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On the integers not of the form $p + 2^a + 2^b$

by

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1. Introduction. As early as 1849, Polignac conjectured that every odd integer greater than 3 is the sum of a prime and a power of 2. Of course, Polignac's conjecture is not true, since 127 is an evident counterexample. In 1934, Romanoff [11] proved that the sumset

$$\{p + 2^b : p \text{ is prime}, b \in \mathbb{N}\}$$

has positive lower density. In the other direction, van der Corput [2] proved that the set

$$\{n \geq 1 : n \text{ is odd and not of the form } p + 2^b\}$$

also has positive lower density. In fact, with the help of covering congruences, Erdős [4] found that no positive integer n with $n \equiv 7629217 \pmod{11184810}$ is no representable as the sum of a prime and a power of 2.

In [3], Crocker proved that there exist infinitely many odd positive integers x not of the form $p + 2^a + 2^b$. One key to Crocker's proof is the following observation: If $b - a = 2^s t$ with $s \geq 0$ and $2 \nmid t$, then $2^a + 2^b \equiv 0 \pmod{2^{2^s} + 1}$. Crocker also constructed a suitable covering system to deal with the case $a = b$. In [13], Sun and Le considered integers not of the form $p^\alpha + c(2^a + 2^b)$. Subsequently, Yuan [15] proved that there exist infinitely many positive odd integers x not of the form $p^\alpha + c(2^a + 2^b)$.

Let

$$\mathcal{N} = \{n \geq 1 : n \text{ is odd and not of the form } p + 2^a + 2^b\},$$

$$\mathcal{N}_* = \{n \geq 1 : n \text{ is odd and not of the form } p^\alpha + 2^a + 2^b\}.$$

Erdős asked whether $|\mathcal{N} \cap [1, x]| \gg x^\epsilon$ for some $\epsilon > 0$. Granville and Soundararajan [6] mentioned that this is true under the assumption that there exist infinitely many $m_1 < m_2 < \cdots$ such that all $2^{2^{m_i}} + 1$ are composite and $\{m_{i+1} - m_i\}$ is bounded. Erdős even suggested [7, A19] that

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$|\mathcal{N}\cap[1,x]|\ge Cx$ for a constant $C>0$, though it seems that the covering congruences could not help here. In [1], Chen, Feng and Templier proved that

$$\limsup_{x\to\infty}\frac{|\mathcal{N}_*\cap[1,x]|}{x^{1/4}}=+\infty$$

if there exist infinitely many m such that $2^{2^m}+1$ is composite, and

$$\limsup_{x\to\infty}\frac{|\mathcal{N}_*\cap[1,x]|}{\sqrt{x}}>0$$

if there are only finitely many m such that $2^{2^m}+1$ is prime. Recently, in his answer to a conjecture of Sun, Poonen [10] gave a heuristic argument which suggests that for each odd $k>0$,

$$|\{1\le n\le x: n \text{ is odd and not of the form } p+2^a+k\cdot 2^b\}|\gg_{k,\epsilon}x^{1-\epsilon}$$

for any $\epsilon>0$, where $\gg_{k,\epsilon}$ means the implied constant only depends on k and ϵ.

On the other hand, using Selberg's sieve method, Tao [14] proved that for any $\mathcal{K}\ge 1$ and sufficiently large x, the number of primes $p\le x$ such that $|kp\pm ja^i|$ is composite for all $1\le a,j,k\le\mathcal{K}$ and $1\le i\le\mathcal{K}\log x$, is at least $C_{\mathcal{K}}x/\log x$, where $C_{\mathcal{K}}$ is a constant only depending on $\mathcal{K}$. Motivated by Tao's idea, in this short note, we shall unconditionally prove

Theorem 1.1.

$$|\mathcal{N}_*\cap[1,x]|\gg x\cdot\exp\biggl(-C\log x\cdot\frac{\log\log\log\log x}{\log\log\log x}\biggr),$$

where $C>0$ is an absolute constant.

Clearly Theorem 1.1 implies $|\mathcal{N}_*\cap[1,x]|\gg_\epsilon x^{1-\epsilon}$ for any $\epsilon>0$. The proof of Theorem 1.1 will be given in the next section. Unless indicated otherwise, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ are always absolute.

2. Proof of Theorem 1.1. Since

$$|\{1\le n\le x: n \text{ is of the form } p^\alpha+2^a+2^b \text{ with } \alpha\ge 2\}|=O(\sqrt{x}\log x),$$

we only need to show that

$$|\mathcal{N}\cap[1,x]|\gg x\cdot\exp\biggl(-C\log x\cdot\frac{\log\log\log\log x}{\log\log\log x}\biggr).$$

We need several auxiliary lemmas. The first lemma is a special case of the Brun–Titchmarsh theorem (cf. [8, Theorem 3.7]).

Lemma 2.1. *Suppose that $W\ge 1$ and β are integers with $(\beta,W)=1$. Then*

$$|\{1\le n\le x: Wn+\beta \textit{ is prime}\}|\le\frac{C_1x}{\log x}\prod_{p\mid W}\biggl(1-\frac{1}{p}\biggr)^{-1},$$

where C_1 is an absolute constant.

The next lemma is an easy application of the Selberg sieve method (cf. [8, Theorems 3.2 and 4.1], [9, Theorem 7.1]).

Lemma 2.2. *Suppose that x is a sufficiently large integer. Suppose that $p_1, \ldots, p_h$ are distinct primes less than $x^{1/8}$. Then*

$$|\{1 \le n \le x : n \not\equiv 0 \pmod{p_j} \text{ for every } 1 \le j \le h\}| \le C_2 x \prod_{j=1}^{h}\left(1 - \frac{1}{p_j}\right),$$

where C_2 is an absolute constant.

The third lemma is due to Ford, Luca and Shparlinski [5, Theorem 1].

Lemma 2.3. *The series*

$$\sum_{n=1}^{\infty} \frac{(\log n)^{\gamma}}{P(2^n - 1)}$$

converges for any $\gamma < 1/2$, where $P(n)$ denotes the largest prime factor of n.

Now we are ready to prove Theorem 1.1. Let

$$C_3 = \sum_{p \text{ prime}} \frac{1}{P(2^p - 1)}.$$

Suppose that x is sufficiently large. Let

$$K = \left\lfloor \frac{\log\log\log x}{100 \log\log\log\log x} \right\rfloor$$

and $L = \log(2^9 C_1 C_2 K) + 2C_3$, where $\lfloor \theta \rfloor = \max\{z \in \mathbb{Z} : z \le \theta\}$.

Let $u = e^{e^{K(L+1)}}$. By the Mertens theorem (cf. [9, Theorem 6.7]), we know that

$$\sum_{\substack{p \le u \\ p \,\text{prime}}} \frac{1}{p} = \log\log u + B + O\left(\frac{1}{\log u}\right) = K(L+1) + O(1),$$

where $B = 0.2614972\ldots$ is a constant. So we may choose some distinct odd primes less than u,

$$p_{1,1}, \ldots, p_{1,h_1}; p_{2,1}, \ldots, p_{2,h_2}; \ldots; p_{K,1}, \ldots, p_{K,h_K},$$

such that

$$\sum_{j=1}^{h_i} \frac{1}{p_{i,j}} \ge L$$

for $1 \le i \le K$. Let $q_{i,j} = P(2^{p_{i,j}} - 1)$ for $1 \le i \le K$ and $1 \le j \le h_i$. Clearly these $q_{i,j}$ are all distinct. Now,

$$\sum_{j=1}^{h_i} \log\left(1 - \frac{1}{p_{i,j}}\right) \le -\sum_{j=1}^{h_i} \frac{1}{p_{i,j}},$$

whence

$$\prod_{j=1}^{h_i}\left(1-\frac{1}{p_{i,j}}\right)\le e^{-L}.$$

Noting that $1+\theta\le e^{\theta}$ for $\theta\ge 0$, we have

$$\prod_{i=1}^{K}\prod_{j=1}^{h_i}\left(1-\frac{1}{q_{i,j}}\right)^{-1}\le\prod_{i=1}^{K}\prod_{j=1}^{h_i}\left(1+\frac{2}{q_{i,j}}\right)\le\exp\left(\sum_{i=1}^{K}\sum_{j=1}^{h_i}\frac{2}{q_{i,j}}\right)\le e^{2C_3}.$$

Let

$$W_{1,i}=\prod_{j=1}^{h_i}q_{i,j}\quad\text{for } 1\le i\le K,\qquad W_1=\prod_{i=1}^{K}W_{1,i}.$$

Then

$$W_1\le 2^{\sum_{i=1}^{K}\sum_{j=1}^{h_i}p_{i,j}}\le 2^{u^2/\log u},$$

since (cf. [12])

$$\sum_{\substack{p\le u\\ p\text{ prime}}}p=\left(\frac{1}{2}+o(1)\right)\frac{u^2}{\log u}.$$

Noting that for sufficiently large x,

$$\frac{\log\log\log(2^{u^2/\log u})}{\log\log\log(x^{1/K})}\le\frac{2K(L+1)}{\log(\log\log x-\log K)}\le 1,$$

we have $W_1\le x^{1/K}$.

Let $m=\lfloor\log_2\log_2(x^{2/(K-1)})\rfloor$ and $K'=1+\lfloor 2^{-m}\log_2 x\rfloor$, where $\log_2 x=\log x/\log 2$. We have

$$K'\le 1+\frac{\log_2 x}{2^m}\le 1+\frac{2\log_2 x}{2^{\log_2\log_2(x^{2/(K-1)})}}=1+\frac{2\log_2 x}{\frac{2}{K-1}\cdot\log_2 x}=K.$$

For each $k\ge 0$, let γ_k be the smallest prime factor of $2^{2^k}+1$. Let

$$W_2=\prod_{k=0}^{m-1}\gamma_k$$

and $W=W_1W_2$. It is not difficult to see that $(W_1,W_2)=1$. Moreover,

$$W\le W_1\prod_{k=0}^{m-1}(1+2^{2^k})\le x^{1/K}\cdot x^{2/(K-1)}\le x^{3/(K-1)}.$$

Let β be an odd integer such that

$$\beta\equiv 2^{2^m(i-1)}+1\ \left(\operatorname{mod}\prod_{j=1}^{h_i}q_{i,j}\right)\quad\text{and}\quad\beta\equiv 0\ (\operatorname{mod}\gamma_k)$$

for $1\le i\le K'$ and $0\le k\le m-1$.

Let

$$\mathcal{S} = \{1 \le n \le x : n \equiv \beta \pmod{2W}\}.$$

Clearly,

$$\frac{x}{2W} - 1 \le |\mathcal{S}| \le \frac{x}{2W} + 1.$$

Let

$$\mathcal{T}_1 = \{n \in \mathcal{S} : n \text{ is of the form } p + 2^a + 2^b \text{ with } p \mid W\},$$
$$\mathcal{T}_2 = \{n \in \mathcal{S} \setminus \mathcal{T}_1 : n \text{ is of the form } p + 2^a + 2^b \text{ with } p \nmid W\}.$$

Clearly $|\mathcal{T}_1| = O(W(\log x)^2)$.

Suppose that $n \in \mathcal{S}$ and $n = p + 2^a + 2^b$ with p prime and $0 \le a \le b$. If $a \not\equiv b \pmod{2^m}$, then $b = a + 2^s t$ where $0 \le s \le m - 1$ and $2 \nmid t$. Thus

$$p = n - 2^a(2^{2^s t} + 1) \equiv \beta - 2^a(2^{2^s} + 1)\sum_{j=0}^{t-1}(-1)^j 2^{2^s j} \equiv 0 \pmod{\gamma_s}.$$

Since p is prime, we must have $p = \gamma_s$, i.e., $n \in \mathcal{T}_1$.

Below we assume that $a \equiv b \pmod{2^m}$. Write $b - a = 2^m(t-1)$ where $1 \le t \le K'$. If $a \equiv 0 \pmod{p_{t,j}}$ for some $1 \le j \le h_t$, then recalling $2^{p_{t,j}} \equiv 1 \pmod{q_{t,j}}$, we have

$$p = n - 2^a(2^{2^m(t-1)} + 1) \equiv \beta - (2^{2^m(t-1)} + 1) \equiv 0 \pmod{q_{t,j}}.$$

So $p = q_{t,j}$ and $n \in \mathcal{T}_1$. On the other hand, for any $a \ge 0$ satisfying $a \not\equiv 0 \pmod{p_{t,j}}$ for all $1 \le j \le h_t$, i.e., $(a, W_{1,t}) = 1$, by Lemma 2.1, we have

$$\begin{aligned}&|\{n \in \mathcal{S} : n - 2^a(2^{2^m(t-1)} + 1) \text{ is prime}\}| \\ &\quad \le \frac{2C_1|\mathcal{S}|}{\log|\mathcal{S}|}\prod_{k=0}^{m-1}\left(1 - \frac{1}{\gamma_k}\right)^{-1}\prod_{i=1}^{K}\prod_{j=1}^{h_i}\left(1 - \frac{1}{q_{i,j}}\right)^{-1} \le \frac{2^5 C_1 e^{2C_3}}{W} \cdot \frac{x}{\log x}\end{aligned}$$

since $\gamma_k \equiv 1 \pmod{2^{k+1}}$ and $\gamma_k > 2^{k+1}$. Noting that

$$\frac{\log\log u}{\log\log((\log_2 x)^{1/8})} \le \frac{K(L+1)}{\log(\log\log x - \log\log 2 - \log 8)} < 1,$$

we have $u < (\log_2 x)^{1/8}$. By Lemma 2.2,

$$\begin{aligned}&|\{0 \le a \le \log_2 x : a \not\equiv 0 \pmod{p_{t,j}} \text{ for all } 1 \le j \le h_t\}| \\ &\qquad\qquad \le C_2\frac{\log x}{\log 2}\prod_{j=1}^{h_t}\left(1 - \frac{1}{p_{t,j}}\right) \le 2C_2 e^{-L}\log x.\end{aligned}$$

Thus

$$|\mathcal{T}_2| \le \sum_{t=1}^{K'} \sum_{\substack{0\le a\le \log_2 x\\ (a,W_{1,t})=1}} |\{n\in\mathcal{S} : n-2^a(2^{2^m(t-1)}+1) \text{ is prime}\}|$$
$$\le K\cdot \frac{2^5C_1e^{2C_3}}{W}\cdot\frac{x}{\log x}\cdot 2C_2e^{-L}\log x \le \frac{x}{4W}.$$

It follows that

$$|\{n\in\mathcal{S} : n \text{ is not of the form } p+2^a+2^b\}|$$
$$= |\mathcal{S}|-|\mathcal{T}_1|-|\mathcal{T}_2| \ge \frac{x}{2W}-1-O(W(\log x)^2)-\frac{x}{4W} \gg x^{1-4/K}.$$

The proof of Theorem 1.1 is complete. ∎

Remark. Using a similar discussion, it is not difficult to prove that for any given $\mathcal{K}\ge 1$,

$$|\{1\le n\le x : n \text{ is odd and } n\ne p+c(2^a+2^b) \text{ with } p \text{ prime},$$
$$a,b\ge 0,\ 1\le c\le \mathcal{K}\}|$$
$$\gg_{\mathcal{K}} x\cdot\exp\left(-C_{\mathcal{K}}\log x\cdot\frac{\log\log\log\log x}{\log\log\log x}\right),$$

where the constant $C_{\mathcal{K}}>0$ only depends on $\mathcal{K}$.

Recently, Professor Y.-G. Chen asked the author the following question:

Are there infinitely many positive odd numbers which are not divisible by 3 *and cannot be represented as* $p+2^a+2^b$*? Similar problems can be posed with* 3 *replaced by* 5, 17 *etc.*

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