$$\begin{aligned} \sigma_a(\lambda) &= \sigma^*(\lambda_r) + 2\sigma^*(\lambda_b), \\ \mathrm{wt}_a(\lambda) &= \mathrm{wt}(\lambda_r) \cdot \mathrm{wt}(\lambda_b), \\ c_a(\lambda) &= c^*(\lambda_r) + c^*(\lambda_b). \end{aligned}$$

Hence we can deduce that

$$\begin{aligned}
&\sum_{\lambda\in\mathcal{P}^*\times\mathcal{P}^*} \mathrm{wt}_a(\lambda)x^{c_a(\lambda)}q^{\sigma_a(\lambda)}\\
&\qquad= \sum_{\lambda_r\in\mathcal{P}^*} \mathrm{wt}(\lambda_r)x^{c^*(\lambda_r)}q^{\sigma^*(\lambda_r)}\cdot \sum_{\lambda_b\in\mathcal{P}^*} \mathrm{wt}(\lambda_b)x^{c^*(\lambda_b)}q^{2\sigma^*(\lambda_b)}\\
&\qquad= \Big(1+(-1+x+x^{-1})q+\sum_{\lambda_r\in\mathcal{P}} x^{c(\lambda_r)}q^{\sigma(\lambda_r)}\Big)\\
&\qquad\quad\times\Big(1+(-1+x+x^{-1})q^2+\sum_{\lambda_b\in\mathcal{P}} x^{c(\lambda_b)}q^{2\sigma(\lambda_b)}\Big)\\
&\qquad= \Big(1+(-1+x+x^{-1})q+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)x^mq^n\Big)\\
&\qquad\quad\times\Big(1+(-1+x+x^{-1})q^2+\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)x^mq^{2n}\Big)\\
&\qquad= \frac{(q;q)_\infty}{(xq;q)_\infty\,(x^{-1}q;q)_\infty}\cdot\frac{(q^2;q^2)_\infty}{(xq^2;q^2)_\infty\,(x^{-1}q^2;q^2)_\infty} = F(x,q).
\end{aligned}$$

We let $M'(m,n)$ be the number of extended cubic partitions of n with crank m counted according to the weight wt_a, so that

$$M'(m,n) = \sum_{\substack{\lambda\in\mathcal{P}^*\times\mathcal{P}^*\\ c_a(\lambda)=m,\,\sigma_a(\lambda)=n}} \mathrm{wt}_a(\lambda),$$

and

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M'(m,n)x^mq^n = F(x,q). \tag{2.3}$$

In summary, we have proven the following theorem.

THEOREM 2.2. *For all $n\geq 1$, $M'(m,n) = N_V^a(m,n)$.*

We let $M'(m,N,n)$ be the number of extended cubic partitions of n with crank congruent to m modulo N counted according to the weight wt_a, so that

$$M'(m,N,n) = \sum_{r\equiv m \pmod N} M'(r,n) = \sum_{\substack{\lambda\in\mathcal{P}^*\times\mathcal{P}^*\\ c_a(\lambda)\equiv m \pmod N,\,\sigma_a(\lambda)=n}} \mathrm{wt}_a(\lambda).$$

By letting $x=1$ in (2.3) we find that

$$\sum_{m=-\infty}^{\infty} M'(m,n) = a(n)$$

for all n.

Now, we are ready to give the proof for our first theorem.

Proof of Theorem 1.1. By a simple argument, we find that

$$\begin{aligned} F(\zeta,q) &= \frac{(q;q)_\infty(q^2;q^2)_\infty}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty(\zeta q^2;q^2)_\infty(\zeta^{-1}q^2;q^2)_\infty} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{2} M'(k,3,n)\zeta^k q^n, \end{aligned}$$

where ζ is a primitive third root of unity.

To find the coefficient of q^{3n+2} of $F(\zeta,q)$, we multiply the numerator and the denominator by $(q;q)_\infty(q^2;q^2)_\infty$. Then, we have

$$F(\zeta,q) = \frac{(q;q)_\infty^2(q^2;q^2)_\infty^2}{(q^3;q^3)_\infty(q^6;q^6)_\infty} \tag{2.4}$$

$$= \frac{(q;q^2)_\infty^2(q^2;q^2)_\infty(q^2;q^2)_\infty^3}{(q^3;q^3)_\infty(q^6;q^6)_\infty} \tag{2.5}$$

$$= \frac{(\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2})(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)})}{(q^3;q^3)_\infty(q^6;q^6)_\infty}.$$

For the last equality, we used the Jacobi triple product identity and Jacobi's identity. (See [3, pp. 12–14] for the proof of these identities.) Since $n^2 \equiv 0$ or $1 \pmod 3$ and $m(m+1) \equiv 0$ or $2 \pmod 3$, the coefficient of q^{3n+2} in $F(\zeta,q)$ is the same as the coefficient of q^{3n+2} in

$$\frac{(\sum_{n=-\infty}^{\infty}(-1)^n q^{9n^2})(\sum_{m=0}^{\infty}(-1)^{3m+1}(6m+3)q^{9m^2+9m+2})}{(q^3;q^3)_\infty(q^6;q^6)_\infty}. \tag{2.6}$$

Note that the coefficients of (2.6) are multiples of 3. Thus,

$$\sum_{k=0}^{2} M'(k,3,3n+2)\zeta^k = 3N$$

for some integer N. Since $1+\zeta+\zeta^2$ is a minimal polynomial in $\mathbb{Z}[\zeta]$, we must have

$$M'(0,3,3n+2) \equiv M'(1,3,3n+2) \equiv M'(2,3,3n+2) \pmod 3.$$

This completes the proof of Theorem 1.1. ∎

Before proceeding, we give an example pertaining to Theorem 1.1.

Example 2.3. There are five extended cubic partitions of 2. We represent these as elements (λ_r,λ_b) of $\mathcal{P}^* \times \mathcal{P}^*$:

(λ_r, λ_b)	$\mathrm{wt}_a(\lambda)$	$c_a(\lambda) \pmod 3$
$((1,1),\emptyset)$	$+1$	$-2 \equiv 1$
$((2),\emptyset)$	$+1$	$2 \equiv 2$
$(\emptyset,(1))$	-1	$0 \equiv 0$
$(\emptyset,(1^*))$	$+1$	$1 \equiv 1$
$(\emptyset,(1^{**}))$	$+1$	$-1 \equiv 2$

We see that

$$M'(0,3,2) = -1, \quad M'(1,3,2) = M'(2,3,2) = 2,$$
$$M'(0,3,2) \equiv M'(1,3,2) \equiv M'(2,3,2) \pmod 3.$$

From (2.4), we observe that

$$M'(1,3,n) = M'(2,3,n) \quad \text{for all } n \geq 1. \tag{2.7}$$

Thus, by (2.5), we arrive at

$$\sum_{n=0}^{\infty}(M'(0,3,n) - M'(1,3,n))q^n = \frac{(q;q)_\infty^2(q^2;q^2)_\infty^2}{(q^3;q^3)_\infty(q^6;q^6)_\infty}. \tag{2.8}$$

By (2.6) and the Jacobi triple product identity, we obtain

$$\sum_{n=0}^{\infty}(M'(0,3,3n+2) - M'(1,3,3n+2))q^n = -3\frac{(q^3;q^3)_\infty^2(q^6;q^6)_\infty^2}{(q;q)_\infty(q^2;q^2)_\infty}. \tag{2.9}$$

Moreover, by using [8, (33.124)], we can deduce that

$$\sum_{n=0}^{\infty}(M'(0,3,9n+8) - M'(1,3,9n+8))q^n = -9\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q;q)_\infty^2(q^2;q^2)_\infty^2}.$$

These identities illuminate the possibility that there are further congruences modulo powers of 3 for cubic crank differences.

3. Preliminary results. This section contains the basic definitions and properties of modular forms that we will use in Section 4. For additional basic properties of modular forms, see [16, Chaps. 1, 2, and 3].

Define $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a \equiv d \equiv 1 \pmod N \text{ and } c \equiv 0 \pmod N \right\}.$$

For a meromorphic function f on the complex upper half plane $\mathbb{H}$, define the Slash operator by

$$f|_k\begin{pmatrix} a & b \\ c & d\end{pmatrix} := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Let $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$) denote the vector space of weakly holomorphic forms (resp. cusp forms) of weight k. Let $\mathcal{M}_k(\Gamma_0(N),\chi)$ (resp. $\mathcal{S}_k(\Gamma_0(N),\chi)$) denote the vector space of weakly holomorphic forms (resp. cusp forms) on $\Gamma_0(N)$ with character χ. For a prime p and a positive integer m, we need to define the Hecke operators T_p, the U_m-operator and the V_m-operator on $\mathcal{M}_k(\Gamma_0,\chi)$. If $f(q)$ has a Fourier expansion $f(q)=\sum a(n)q^n$, then

$$f|T_p := \sum\left(a(pn)+\chi(p)p^{k-1}a\left(\frac{n}{p}\right)\right)q^n,$$

$$f|U_m := \sum a(mn)q^n = m^{k/2-1}\sum_{v=0}^{m-1} f|_k\begin{pmatrix} 1 & v \\ 0 & m\end{pmatrix},$$

$$f|V_m := \sum a(n)q^{mn}.$$

Recall that the *Dedekind eta function* $\eta(z)$ is defined by

$$\eta(z) = q^{1/24}(q;q)_\infty, \tag{3.1}$$

where $q=\exp(2\pi i z)$ and $z\in\mathbb{H}$. For a fixed N and integers r_i, a function of the form

$$f(z) := \prod_{\substack{n|N\\ n>0}} \eta(nz)^{r_n}$$

is called an *η-quotient*. The following theorem of [15] shows when an η-quotient becomes a modular function.

THEOREM 3.1. *The η-quotient is in $\mathcal{M}_0(\Gamma_0(N))$ if and only if*

(1) $\sum_{n|N} r_n = 0$,
(2) $\sum_{n|N} nr_n \equiv 0 \pmod{24}$,
(3) $\sum_{n|N}(N/n)r_n \equiv 0 \pmod{24}$,
(4) $\prod_{n|N} n^{r_n}$ *is a square of a rational number.*

The following theorem of [13] gives the order of the η-quotient f at the cusps c/d of $\Gamma_0(N)$ provided $f\in\mathcal{M}_0(\Gamma_0(N))$.

THEOREM 3.2. *If the η-quotient f is in $\mathcal{M}_0(\Gamma_0(N))$, then its order at the cusp c/d of $\Gamma_0(N)$ is*

$$\frac{1}{24}\sum_{n|N}\frac{N(d,n)^2 r_n}{(d,N/d)dn}.$$

Recall that if $p \mid N$ and $f \in \mathcal{M}_0(\Gamma_0(pN))$, then $U_p f \in \mathcal{M}_0(\Gamma_0(N))$. Also, the following theorem of [10] gives bounds on the order of $f|U_p$ at cusps of $\Gamma_0(N)$ in terms of the order of f at cusps of $\Gamma_0(pN)$.

THEOREM 3.3. *Let p be a prime and $\pi(n)$ be the highest power of p dividing n. Suppose that $f \in \mathcal{M}_0(\Gamma_0(pN))$, where $p \mid N$ and $\alpha = c/d$ is a cusp of $\Gamma_0(N)$. Then*

$$\operatorname{ord}_\alpha f|U_p \geq \begin{cases} \dfrac{1}{p} \operatorname{ord}_{\alpha/p} f & \text{if } \pi(d) \geq \pi(N)/2, \\ \operatorname{ord}_{\alpha/p} f & \text{if } 0 < \pi(d) < \pi(N)/2, \\ \min\limits_{0 \leq \beta \leq p-1} \operatorname{ord}_{(\alpha+\beta)/p} f & \text{if } \pi(d) = 0. \end{cases}$$

The following eta-quotient $E_{\ell,t}(z)$ will play an important role in our proof. Given a prime $\ell \geq 5$ and a positive integer t, we define

$$E_{\ell,t}(z) = \frac{\eta^{\ell^t}(z)}{\eta(\ell^t z)}.$$

The following lemma summarizes necessary and well-known properties of $E_{\ell,t}(z)$.

LEMMA 3.4. *The eta-quotient $E_{\ell,t}$ satisfies:*

(i) *For a prime $\ell \geq 5$,*

$$E_{\ell,t}(z) \in \mathcal{M}_{(\ell^t-1)/2}(\Gamma_0(\ell^t), \chi_{\ell,t}),$$

where $\chi_{\ell,t}(\cdot) = \big(\frac{(-1)^{(\ell^t-1)/2}\ell^t}{\cdot}\big)$ denotes the Legendre–Jacobi symbol,

(ii) $E_{\ell,t}(z)^{\ell^j} \equiv 1 \pmod{\ell^{j+1}}$ *for $j \geq 0$,*

(iii) *$E_{\ell,t}(z)$ vanishes at every cusp a/c with $\ell^t \nmid c$.*

The following theorem is a slightly modified version of Serre's famous theorem of [19], which is an integer weight version of Theorem 2.2 of [14].

THEOREM 3.5. *For $0 \leq i \leq r$, let N_i and k_i be positive integers and let $g_i \in \mathcal{S}_{k_i}(\Gamma_1(N_i))$, where the Fourier coefficients of g_i are algebraic integers. If $M \geq 1$, then a positive proportion of primes $p \equiv -1 \pmod{N_1 \cdots N_r M}$ have the property that for every i,*

$$g_i(z)|T_p \equiv 0 \pmod{M}.$$

If $\zeta = \exp(2\pi i/N)$, then for $1 \leq s \leq N-1$, we define the $(0, s)$-*Klein form* by

$$t_{0,s}(z) = \frac{\omega_s}{2\pi i} \frac{(\zeta^s q; q)_\infty (\zeta^{-s} q; q)_\infty}{(q;q)_\infty^2} \quad \text{for } 1 \leq s \leq N-1, \tag{3.2}$$

where $\omega_s := \zeta^{s/2}(1-\zeta^{-s})$.

The following proposition gives a transformation formula under $\Gamma_0(N)$.

PROPOSITION 3.6 (Proposition 3.2 in [14], eqn. **K2** (p. 28) in [12]). *If* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, *then*

$$t_{0,s}(z)|_{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \cdot t_{0,\overline{ds}}(z), \tag{3.3}$$

where β *is given by* $e\Big(\frac{cs+(ds-\overline{ds})}{2N} - \frac{cds^2}{2N^2}\Big)$ *and* $e(z) = \exp(2\pi i z)$. *Here* $\overline{ds}$ *is the least nonnegative residue of* ds *modulo* N.

For certain congruence subgroups, a Klein form is a weakly holomorphic modular form.

LEMMA 3.7 (Corollary 3.3 of [14]). *If* $1 \le s \le N-1$, *then* $t_{0,s}(z) \in \mathcal{M}_{-1}(\Gamma_1(2N^2))$.

4. Proof of Theorems 1.4 and 1.5. Since we will follow the argument of B. Gordon and K. Hughes [10] for the proof of Theorem 1.4, we do not give every detail for the proof.

Proof of Theorem 1.4. Let

$$C(q) = \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^2}{(q^3;q^3)_\infty (q^6;q^6)_\infty}$$

and for $k \ge 0$ define

$$D_k(q) = \sum_{n=0}^{\infty} \gamma_3(3^k n + c_k) q^n, \quad \text{where} \quad \gamma_3(n) = M'(0,3,n) - M'(1,3,n),$$

so that equation (2.9) can be written as $D_1(q) = C(q)$. Define

$$F(z) := \frac{\eta^2(z)\eta^2(2z)\eta(27z)\eta(54z)}{\eta(3z)\eta(6z)\eta^2(9z)\eta^2(18z)}, \tag{4.1}$$

$$G(z) := \frac{\eta(9z)\eta(18z)}{\eta(z)\eta(2z)}. \tag{4.2}$$

Then, by Theorem 3.1, $F(z) \in \mathcal{M}_0(\Gamma_0(54))$, and $G(z) \in \mathcal{M}_0(\Gamma_0(18))$.

We also define a sequence of functions L_k ($k \ge 0$) inductively, by

$$L_0 := 1, \quad L_{2k+1} = FL_{2k}|U_3, \quad L_{2k+2} = L_{2k+1}|U_3.$$

Then one can show easily by induction that

$$D_{2k}(q) = C(q)L_{2k}(q), \quad D_{2k+1}(q) = C(q^3)L_{2k+1}(q).$$

By Theorem 3.2, the orders for $F(z)$ and $G(z)$ as a modular function of level 54 at the cusps are as follows.

d	1	2	3	6	9	18	27	54
$\operatorname{ord} F$	5	5	-1	-1	-2	-2	1	1
$\operatorname{ord} G$	-3	-3	0	0	1	1	1	1

Note that $G^i|U_3$, $FG^i|U_3$, $G(z) \in \mathcal{M}_0(\Gamma_0(18))$. By Theorem 3.3, their orders at the cusps are as follows.

d	1	2	3	6	9	18
$\operatorname{ord} G$	-1	-1	0	0	1	1
$\operatorname{ord} G^i\|U_3 \geq$	$-3i$	$-3i$	$\frac{i}{3}$	$\frac{i}{3}$	$\frac{i}{3}$	$\frac{i}{3}$
$\operatorname{ord} FG^i\|U_3 \geq$	$\min\{5-3i,-1\}$	$\min\{5-3i,-1\}$	$\frac{i-2}{3}$	$\frac{i-2}{3}$	$\frac{i+1}{3}$	$\frac{i+1}{3}$

By comparing the order at the cusps, we can see that $(F|U_3)/G$ is a holomorphic modular function, i.e. a constant. Hence,

$$F|U_3 = -3G.$$

Remark 4.1. This can be proved by an elementary argument by using (2.8) and (2.9).

By using a similar argument, we can see the following:

$$\begin{aligned}
G|U_3 &= 3G + 9G^2 + 27G^3,\\
G^2|U_3 &= 2G + 33G^2 + 180G^3 + 729G^4 + 1458G^5 + 2187G^6,\\
G^3|U_3 &= G + 30G^2 + 414G^3 + 2916G^4 + 14580G^5 + 48114G^6\\
&\qquad + 118098G^7 + 177147G^8 + 177147G^9,\\
FG|U_3 &= -G,\\
FG^2|U_3 &= G,\\
FG^3|U_3 &= 3G^2 + 9G^3 + 27G^4.
\end{aligned}$$

By using Newton's formula, we obtain, for $i \geq 3$, a recurrence formula for $G^i|U_3$,

$$G^i|U_3 = \sigma_1 G^{i-1}|U_3 - \sigma_2 G^{i-2}|U_3 + \sigma_3 G^{i-3}|U_3,$$

where $\sigma_1 = 9G + 27G^2 + 81G^3$, $\sigma_2 = -3G - 9G^2 - 27G^3$, and $\sigma_3 = G + 3G^2 + 9G^3$. Since $FG^i|U_3$ satisfies the same recurrence formula, for all $i \geq 1$, we can write $G^i|U_3$ and $FG^i|U_3$ as linear sums of G^i's,

$$G^i|U_3 = \sum_{j=1}^{\infty} a_{i,j} G^j \quad \text{and} \quad FG^i|U_3 = \sum_{j=1}^{\infty} b_{i,j} G^j, \tag{4.3}$$

where $a_{i,j}$ and $b_{i,j}$ are integers. Thus, by (4.3), each L_k for $k \geq 1$ is a linear sum of G^i. If $L_k = \sum_{j=1}^{\infty} l_j(k) G^j$, we will write $L_k = (l_1(k), l_2(k), \ldots)$. By setting $A := (a_{i,j})$ and $B := (b_{i,j})$, we obtain

$$\begin{aligned}
L_1 &= -3G = (-3, 0, 0, \ldots),\\
L_{2k+1} &= (-3, 0, 0, \ldots)(AB)^k,\\
L_{2k+2} &= (-3, 0, 0, \ldots)(AB)^k A.
\end{aligned}$$

Thus Theorem 1.4 will follow once we show that for all $k \geq 0$

$$(4.4)\qquad \pi(l_j(2k+1)) \geq k+1+\left\lfloor \frac{j}{2} \right\rfloor, \qquad \pi(l_j(2k+2)) \geq k+1+\left\lfloor \frac{j+1}{2} \right\rfloor,$$

where $\pi(n)$ is the 3-adic order of n. By using recurrence formulas for $G^i|U_3$ and $FG^i|U_3$ and induction, we find that

$$\pi(a_{i,j}) \geq \left\lfloor \frac{3j-i+1}{3} \right\rfloor \quad \text{and} \quad \pi(b_{i,j}) \geq \left\lfloor \frac{3j-i+1}{3} \right\rfloor.$$

From this, again by induction, we can derive (4.4), which completes the proof of Theorem 1.4. ■

Now we turn to the proof of Theorem 1.5. For the rest of this section, we set $N := \ell^j$, where ℓ is a fixed prime ≥ 5, and j is a fixed positive integer. Since our proof follows the works of K. Ono and S. Ahlgren ([1], [16]) and Mahlburg [14], we will not give every detail of each step.

Recall that

$$F(x,q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M'(m,n) x^m q^n,$$

where $q = \exp(2\pi i z)$ and $z \in \mathbb{H}$. Then, by a simple argument,

$$(4.5)\qquad \sum_{n=0}^{\infty} M'(m,N,n) q^n = \frac{1}{N} \sum_{s=0}^{N-1} F(\zeta^s, z) \zeta^{-ms},$$

where $\zeta = \exp(2\pi i/N)$.

From (3.1) and (3.2), we deduce that

$$(4.6)\qquad F(\zeta^s, z) = \frac{-\omega_s^2 q^{1/8}}{4\pi^2} \frac{1}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)}.$$

Therefore, by (4.5) and (4.6),

$$\sum_{n=0}^{\infty} N \cdot M'(m,N,n) q^n = \frac{-1}{4\pi^2} \sum_{s=1}^{N-1} \frac{\omega_s^2 \zeta^{-ms} q^{1/8}}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)} + \sum_{n=0}^{\infty} a(n) q^n.$$

Remark 4.2. We have multiplied (4.5) by N to ensure that the Fourier coefficients of

$$\frac{-1}{4\pi^2} \sum_{s=1}^{N-1} \frac{\omega_s^2 \zeta^{-ms} q^{1/8}}{\eta(z)\eta(2z)t_{0,s}(z)t_{0,s}(2z)}$$

are algebraic integers with a view to applying Theorem 3.5.

Define $\delta_\ell = (\ell^2-1)/24$ and $\overline{\delta}_\ell = 3\delta_\ell$. We also define

$$(4.7)\qquad g_m(z) = \Big(\sum_{n=0}^{\infty} N \cdot M'(m,N,n) q^{n+\overline{\delta}_\ell}\Big) (q^\ell; q^\ell)_\infty^\ell (q^{2\ell}; q^{2\ell})_\infty^\ell.$$

Then

$$\begin{aligned} g_m(z) &= \frac{-1}{4\pi^2}\sum_{s=1}^{N-1}\frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)}\,\frac{\omega_s^2\zeta^{-ms}}{t_{0,s}(z)t_{0,s}(2z)}+\frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)} \\ &=: \frac{1}{4\pi^2}\sum_{s=1}^{N-1}G_{m,s}(z)+P(z). \end{aligned}$$

In [6], Chan proved that, for sufficiently large τ,

$$\left(\frac{P(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)}E_{\ell,1}^{\ell^\tau}\right)\bigg|V_8 \in S_k(\Gamma_0(128\ell,\chi)) \tag{4.8}$$

for some positive integer k and Dirichlet character χ. Here, we prove the following similar result.

THEOREM 4.3. *For sufficiently large τ, there is a positive integer k' such that*

$$\left(\frac{G_{m,s}(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)}E_{\ell,j+1}^{\ell^\tau}\right)\bigg|V_8 \in S_{k'}(\Gamma_1(128N^2)) \quad \textit{for all } 1\le s\le N-1. \tag{4.9}$$

Throughout the proof, we will use the notation

$$q_m=e^{2\pi iz/m}=q^{1/m} \quad\text{and}\quad \lambda=e^{2\pi i/\ell}.$$

Proof. First, note that $\eta^\ell(\ell z)/\eta(z)\in\mathcal{M}_{(\ell-1)/2}(\Gamma_0(\ell),(\frac{\cdot}{\ell}))$. Thus, by Lemma 3.7, $G_{m,s}(z)\in M_{\ell+1}(\Gamma_1(4N^2))$. Since $\eta(8z)\eta(16z)\in\mathcal{S}_1(\Gamma_1(128))$, the left side of (4.9) transforms correctly on $\Gamma_1(128N^2)$. By Lemma 3.4, if τ is sufficiently large, then we only need to show that $G_{m,s}(z)|U_\ell/(\eta^\ell(z)\eta^\ell(2z))$ vanishes at each cusp a/c with $\ell N\mid c$. Suppose that $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(\ell N)$. If c is even, then the result is a straightforward exercise and basically reduces to considering the cusp at infinity. At infinity, we have

$$G_{m,s}=\sum_{n=0}^{\infty}C(n)q^{n+(\ell^2-1)/8},$$

where $C(n)$'s are complex numbers. Thus the order of $G_{m,s}|U_\ell$ is greater than the order of $\eta^\ell(z)\eta^\ell(2z)$, which is $\ell/8$. Thus, from now on, we assume that c is odd. Since the Fourier expansion of $\eta^\ell(z)\eta^\ell(2z)$ at such cusps is of the form $B_0q_2^{\ell/8}+\cdots$, where B_0 is a nonzero constant, it suffices to show that the Fourier expansion of $G_{m,s}|U_\ell$ at such cusps is of the form $B_1q_2^h+\cdots$, where B_1 is a constant and $h>\ell/8$. Then

$$(G_{m,s}(z)|U_\ell)|_{\ell+1}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\ell^{(\ell-1)/2}\sum_{j=0}^{\ell-1}G_{m,s}(z)|_{\ell+1}\begin{pmatrix} 1 & j\\ 0 & \ell\end{pmatrix}|_{\ell+1}\begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

Note that, for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix},$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + cj & (-aj' - cjj' + b + dj)/\ell \\ c\ell & -cj' + d \end{pmatrix}.$$

By choosing $j' \in \{0, 1, \ldots, \ell - 1\}$ such that $-aj' + b + dj \equiv 0 \pmod{\ell}$, we have $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma_0(\ell N)$. Note that as j runs over a complete residue system modulo ℓ, so does j'. Thus,

$$(G_{m,s}(z)|U_\ell)|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \ell^{(\ell-1)/2}\sum_{j'=0}^{\ell-1} G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix}.$$

From the fact that

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 2a' & -a'v + b' \\ c' & (d' - c'v)/2 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & 2 \end{pmatrix}, \tag{4.10}$$

where

$$v = \begin{cases} 0 & \text{if } d' \text{ is even,} \\ 1 & \text{if } d' \text{ is odd,} \end{cases}$$

we deduce that, by setting $u = (z + v)/2$,

$$\begin{aligned} G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} &= \left(\frac{\eta^\ell(\ell z)\eta^\ell(2\ell z)}{\eta(z)\eta(2z)}\,\frac{\omega_s^2\zeta^{-ms}}{t_{0,s}(z)t_{0,s}(2z)}\right)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \\ &= \chi(d')\chi((d' - c'v)/2)\frac{\eta^\ell(\ell z)}{\eta(z)}\,\frac{\eta^\ell(\ell u)}{\eta(u)}\,\frac{\omega_s^2\zeta^{-ms}}{\beta t_{0,\overline{d's}}(z)\beta' t_{0,\overline{(d'-c'v)s/2}}(u)}, \end{aligned}$$

where β and β' are the roots of unity defined in Proposition 3.6, and $\chi(d) = \left(\frac{d}{\ell}\right)$. Since $\ell N \mid c$, by calculation we can check that $\chi(d')$, $\chi((d' - c'v)/2)$ and the products $\beta t_{0,\overline{d's}}(z)$, $\beta' t_{0,\overline{(d'-c'v)s/2}}(u)$ do not depend on j'. In summary, we obtain

$$G_{m,s}(z)|_{\ell+1}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = A_1 q_2^{\overline{\delta}_\ell}(-1)^{\overline{\delta}_\ell v}\Big(1 + \sum_{n\ge1} c_1(n, j')q_2^n\Big), \tag{4.11}$$

where A_1 is a nonzero constant not depending on j'.

Thus, we finally arrive at

$$
\begin{aligned}
&(G_{m,s}(z)|U_\ell)|_{\ell+1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&\qquad = A_1 \sum_{j'=0}^{\ell-1} \Big(q_2^{\overline{\delta}_\ell} (-1)^{\overline{\delta}_\ell v} \Big(1 + \sum_{n\geq 1} c_1(n,j') q_2^n \Big) \Big) \Big| \begin{pmatrix} 1 & j' \\ 0 & \ell \end{pmatrix} \\
&\qquad = A_2 q_{2\ell}^{\overline{\delta}_\ell} \sum_{j'=0}^{\ell-1} \lambda^{\overline{\delta}_\ell j'/2} (-1)^{\overline{\delta}_\ell v} \Big(1 + \sum_{n\geq 1} c_2(n,j') q_{2\ell}^n \Big) \\
&\qquad = A_2 q_{2\ell}^{\overline{\delta}_\ell} \sum_{j'=0}^{\ell-1} \lambda^{\overline{\delta}_\ell j'/2} (-1)^{\overline{\delta}_\ell v} + q_{2\ell}^{\overline{\delta}_\ell} \Big(\sum_{n\geq 1} c_3(n) q_{2\ell}^n \Big),
\end{aligned}
$$

where $\lambda = \exp(2\pi i/\ell)$. Since $1 + \overline{\delta}_\ell - \ell^2/8 > 0$, it suffices to show that

$$S := \sum_{j'=0}^{\ell-1} \lambda^{\overline{\delta}_\ell j'/2} (-1)^{\overline{\delta}_\ell v} = 0.$$

When $\overline{\delta}_\ell$ is even, it is obvious that $S = 0$. Now we assume that $\overline{\delta}_\ell$ is odd. Since $v \equiv -cj' + d \pmod 2$, we find that

$$S = \sum_{j'=0}^{\ell-1} \lambda^{\overline{\delta}_\ell j'/2} (-1)^{j} = \sum_{j'=0}^{\ell-1} \lambda^{j'(\overline{\delta}_\ell + \ell)/2}.$$

Since $(\overline{\delta}_\ell + \ell)/2$ is an integer that is relatively prime to ℓ, we have $S = 0$. This completes the proof. ∎

Now, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. We see that

$$g_m(z)|U_\ell = \Big(\sum_{n=0}^{\infty} N \cdot M'(m,N,n) q^{n+\overline{\delta}_\ell} \Big) \Big| U_\ell \cdot (q;q)_\infty^\ell (q^2;q^2)_\infty^\ell$$

and so

$$\frac{g_m(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)} = \sum_{n=0}^{\infty} N \cdot M'(m,N,\ell n - \overline{\delta}_\ell) q^{n-\ell/8}.$$

Thus, by Theorem 4.3, for sufficiently large t,

$$
\begin{aligned}
\Big(\frac{g_m(z)|U_\ell}{\eta^\ell(z)\eta^\ell(2z)} E_{\ell,j+1}^{\ell^t} \Big) \Big| V_8 &\equiv \sum_{\substack{n\geq 0 \\ \ell n \equiv -1 \pmod 8}} N \cdot M'\Big(m, N, \frac{\ell n + 1}{8} \Big) q^n \pmod{\ell^{\tau+j}} \\
&\equiv H_1 + H_2 \pmod{\ell^{\tau+j}},
\end{aligned}
$$

where $H_1 \in \mathcal{S}_{k'}(\Gamma_1(128N^2))$ and $H_2 \in \mathcal{S}_k(\Gamma_0(128\ell), \chi)$. Then, by Theorem 3.5, a positive proportion of primes $Q \equiv -1 \pmod{128N^2}$ have the property that

$$H_1|T_Q = H_2|T_Q \equiv 0 \pmod{\ell^{\tau+j}}.$$

This implies that

$$N \cdot M'\left(m, N, \frac{\ell n Q + 1}{8}\right) \equiv 0 \pmod{\ell^{\tau+j}} \quad \text{whenever } (n, Q) = 1.$$

This completes the proof of Theorem 1.5. ∎

5. Concluding remarks. From the numerical data, the following inequalities seem true:

$$\begin{aligned} M'(0,3,3n+1) &\leq M'(1,3,3n+1),\\ M'(0,3,3n+2) &\leq M'(1,3,3n+2),\\ M'(0,3,3n+3) &\geq M'(1,3,3n+3). \end{aligned}$$

These inequalities and other similar inequalities will be discussed in a forthcoming paper [11]. It would be nice to find a more natural combinatorial interpretation for the coefficients of $F(x,q)$ as in (1.4). It is also worthwhile to seek another crank which can explain cubic partition congruences modulo every power of 3. After the author completed writing his paper, F. Garvan informed him that another analog of crank for $a(n)$ was also studied by Z. Reti in his unpublished thesis [18], which explains cubic partition congruences modulo 3 and 9.

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Byungchan Kim
School of Liberal Arts
Seoul National University of Science and Technology
172 Gongreung 2 dong, Nowongu
Seoul, 139-743, Korea
E-mail: bkim4@seoultech.ac.kr

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