On the number of solutions of exponential congruences

by

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1. Introduction. For a prime $p$ and an integer $a \in \mathbb{Z}$ we denote by $N(p; a)$ the number of solutions to the congruence

$$x^x \equiv a \pmod{p}, \quad 1 \leq x \leq p-1.$$

 Obviously only the case of $\gcd(a, p) = 1$ is of interest.

We note that other than the results of Crocker [3] and Somer [10] showing that there are at least $\lfloor \sqrt{(p-1)/2} \rfloor$ and at most $3p/4 + O(p^{1/2+o(1)})$, respectively, incongruent values of $x^x \pmod{p}$ when $1 \leq x \leq p-1$, little has been known about the solutions to (1). The function $x \mapsto x^x \pmod{p}$ is also used in some cryptographic protocols (see [9, Sections 11.70 and 11.71]), so certainly deserves further investigation; see also [8] for various conjectures concerning this function. We note that the function $x^x$ is periodic modulo $p$ with period $p(p-1)$, which is much larger than the range of $x$ in the congruence (1) and which explains why it is so difficult to study.

Here we suggest several approaches to studying this congruence and derive some upper bounds for $N(p; a)$.

Our first bound is nontrivial if $a$ is of small multiplicative order, which in the particular case when $a = 1$, takes the form $N(p; a) \leq p^{1/3+o(1)}$ as $p \to \infty$. The second bound is nontrivial if $a$ is of large multiplicative order, which in the particular case when $a$ is a primitive root modulo $p$, takes the form $N(p; a) \leq p^{11/12+o(1)}$ as $p \to \infty$.

Furthermore, both bounds combined imply that as $p \to \infty$, we have the uniform estimate

$$N(p; a) \leq p^{12/13+o(1)}.$$
Finally, we estimate the number of solutions $M(p)$ to the symmetric congruence
\begin{equation}
x^x \equiv y^y \pmod{p}, \quad 1 \leq x, y \leq p - 1,
\end{equation}
which has been considered by Holden & Moree [8] in their study of short cycles in the iterations of the discrete logarithm modulo $p$ (see also [6], [7]). However, no nontrivial estimate of $M(p)$ has been known prior to this work. Clearly
\begin{equation}
M(p) = \sum_{a=1}^{p-1} N(p; a)^2.
\end{equation}
Thus using the bound (2) and the identity
\begin{equation}
\sum_{a=1}^{p-1} N(p; a) = p - 1,
\end{equation}
we immediately derive
\begin{equation}
M(p) \leq p^{25/13+o(1)}.
\end{equation}
However here we obtain a slightly stronger bound, namely
\begin{equation}
M(p) \leq p^{48/25+o(1)}.
\end{equation}

Surprisingly enough, besides elementary number theory arguments, the bounds derived here rely on some results and arguments from additive combinatorics, in particular on results of Garaev [4].

For an integer $m \geq 1$ we use $\mathbb{Z}_m$ to denote the residue ring modulo $m$ and we use $\mathbb{Z}_m^*$ to denote the unit group of $\mathbb{Z}_m$.

Note that without the condition $1 \leq x \leq p - 1$ (needed in the cryptographic application) there are always many solutions. Let $g$ be a primitive root modulo $p$. For any element $a \in \mathbb{Z}_p^*$ (and so for any integer $a \not\equiv 0 \pmod{p}$) we use $\text{ind} \ a$ for its discrete logarithm modulo $p$, that is, the unique residue class $v$ modulo $p - 1$ with
\begin{equation}
g^v \equiv a \pmod{p}.
\end{equation}
Now, if for a primitive root $g$ we have
\begin{equation}
x \equiv p \text{ind} \ a - (p - 1)g \pmod{p(p - 1)},
\end{equation}
then
\begin{equation}
x^x \equiv g^{p \text{ind} \ a -(p - 1)g} \equiv (g^p)^{\text{ind} \ a} \cdot (g^{-g})^{p-1} \equiv a \pmod{p}.
\end{equation}

2. Elements of small order. We need to recall some notions and results from additive combinatorics.

For a prime $p$ and a set $\mathcal{A} \subseteq \mathbb{Z}_p^*$ we define the sets
\begin{equation}
\mathcal{A} + \mathcal{A} = \{a_1 + a_2 : a_1, a_2 \in \mathcal{A}\}, \quad \mathcal{A} \cdot \mathcal{A} = \{a_1 a_2 : a_1, a_2 \in \mathcal{A}\}.
\end{equation}
Our bound on $N(p; a)$ makes use of the following estimate of Garaev [4, Theorem 1].

**Lemma 1.** For any set $\mathcal{A} \subseteq \mathbb{Z}_p^*$,

$$\#(\mathcal{A} + \mathcal{A}) \cdot \#(\mathcal{A} \cdot \mathcal{A}) \gg \min \left\{ p\#\mathcal{A}, \frac{(\#\mathcal{A})^4}{p} \right\}.$$  

Let ord $a$ denote the multiplicative order of $a \in \mathbb{Z}_p^*$.

**Theorem 2.** Uniformly over $t | p - 1$, we have, as $p \to \infty$,

$$\sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a | t} N(p; a) \leq \max\{t, p^{1/2}t^{1/4}\} p^{o(1)}.$$  

**Proof.** Fix a primitive root $g \mod p$. The union of the nonzero residue classes $x$ satisfying (11) with ord $a | t$ is precisely the set of solutions to (7) $x^{tx} \equiv 1 \pmod{p}$, $1 \leq x \leq p - 1$.

This congruence is equivalent to $tx \text{ ind } x \equiv 0 \pmod{p - 1}$, or if we put $$T = \frac{p - 1}{t}$$ to $x \text{ ind } x \equiv 0 \pmod{T}$, or after fixing $d | T$ and considering only the solutions to (7) with $\gcd(x, T) = d$,

$$\gcd(x, T) = d,$$

they can be written as $x = dy$ and seen to satisfy (8) $\text{ ind}(dy) \equiv 0 \pmod{T_d}$, $1 \leq y \leq D$, $\gcd(y, T_d) = 1$,

where $$T_d = \frac{T}{d}, \quad D = \frac{p - 1}{d}.$$  

Let us denote by $\mathcal{Y}_d$ the set of integers $y$ satisfying (8), and by $\mathcal{W}_d$ the set of residue classes modulo $p$ represented by the elements of $\mathcal{Y}_d$. Obviously $\#\mathcal{Y}_d = \#\mathcal{W}_d$, and we have

$$\sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a | t} N(p; a) = \sum_{d | T} \#\mathcal{Y}_d = \sum_{d | T} \#\mathcal{W}_d.$$  

First note that

$$\#(\mathcal{W}_d + \mathcal{W}_d) \leq \#(\mathcal{Y}_d + \mathcal{Y}_d) \leq 2D$$  

from the second condition in (8).
Furthermore, the product set of $W_d$ is contained in
$$\{w \in \mathbb{Z}_p^* : \text{ind}(d^2w) \equiv 0 \pmod{T_d}\},$$
and so
$$\#(W_d \cdot W_d) \leq \frac{p - 1}{T_d} = dt.$$ (11)

Hence, applying Lemma 1 and using the bounds (10) and (11) we see that
$$\min\left\{p\#W_d, \frac{(\#W_d)^4}{p}\right\} \ll pt.$$

Therefore
$$\#W_d \ll \max\{t, p^{1/2}t^{1/4}\}.$$ (12)

Recalling the bound on the divisor function $\tau(k)$ given by
$$\tau(k) = \sum_{d|k} 1 = k^{o(1)}$$ (13)
(see [5, Theorem 315]), and using (12) in (9), we conclude the proof. ■

**Corollary 3.** Uniformly over $t | p - 1$ and all integers $a$ with $\gcd(a, p) = 1$ of multiplicative order $\text{ord} a = t$, we have, as $p \to \infty$,
$$N(p; a) \leq \max\{t, p^{1/2}t^{1/4}\}p^{o(1)}.$$ 

Next we show that if $t$ is very small then the bound of Theorem 2 can be improved. For example, this applies to the most interesting special case of the congruence (11), namely the case $a = 1$.

**Theorem 4.** Uniformly over $t | p - 1$, we have, as $p \to \infty$,
$$\sum_{a \in \mathbb{Z}_p^* \atop \text{ord} a | t} N(p; a) \leq p^{1/3 + o(1)}t^{2/3}.$$ 

**Proof.** We follow the proof of Theorem 2 up to (11), but finish the argument in a different way to derive a new bound for $\#\mathcal{Y}_d$. Let us define
$$s(b) = \#\{(y_1, y_2) : y_1, y_2 \in \mathcal{Y}_d, y_1y_2 \equiv b \pmod{p}\}.$$ 
First note that $s(b) > 0$ only when $b \in W_d \cdot W_d$, and so
$$\#\mathcal{Y}_d^2 = \sum_{b \in \mathbb{Z}_p^*} s(b) \leq \#(W_d \cdot W_d) \max_{b \in \mathbb{Z}_p^*} s(b).$$ (14)

If $(y_1, y_2)$ is counted in $s(b)$ then on the one hand $y_1y_2 \equiv b \pmod{p}$, on the other hand $1 \leq y_1y_2 \leq D^2$ (where as before $D = (p - 1)/d$), therefore $y_1y_2 = b + kp$, where $0 \leq k < p/d^2$. Thus the product $y_1y_2$ can take at
most \( p/d^2 + 1 \) possible values \( y_1 y_2 = z \) and once \( z \) is fixed, there are \( \tau(z) = z^{o(1)} = p^{o(1)} \) possibilities for the pair \((y_1, y_2)\) (see \([13]\)). Thus

\[
s(b) \leq (p/d^2 + 1)p^{o(1)},
\]

which after inserting in \((14)\) and recalling \((11)\) yields

\[
\#Y_d \leq ((pt/d)^{1/2} + (td)^{1/2})p^{o(1)}.
\]

For \( d \leq p^{1/3}t^{-1/3} \) we use \( \#Y_d \leq dt \) from the first condition of \((8)\) and for \( d \geq p^{2/3}t^{-1/3} \) we use \( \#Y_d \leq D \) from the second condition of \((8)\). Therefore we obtain

\[
\#Y_d \ll p^{1/3}t^{2/3} \quad \text{and} \quad \#Y_d \ll p^{1/3}t^{1/3},
\]

respectively.

Finally, for \( p^{1/3}t^{-1/3} \leq d \leq p^{2/3}t^{-1/3} \) we use \((15)\) to derive

\[
\#Y_d \leq (p^{1/3}t^{2/3} + p^{1/3}t^{1/3})p^{o(1)} = p^{1/3 + o(1)}t^{2/3}.
\]

Using these bounds with \((13)\) in \((9)\) we conclude the proof. 

**Corollary 5.** Uniformly over \( t \mid p-1 \) and all integers \( a \) with \( \gcd(a, p) = 1 \) of multiplicative order \( \text{ord } a = t \), we have, as \( p \to \infty \),

\[
N(p; a) \leq p^{1/3 + o(1)}t^{1/3}.
\]

### 3. Elements of large order.

Here we use a different argument, which is similar to the one used in \([1]\), and a bound of \([2]\), on the number of solutions of an exponential congruence, plays the crucial role. However, this approach is effective only for values of \( a \) of sufficiently large order.

We recall the following estimate, given in \([2\) Lemma 7]”, on the number of zeros of sparse polynomials over a finite field \( \mathbb{F}_q \) of \( q \) elements.

**Lemma 6.** For \( n \geq 2 \) given elements \( a_1, \ldots, a_n \in \mathbb{F}_q^* \) and integers \( k_1, \ldots, k_n \) in \( \mathbb{Z} \) let us denote by \( Q \) the number of solutions of the equation

\[
\sum_{i=1}^{n} a_i X^{k_i} = 0, \quad X \in \mathbb{F}_q^*.
\]

Then

\[
Q \leq 2q^{1 - 1/(n-1)} \Delta^{1/(n-1)} + O(q^{1-2/(n-1)} \Delta^{2/(n-1)}),
\]

where

\[
\Delta = \min_{1 \leq i < n} \max_{j \neq i} \gcd(k_j - k_i, q - 1).
\]

We are now ready to prove the main result of this section.

**Theorem 7.** Uniformly over \( t \mid p-1 \) and all integers \( a \) with \( \gcd(a, p) = 1 \) of multiplicative order \( \text{ord } a = t \), we have, as \( p \to \infty \),

\[
N(p; a) \leq p^{1 + o(1)}t^{-1/12}.
\]
Proof. Let $a$ be a nonzero residue class modulo $p$ of multiplicative order $t | p - 1$. As before, we put
$$T = \frac{p - 1}{t}.$$ 

Clearly, there is a primitive root $g$ modulo $p$ with $a \equiv g^T \pmod{p}$. Using the discrete logarithm to base $g$, the congruence (1) is equivalent to
$$x \text{ ind } x \equiv T \pmod{p - 1}.$$ 

Note the condition $\gcd(x, p - 1) | T$. After fixing $d | T$ and considering only the solutions to (1) with $\gcd(x, p - 1) = d$, they can be written as $x = dy$ and satisfy
$$y \text{ ind}(dy) \equiv T_d \pmod{D}, \quad 1 \leq y \leq D, \quad \gcd(y, D) = 1,$$
where, as before,
$$T_d = \frac{T}{d}, \quad D = \frac{p - 1}{d}.$$ 

Note that $t | D$. The congruence $yz \equiv 1 \pmod{D}$ defines a one-to-one correspondence between the integers $\{1 \leq y \leq D : \gcd(y, D) = 1\}$ and $z \in \mathbb{Z}_D^*$. 

Furthermore, the relation $yz \equiv 1 \pmod{D}$ defines a one-to-$M_d$ correspondence between the set $\{1 \leq y \leq D : \gcd(y, D) = 1\}$ and $z \in \mathbb{Z}_{p-1}^*$, where $M_d$ is the number of residue classes in $\mathbb{Z}_{p-1}^*$ of the form $z + kD$. These residue classes are automatically coprime to $D$, but we have to ensure that they are coprime to $d$ as well (and thus belong to $\mathbb{Z}_{p-1}^*$). Thus using $\mu(k)$ to denote the Möbius function, by [5, Theorem 263] (which is essentially the inclusion-exclusion principle) we obtain
$$M_d = \sum_{k=1}^{d} \sum_{f | \gcd(z+kD,d)} \mu(f) = \sum_{f | d} \mu(f) \sum_{k=1}^{d} 1_{z+kD \equiv 0 \pmod{f}}$$
$$= \sum_{f | d} \mu(f) \frac{d}{f} = d \frac{\varphi(m)}{m},$$
where $\varphi(k)$ is the Euler function and $m$ is the product of primes $q$ with $q | d$ and $q : D$, see [5, equation (16.3.1)]. In particular $m \leq d \leq p$ and recalling the well-known estimate on the Euler function (see [5, Theorem 328]) we obtain
$$M_d = dp^{o(1)}.$$ 

From now on the integer $1 \leq y \leq D$ and the residue class $z \in \mathbb{Z}_{p-1}^*$ with or without subscripts are always connected by $yz \equiv 1 \pmod{D}$, even if this is not explicitly stated.
Let us define
\[ \mathcal{Z}_d = \{ z \in \mathbb{Z}_{p-1}^* : \text{ind}(dy) \equiv Dz/t \pmod{D}, 1 \leq y \leq D \} \]
(we recall our convention that we always have \( yz \equiv 1 \pmod{D} \)). We have
\[ N(p; a) = \sum_{d \mid T} \frac{1}{M_d} \# \mathcal{Z}_d \leq p^{\rho(1)} \sum_{d \mid T} \frac{1}{d} \# \mathcal{Z}_d. \]

The congruence \( \text{ind}(dy) \equiv Dz/t \pmod{D} \) is equivalent to
\[ dy \equiv \rho g^{Dz/t} \pmod{p} \]
for some \( \rho \in \mathbb{Z}_p^* \) with \( \rho^d \equiv 1 \pmod{p} \). Thus we split \( \mathcal{Z}_d \) into subsets \( \mathcal{Z}_{d,\rho} \) getting
\[ \# \mathcal{Z}_d = \sum_{\rho^d \equiv 1 \pmod{p}} \# \mathcal{Z}_{d,\rho}, \]
where
\[ \mathcal{Z}_{d,\rho} = \{ z \in \mathbb{Z}_{p-1}^* : dy \equiv \rho g^{Dz/t} \pmod{p}, 1 \leq y \leq D \} \]
(and again we recall our convention that \( yz \equiv 1 \pmod{D} \)).

Clearly,
\[ (\# \mathcal{Z}_{d,\rho})^2 = \# \{ z_1, z_2 \in \mathbb{Z}_{p-1}^* : dy_j \equiv \rho g^{Dz_j/t} \pmod{p}, j = 1, 2 \}. \]

We deduce by adding the two congruences that
\[ (\# \mathcal{Z}_{d,\rho})^4 \leq 2D \# \{ z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* : d(y_1 + y_2) \equiv \rho(g^{Dz_1/t} + g^{Dz_2/t}) \pmod{p} \} \]
\[ = \sum_{v \in \mathbb{Z}} \# \{ z_1, z_2 \in \mathbb{Z}_{p-1}^* : d(y_1 + y_2) = v, \rho(g^{Dz_1/t} + g^{Dz_2/t}) \equiv v \pmod{p} \}. \]
The sum over \( v \in \mathbb{Z} \) is empty unless \( v = dw \), where \( 2 \leq w \leq 2D \) and we find by the Cauchy–Schwarz inequality that
\[ (\# \mathcal{Z}_{d,\rho})^4 \leq 2D \# \{ z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* : d(y_1 + y_2) = d(y_3 + y_4) \equiv \rho(g^{Dz_1/t} + g^{Dz_2/t}) \equiv \rho(g^{Dz_3/t} + g^{Dz_4/t}) \pmod{p} \}. \]

Clearly, when \( z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* \) are fixed, the condition
\[ d(y_1 + y_2) = d(y_3 + y_4) \equiv \rho(g^{Dz_1/t} + g^{Dz_2/t}) \equiv \rho(g^{Dz_3/t} + g^{Dz_4/t}) \pmod{p} \]
defines \( \rho \) uniquely. Hence
\[ \sum_{\rho^d \equiv 1 \pmod{p}} (\# \mathcal{Z}_{d,\rho})^4 \leq 2D \# \{ z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* : y_1 + y_2 = y_3 + y_4, \]
\[ g^{Dz_1/t} + g^{Dz_2/t} \equiv g^{Dz_3/t} + g^{Dz_4/t} \pmod{p} \}. \]
Relaxing the condition \( y_1 + y_2 = y_3 + y_4 \) (mod \( D \)) only increases the number of solutions (but allows us to think about \( y_j \) as a residue class modulo \( D \) defined by \( y_j z_j \equiv 1 \) (mod \( D \)), \( j = 1, 2, 3, 4 \)). Thus

\[
\sum_{\rho^d \equiv 1 \pmod{p}} (\# \mathbb{Z}_{d, \rho})^4 \leq 2DT
\]

where

\[
T = \# \{ z_1, z_2, z_3, z_4 \in \mathbb{Z}_{p-1}^* : y_1 + y_2 \equiv y_3 + y_4 \pmod{D},
\]

\[
g^{Dz_1/t} + g^{Dz_2/t} \equiv g^{Dz_3/t} + g^{Dz_4/t} \pmod{p}\}.
\]

Finally, after the substitution \( z_j \mapsto wz_j \) for \( w \in \mathbb{Z}_{p-1}^* \) (and thus \( y_j \mapsto w^{-1}y_j \)), \( j = 1, 2, 3, 4 \), where \( w^{-1} \) is defined modulo \( D \), we deduce that any solution is counted with multiplicity \( \varphi(p-1) \), that is,

\[
\sum_{\rho^d \equiv 1 \pmod{p}} (\# \mathbb{Z}_{d, \rho})^4 \leq \frac{2D}{\varphi(p-1)} \tilde{T}
\]

where

\[
\tilde{T} = \# \{ z_1, z_2, z_3, z_4, w \in \mathbb{Z}_{p-1}^* : y_1 + y_2 \equiv y_3 + y_4 \pmod{D},
\]

\[
(g^w)^{Dz_1/t} + (g^w)^{Dz_2/t} \equiv (g^w)^{Dz_3/t} + (g^w)^{Dz_4/t} \pmod{p}\}.
\]

Writing \( X \equiv g^w \pmod{p} \) and \( k_j = Dz_j/t = (p-1)z_j/dt = T_dz_j \), after fixing \( z_1, z_2, z_3, z_4 \), the number of \( w \in \mathbb{Z}_{p-1}^* \) satisfying the congruence in (18) is bounded by the number of solutions to the congruence \( X^{k_1} + X^{k_2} \equiv X^{k_3} + X^{k_4} \pmod{p} \), and this is bounded in Lemma 6 applied with \( n = 4 \), by \( O(p^{2/3} \Delta^{1/3}) \), where

\[
\Delta = \min_{1 \leq i \leq 4} \max_{1 \leq j \leq 4, j \neq i} \gcd(T_d(z_i - z_j), p - 1)
\]

\[
= T_d \min_{1 \leq i < j \leq 4} \max_{1 \leq k \leq 4, k \neq i} \gcd(z_i - z_j, dt).
\]

For every fixed \( i \neq j, 1 \leq i, j \leq 4 \) and \( \delta \mid dt \) there are \( (p-1)^2/\delta \) choices for \((z_i, z_j)\) with

\[
\gcd(z_i - z_j, dt) = \delta.
\]

When \( z_i \) and \( z_j \) are fixed the congruence \( y_1 + y_2 \equiv y_3 + y_4 \pmod{D} \) implies that there are \( dp^{1+o(1)} \) choices for the remaining two variables. (Recall that each \( y \) determines \( M_d = dp^{o(1)} \) different choices of \( z \).) Thus, putting everything together in (18) and recalling (13), we obtain
Putting this into (17), by the Hölder inequality we get

$$\# \mathcal{Z}_d \leq d^{3/4} \left( \sum_{\rho \equiv 1 \pmod{p}} (\# \mathcal{Z}_{d, \rho})^4 \right)^{1/4} \leq \frac{p^{1+o(1)}}{t^{1/12}} d^{2/3}. $$

Finally (16) and (13) give

$$N(p; a) \leq \sum_{d \mid (p-1)/t} \frac{p^{1+o(1)}}{t^{1/12} d^{1/3}} \leq \frac{p^{1+o(1)}}{t^{1/12}},$$

and we conclude the proof. ■

4. Symmetric congruence. We now improve the bound (6) on the number of solutions to the symmetric congruence (3).

THEOREM 8. We have, as $p \to \infty$,

$$M(p) \leq p^{48/25+o(1)}. $$

Proof. From (4) we obtain

$$M(p) \leq \sum_{t \mid p-1} \sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a = t} N(p; a)^2.$$

We fix some parameter $\vartheta$ and for $t \leq \vartheta$ we use Theorem 2 to estimate

$$\sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a = t} N(p; a)^2 \leq \left( \sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a = t} N(p; a) \right)^2 \leq \max\{t^2 p^{o(1)}, p^{1+o(1)} t^{1/2}\} \leq \max\{\vartheta^2 p^{o(1)}, p^{1+o(1)} \vartheta^{1/2}\}. $$

For $t \geq \vartheta$ we use Theorem 7 together with (5) to estimate

$$\sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a = t} N(p; a)^2 \leq p^{1+o(1)} t^{-1/12} \sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a = t} N(p; a) \leq p^{2+o(1)} \vartheta^{-1/12}. $$
Taking $\vartheta = p^{24/25}$ to balance the above estimates, we obtain the bound

$$\sum_{a \in \mathbb{Z}_p^* \atop \text{ord } a = t} N(p; a)^2 \leq p^{48/25 + o(1)},$$

and using (13), we conclude the proof. ■

5. Concluding remarks. Clearly Theorem 2 is nontrivial provided that $t \leq p^{1-\varepsilon}$ for some $\varepsilon > 0$, while Theorem 7 is nontrivial provided $t \geq p^\varepsilon$ for an arbitrary $\varepsilon > 0$ and a sufficiently large $p$. In particular, using Corollary 3 for $t \leq p^{12/13}$ and Theorem 7 for $t > p^{12/13}$, we derive (2).

It is also easy to see that all but $o(p)$ elements $a \in \mathbb{Z}_p^*$ are of multiplicative order $t = p^{1+o(1)}$. Thus for almost all $a \in \mathbb{Z}_p^*$ we have

$$N(p; a) \leq p^{11/12 + o(1)}$$

by Theorem 7.

Similar results can also be established for several other congruences. For example, the same arguments as those used in the proof of Theorem 4 imply that the congruence

$$x^{x-1} \equiv 1 \pmod{p}, \quad 1 \leq x \leq p - 1,$$

has $O(p^{1/3 + o(1)})$ solutions. This means that the function $x \mapsto x^x \pmod{p}$ has $O(p^{1/3 + o(1)})$ fixed points in the interval $1 \leq x \leq p - 1$.

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