On the integers not of the form $p + 2^a + 2^b$

by

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1. Introduction. As early as 1849, Polignac conjectured that every odd integer greater than 3 is the sum of a prime and a power of 2. Of course, Polignac’s conjecture is not true, since 127 is an evident counterexample. In 1934, Romanoff [11] proved that the sumset
\[ \{ p + 2^b : p \text{ is prime}, b \in \mathbb{N} \} \]
has positive lower density. In the other direction, van der Corput [2] proved that the set
\[ \{ n \geq 1 : n \text{ is odd and not of the form } p + 2^b \} \]
also has positive lower density. In fact, with the help of covering congruences, Erdős [4] found that no positive integer $n$ with $n \equiv 7629217 \pmod{11184810}$ is no representable as the sum of a prime and a power of 2.

In [3], Crocker proved that there exist infinitely many odd positive integers $x$ not of the form $p + 2^a + 2^b$. One key to Crocker’s proof is the following observation: If $b - a = 2^s t$ with $s \geq 0$ and $2 \nmid t$, then $2^a + 2^b \equiv 0 \pmod{2^{2^s} + 1}$. Crocker also constructed a suitable covering system to deal with the case $a = b$. In [13], Sun and Le considered integers not of the form $p^\alpha + c(2^a + 2^b)$. Subsequently, Yuan [15] proved that there exist infinitely many positive odd integers $x$ not of the form $p^\alpha + c(2^a + 2^b)$.

Let
\[ \mathcal{N} = \{ n \geq 1 : n \text{ is odd and not of the form } p + 2^a + 2^b \}, \]
\[ \mathcal{N}_* = \{ n \geq 1 : n \text{ is odd and not of the form } p^\alpha + 2^a + 2^b \}. \]
Erdős asked whether $|\mathcal{N} \cap [1, x]| \gg x^\epsilon$ for some $\epsilon > 0$. Granville and Soundararajan [6] mentioned that this is true under the assumption that there exist infinitely many $m_1 < m_2 < \cdots$ such that all $2^{2 m_i} + 1$ are composite and $\{m_{i+1} - m_i\}$ is bounded. Erdős even suggested [7, A19] that
$|N \cap [1, x]| \geq Cx$ for a constant $C > 0$, though it seems that the covering congruences could not help here. In [1], Chen, Feng and Templier proved that

$$\limsup_{x \to \infty} \frac{|N_* \cap [1, x]|}{x^{1/4}} = +\infty$$

if there exist infinitely many $m$ such that $2^{2m} + 1$ is composite, and

$$\limsup_{x \to \infty} \frac{|N_* \cap [1, x]|}{\sqrt{x}} > 0$$

if there are only finitely many $m$ such that $2^{2m} + 1$ is prime. Recently, in his answer to a conjecture of Sun, Poonen [10] gave a heuristic argument which suggests that for each odd $k > 0$,

$$|\{1 \leq n \leq x : n \text{ is odd and not of the form } p + 2^a + k \cdot 2^b\}| \gg_{k, \epsilon} x^{1-\epsilon}$$

for any $\epsilon > 0$, where $\gg_{k, \epsilon}$ means the implied constant only depends on $k$ and $\epsilon$.

On the other hand, using Selberg’s sieve method, Tao [14] proved that for any $K \geq 1$ and sufficiently large $x$, the number of primes $p \leq x$ such that $|kp \pm ja^i|$ is composite for all $1 \leq a, j, k \leq K$ and $1 \leq i \leq K \log x$, is at least $C_K x / \log x$, where $C_K$ is a constant only depending on $K$. Motivated by Tao’s idea, in this short note, we shall unconditionally prove

**Theorem 1.1.**

$$|N_* \cap [1, x]| \gg x \cdot \exp\left(-C \log x \cdot \frac{\log \log \log \log x}{\log \log \log x}\right),$$

where $C > 0$ is an absolute constant.

Clearly Theorem 1.1 implies $|N_* \cap [1, x]| \gg x^{1-\epsilon}$ for any $\epsilon > 0$. The proof of Theorem 1.1 will be given in the next section. Unless indicated otherwise, the constants implied by $\ll, \gg$ and $O(\cdot)$ are always absolute.

**2. Proof of Theorem 1.1.** Since

$$|\{1 \leq n \leq x : n \text{ is of the form } p^a + 2^a + 2^b \text{ with } \alpha \geq 2\}| = O(\sqrt{x} \log x),$$

we only need to show that

$$|N \cap [1, x]| \gg x \cdot \exp\left(-C \log x \cdot \frac{\log \log \log \log x}{\log \log \log x}\right).$$

We need several auxiliary lemmas. The first lemma is a special case of the Brun–Titchmarsh theorem (cf. [8, Theorem 3.7]).

**Lemma 2.1.** Suppose that $W \geq 1$ and $\beta$ are integers with $(\beta, W) = 1$. Then

$$|\{1 \leq n \leq x : Wn + \beta \text{ is prime}\}| \leq \frac{C_1 x}{\log x} \prod_{p | W} \left(1 - \frac{1}{p}\right)^{-1},$$

where $C_1$ is an absolute constant.
The next lemma is an easy application of the Selberg sieve method (cf. [8, Theorems 3.2 and 4.1], [9, Theorem 7.1]).

**Lemma 2.2.** Suppose that $x$ is a sufficiently large integer. Suppose that $p_1, \ldots, p_h$ are distinct primes less than $x^{1/8}$. Then

$$|\{1 \leq n \leq x : n \not\equiv 0 \pmod{p_j} \text{ for every } 1 \leq j \leq h\}| \leq C_2 x \prod_{j=1}^{h} \left(1 - \frac{1}{p_j}\right),$$

where $C_2$ is an absolute constant.

The third lemma is due to Ford, Luca and Shparlinski [5, Theorem 1].

**Lemma 2.3.** The series

$$\sum_{n=1}^{\infty} \frac{(\log n)^\gamma}{P(2^n - 1)}$$

converges for any $\gamma < 1/2$, where $P(n)$ denotes the largest prime factor of $n$.

Now we are ready to prove Theorem 1.1. Let

$$C_3 = \sum_{p \text{ prime}} \frac{1}{P(2^p - 1)}.$$

Suppose that $x$ is sufficiently large. Let

$$K = \left\lfloor \frac{\log \log \log x}{100 \log \log \log \log x} \right\rfloor$$

and $L = \log(2^9 C_1 C_2 K) + 2C_3$, where $[\theta] = \max\{z \in \mathbb{Z} : z \leq \theta\}$.

Let $u = e^{eK(L+1)}$. By the Mertens theorem (cf. [9, Theorem 6.7]), we know that

$$\sum_{p \leq u \atop \text{prime}} \frac{1}{p} = \log \log u + B + O\left(\frac{1}{\log u}\right) = K(L + 1) + O(1),$$

where $B = 0.2614972\ldots$ is a constant. So we may choose some distinct odd primes less than $u$,

$$p_{1,1}, \ldots, p_{1,h_1}; p_{2,1}, \ldots, p_{2,h_2}; \ldots; p_{K,1}, \ldots, p_{K,h_K},$$

such that

$$\sum_{j=1}^{h_i} \frac{1}{p_{i,j}} \geq L$$

for $1 \leq i \leq K$. Let $q_{i,j} = P(2^{p_{i,j}} - 1)$ for $1 \leq i \leq K$ and $1 \leq j \leq h_i$. Clearly these $q_{i,j}$ are all distinct. Now,

$$\sum_{j=1}^{h_i} \log \left(1 - \frac{1}{p_{i,j}}\right) \leq -\sum_{j=1}^{h_i} \frac{1}{p_{i,j}},$$
whence
\[ \prod_{j=1}^{h_i} \left( 1 - \frac{1}{p_{i,j}} \right) \leq e^{-L}. \]

Noting that \( 1 + \theta \leq e^{\theta} \) for \( \theta \geq 0 \), we have
\[ \prod_{i=1}^{K} \prod_{j=1}^{h_i} \left( 1 - \frac{1}{q_{i,j}} \right)^{-1} \leq \prod_{i=1}^{K} \prod_{j=1}^{h_i} \left( 1 + \frac{2}{q_{i,j}} \right) \leq \exp \left( \sum_{i=1}^{K} \sum_{j=1}^{h_i} \frac{2}{q_{i,j}} \right) \leq e^{2C_3}. \]

Let
\[ W_{1,i} = \prod_{j=1}^{h_i} q_{i,j} \quad \text{for } 1 \leq i \leq K, \quad W_1 = \prod_{i=1}^{K} W_{1,i}. \]

Then
\[ W_1 \leq \frac{2^{\sum_{i=1}^{K} \sum_{j=1}^{h_i} p_{i,j}}}{2^{K^2/\log u}}, \]

since (cf. [12])
\[ \sum_{\substack{p \leq u \\ p \text{ prime}}} p = \left( \frac{1}{2} + o(1) \right) \frac{u^2}{\log u}. \]

Noting that for sufficiently large \( x \),
\[ \frac{\log \log (2^{u^2/\log u})}{\log \log \log (x^{1/K})} \leq \frac{2K(L + 1)}{\log(\log \log x - \log K)} \leq 1, \]
we have \( W_1 \leq x^{1/K} \).

Let \( m = \lfloor \log_2 \log_2 (x^{2/(K-1)}) \rfloor \) and \( K' = 1 + \lfloor 2^{-m} \log_2 x \rfloor \), where \( \log_2 x = \log x / \log 2 \). We have
\[ K' \leq 1 + \frac{\log_2 x}{2^m} \leq 1 + \frac{2 \log_2 x}{2 \log_2 (x^{2/(K-1)})} = 1 + \frac{2 \log_2 x}{K-1} \cdot \log_2 x = K. \]

For each \( k \geq 0 \), let \( \gamma_k \) be the smallest prime factor of \( 2^{2^k} + 1 \). Let
\[ W_2 = \prod_{k=0}^{m-1} \gamma_k \]

and \( W = W_1 W_2 \). It is not difficult to see that \( (W_1, W_2) = 1 \). Moreover,
\[ W \leq W_1 \prod_{k=0}^{m-1} (1 + 2^{2^k}) \leq x^{1/K} \cdot x^{2/(K-1)} \leq x^{3/(K-1)}. \]

Let \( \beta \) be an odd integer such that
\[ \beta \equiv 2^{2^m(i-1)} + 1 \pmod{\prod_{j=1}^{h_i} q_{i,j}} \quad \text{and} \quad \beta \equiv 0 \pmod{\gamma_k} \]
for \( 1 \leq i \leq K' \) and \( 0 \leq k \leq m - 1 \).
Let
\[ S = \{1 \leq n \leq x : n \equiv \beta \pmod{2W}\} \].
Clearly,
\[ \frac{x}{2W} - 1 \leq |S| \leq \frac{x}{2W} + 1. \]

Let
\[ T_1 = \{n \in S : n \text{ is of the form } p + 2^a + 2^b \text{ with } p \mid W\}, \]
\[ T_2 = \{n \in S \setminus T_1 : n \text{ is of the form } p + 2^a + 2^b \text{ with } p \nmid W\}. \]
Clearly \(|T_1| = O(W(\log x)^2)\).

Suppose that \(n \in S\) and \(n = p + 2^a + 2^b\) with \(p\) prime and \(0 \leq a \leq b\). If \(a \not\equiv b \pmod{2^m}\), then \(b = a + 2^st\) where \(0 \leq s \leq m - 1\) and \(2 \nmid t\). Thus
\[ p = n - 2^a(2^{2^st} + 1) \equiv \beta - 2^a(2^{2^s} + 1) \sum_{j=0}^{t-1} (-1)^j 2^{2sj} \equiv 0 \pmod{\gamma_s}. \]

Since \(p\) is prime, we must have \(p = \gamma_s\), i.e., \(n \in T_1\).

Below we assume that \(a \equiv b \pmod{2^m}\). Write \(b - a = 2^m(t - 1)\) where \(1 \leq t \leq K'\). If \(a \equiv 0 \pmod{p_{t,j}}\) for some \(1 \leq j \leq h_t\), then recalling \(2^{p_{t,j}} \equiv 1 \pmod{q_{t,j}}\), we have
\[ p = n - 2^a(2^{2^m(t-1)} + 1) \equiv \beta - (2^{2^m(t-1)} + 1) \equiv 0 \pmod{q_{t,j}}. \]

So \(p = q_{t,j}\) and \(n \in T_1\). On the other hand, for any \(a \geq 0\) satisfying \(a \not\equiv 0 \pmod{p_{t,j}}\) for all \(1 \leq j \leq h_t\), i.e., \((a, W_{1,t}) = 1\), by Lemma 2.1 we have
\[ |\{n \in S : n - 2^a(2^{2^m(t-1)} + 1) \text{ is prime}\}| \leq \frac{2C_1|S|}{\log |S|} \prod_{k=0}^{m-1} \left(1 - \frac{1}{\gamma_k}\right)^{-1} \prod_{i=1}^{K} \prod_{j=1}^{h_i} \left(1 - \frac{1}{q_{i,j}}\right)^{-1} \leq \frac{2^5C_1c^{2C_3}}{W} \cdot \frac{x}{\log x} \]
since \(\gamma_k \equiv 1 \pmod{2^{k+1}}\) and \(\gamma_k > 2^{k+1}\). Noting that
\[ \frac{\log \log u}{\log \log((\log_2 x)^{1/8})} \leq \frac{K(L + 1)}{\log(\log \log x - \log \log 2 - \log 8)} < 1, \]
we have \(u < (\log_2 x)^{1/8}\). By Lemma 2.2
\[ |\{0 \leq a \leq \log_2 x : a \not\equiv 0 \pmod{p_{t,j}}\} \text{ for all } 1 \leq j \leq h_t\}| \]
\[ \leq C_2 \frac{\log x}{\log 2} \prod_{j=1}^{h_t} \left(1 - \frac{1}{p_{t,j}}\right) \leq 2C_2e^{-L \log x}. \]
Thus
\[ |T_2| \leq \sum_{t=1}^{K'} \sum_{0 \leq a \leq \log_2 x \atop (a, W_1, t) = 1} |\{ n \in S : n - 2^a (2^{2^m (t-1)} + 1) \text{ is prime} \}| \leq K \cdot \frac{2^5 C_1 e^{2C_3}}{W} \cdot \frac{x}{\log x} \cdot 2C_2 e^{-L} \log x \leq \frac{x}{4W}. \]

It follows that
\[ |\{ n \in S : n \text{ is not of the form } p + 2^a + 2^b \}| = |S| - |T_1| - |T_2| \geq \frac{x}{2W} - 1 - O(W (\log x)^2) - \frac{x}{4W} \gg x^{1-4/K}. \]

The proof of Theorem 1.1 is complete. \( \blacksquare \)

Remark. Using a similar discussion, it is not difficult to prove that for any given \( K \geq 1 \),
\[ |\{ 1 \leq n \leq x : n \text{ is odd and } n \neq p + c(2^a + 2^b) \text{ with } p \text{ prime}, \quad a, b \geq 0, 1 \leq c \leq K \}| \gg_K x \cdot \exp \left( -C_K \log x \cdot \frac{\log \log \log \log x}{\log \log x} \right), \]

where the constant \( C_K > 0 \) only depends on \( K \).

Recently, Professor Y.-G. Chen asked the author the following question:

Are there infinitely many positive odd numbers which are not divisible by 3 and cannot be represented as \( p + 2^a + 2^b \)? Similar problems can be posed with 3 replaced by 5, 17 etc.

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References

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