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Harmonic weak Maass-modular grids in higher level cases

by

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1. Introduction and statement of results. Let $\Gamma = \Gamma_0(N)$ or $\Gamma_0^+(p)$ with $p = 1$ or a prime. Here $\Gamma_0^+(p)$ denotes the group generated by the Hecke group $\Gamma_0(p)$ and the Fricke involution $W_p = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. For $k \in \mathbb{Z}$ we denote by $M_k^!(\Gamma)$ the space of weakly holomorphic modular forms of weight k on Γ. As usual, $M_k(\Gamma)$ and $S_k(\Gamma)$ are the spaces of weight k modular and cusp forms respectively on Γ. Let $H_k(\Gamma)$ be the space of weight k harmonic weak Maass forms on Γ and $H_k^\infty(\Gamma)$ be the subspace of those $g \in H_k(\Gamma)$ whose principal parts at the cusps other than ∞ are constant. Following [16] we call two collections $f_n \in M_k^!(\Gamma)$ and $g_m \in H_{2-k}(\Gamma)$ with q-expansions

$$f_n = q^{-n} + \sum_{m>0} c_{f_n}(m) q^m, \qquad n \geq 0,$$
$$g_m = g_m^- + q^{-m} + \sum_{n\geq 0} c_{g_m}(n) q^n, \quad m > 0,$$

a *harmonic weak Maass-modular grid* of weight k on Γ if the identity of Fourier coefficients

$$c_{f_n}(m) = -c_{g_m}(n)$$

holds. Here $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$.

Guerzhoy [16] showed the existence of grids of integral weight k on the full modular group $\mathrm{SL}_2(\mathbb{Z})$. In [9] Cho and Choie considered vector-valued harmonic weak Maass-modular grids of integral weights and obtained, as a corollary, (scalar-valued) harmonic weak Maass-modular grids of prime level p with a certain Nebentypus.

In this paper we deal with harmonic weak Maass-modular grids of integral weight k for both $\Gamma_0(N)$ and $\Gamma_0^+(p)$. More precisely, we prove

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THEOREM 1.1. *Let $\Gamma = \Gamma_0(N)$ or $\Gamma_0^+(p)$, and assume that $k \geq 2$ is even.*

(i) *If $k > 2$, then there exist unique $f_n \in M_k^\infty(\Gamma) = M_k^!(\Gamma) \cap H_k^\infty(\Gamma)$ and $g_m \in H_{2-k}^\infty(\Gamma)$ with Fourier expansions*

$$f_n(\tau) = q^{-n} + \sum_{m>0} c_{f_n}(m) q^m, \quad n \geq 0,$$

$$g_m(\tau) = g_m^-(\tau) + q^{-m} + \sum_{n \geq 0} c_{g_m}^+(n), q^n, \quad m > 0,$$

such that

$$c_{g_m}^+(n) = -c_{f_n}(m).$$

(ii) *If $k = 2$, then there exists such a grid for all $m, n \geq 1$. In this case f_n is unique and g_m is unique up to constants.*

REMARK 1. (i) For complex s, let

$$\mathcal{M}_s(y) := |y|^{k/2-1} M_{(1-k/2)\mathrm{sgn}(y), s-1/2}(|y|),$$

where $M_{\mu,\nu}(z)$ is the usual M-Whittaker function. Following [8, Section 6] and [3, Section 2.2], we define, for $k \in 2\mathbb{N}$ and integers $m, N \geq 1$,

$$Q(-m, k, N; \tau) := \frac{1}{(k-1)!} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_{-m}^* |_{2-k} \gamma)(\tau)$$

where $\varphi_{-m}^*(\tau) = \mathcal{M}_{k/2}(-4\pi m y) e^{-2\pi i m x}$ and $\tau = x + iy \in \mathbb{H}$. In Theorem 1.1, g_m is given in terms of the Maass Poincaré series $Q(-m, k, N; \tau)$ and $f_n = (-n)^{1-k} D^{k-1} g_n$ for $m, n \geq 1$. And f_0 is constructed from the Eisenstein series $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma$ (see Section 3). We note that the Fourier coefficient of q^n in g_m^+ is expressed in terms of the I-Bessel function and the Kloosterman sum as follows [8, Proposition 6.2]: for all $m, n \geq 1$,

$$c_{g_m}^+(0) = -\frac{2^k \pi^k (-1)^{k/2} m^{k-1}}{(k-1)!} \sum_{\substack{c>0 \\ c \equiv 0\,(N)}} \frac{K(-m, 0, c)}{c^k},$$

$$c_{g_m}^+(n) = -2\pi (-1)^{k/2} \sum_{\substack{c>0 \\ c \equiv 0\,(N)}} (m/n)^{(k-1)/2} \frac{K(-m, n, c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Similarly the Fourier coefficient of q^m in f_0^+ has the following exact formula: for all $m \geq 1$,

$$c_{f_0}^+(m) = \frac{2^k \pi^k (-1)^{k/2} m^{k-1}}{(k-1)!} \sum_{\substack{c>0 \\ c \equiv 0\,(N)}} \frac{K(0, m, c)}{c^k}.$$

(ii) Assume that the genus $g(\Gamma)$ of Γ is zero. Let m_k denote the maximal order of a nonzero $f \in M_k^{\infty}(\Gamma)$ at ∞. For every integer $n \geq -m_k$, there exists a unique weakly holomorphic modular form $f_{k,n} \in M_k^{\infty}(\Gamma)$ with Fourier expansion

$$f_{k,n}(\tau) = q^{-n} + \sum_{m > m_k} a_k(m,n) q^m$$

and together they form a basis for $M_k^{\infty}(\Gamma)$ (see [15, 13, 14, 11, 12]). Another type of grids which do not involve harmonic weak Maass forms appears in [15, 13, 14]. It follows from [15, Corollary 1] and [14, Corollary 3.6] that the two collections $f_{k,n} \in M_k^!(\Gamma_0^+(p))$ $(= M_k^{\infty}(\Gamma_0^+(p)))$ and $f_{2-k,m} \in M_{2-k}^!(\Gamma_0^+(p))$ form a grid. In particular, if $\dim S_k(\Gamma_0^+(p)) = 0$, then the grid given by $f_{k,n}$ and $f_{2-k,m}$ coincides with our grid (f_n, g_m) of Theorem 1.1.

THEOREM 1.2.

(i) *Let (f_n, g_m) be a weight k grid on Γ and let T_n denote the usual weight k Hecke operator. Then $f_n = n^{1-k} f_1 | T_n$ for any positive integer n relatively prime to the level of Γ.*
(ii) *Assume that $g(\Gamma) = 0$. Then there exists a modular form $\psi \in M_k(\Gamma)$ such that if n is a positive integer relatively prime to the level of Γ, then*

$$f_n - n^{1-k} \psi | T_n \in \mathbb{Q}((q^{-1}, q)).$$

REMARK 2. In the proof of Theorem 1.2 in Section 4, ψ is chosen to be $f_1 - f_{k,1}$, which belongs to $S_k(\Gamma)$ when $\Gamma = \Gamma_0^+(p)$.

EXAMPLE 1. Let $\Gamma = \Gamma_0(9)$ and $k = 4$. It follows from the Fourier expansion of $Q^+(-m, 4, 9; \tau)$ that

$$\begin{aligned}
g_1^+(\tau) &= q^{-1} - \tfrac{1}{4}q^2 + \tfrac{49}{125}q^5 + O(q^6),\\
g_2^+(\tau) &= q^{-2} - 2q - q^4 + O(q^6),\\
g_3^+(\tau) &= q^{-3} + 3 - 18q^3 + O(q^6),\\
g_4^+(\tau) &= q^{-4} - 8q^2 - \tfrac{11392}{125}q^5 + O(q^6),\\
g_5^+(\tau) &= q^{-5} + 49q - 178q^4 + O(q^6).\\
&\ \ \vdots
\end{aligned}$$

Since $f_0(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma$ and $f_n = (-n)^{-3} D^3 g_n$ for $n \geq 1$, the Fourier expansion of f_n is given by

$$
\begin{aligned}
f_0(\tau) &= 1 && - 3q^3 && + O(q^6),\\
f_1(\tau) &= q^{-1} && + 2q^2 \qquad - 49q^5 && + O(q^6),\\
f_2(\tau) &= q^{-2} + \frac{1}{4}q && + 8q^4 && + O(q^6),\\
f_3(\tau) &= q^{-3} && + 18q^3 && + O(q^6),\\
f_4(\tau) &= q^{-4} && + q^2 \qquad + 178q^5 && + O(q^6),\\
f_5(\tau) &= q^{-5} - \tfrac{49}{125}q && + \tfrac{11392}{125}q^4 && + O(q^6),\\
&\vdots
\end{aligned}
$$

In the above we can observe that the duality relation $c_{g_m}^+(n) = -c_{f_n}(m)$ holds, as expected from Theorem 1.1.

Let $j_9(\tau) = (\eta(\tau)/\eta(9\tau))^3 + 3$ be the Hauptmodul for $\Gamma_0(9)$, and $\Delta_9 = \eta(9\tau)^6/\eta(3\tau)^2$. It then follows from the Fourier expansion of f_1 and [12] that

$$f_1(\tau) = f_{4,1} + 2f_{4,-2}$$

where $f_{4,-2} = \Delta_9^2 \cdot j_9^2$ and $f_{4,1} = \Delta_9^2(j_9^5 - 29j_9^2)$. From the action of the Hecke operator T_n on the Fourier coefficients of f_1 we can verify that $f_n = \frac{1}{n^3} f_1 | T_n$, as desired in Theorem 1.2.

This paper is organized as follows. We begin with necessary background on harmonic weak Maass forms in Section 2. The proof of Theorem 1.1 is given in Section 3, while Theorem 1.2 is proved in Section 4.

2. Harmonic weak Maass forms. Let $\tau = x + iy \in \mathbb{H}$, the complex upper half-plane, with $x, y \in \mathbb{R}$. Let $k \in \mathbb{Z}$ and N a positive integer. A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a *harmonic weak Maass form of weight k for $\Gamma_0(N)$* if it satisfies:

(i) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau).$$

(ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplace operator defined by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(iii) There is a Fourier polynomial $P_f(\tau) = \sum_{-\infty \ll n \le 0} c_f^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) = P_f(\tau) + O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. Here $q = e^{2\pi i\tau}$ as usual. Analogous conditions are required at all cusps.

We denote the space of all harmonic weak Maass forms by $H_k(\Gamma_0(N))$. The polynomial $P_f \in \mathbb{C}[q^{-1}]$ is called the *principal part* of f at the corresponding cusps. In particular $f \in H_k(\Gamma_0(N))$ has a unique decomposition $f = f^+ + f^-$, where

$$f^+(\tau) = \sum_{n \gg -\infty} c_f^+(n) q^n, \qquad f^-(\tau) = \sum_{n<0} c_f^-(n) \Gamma(1-k, 4\pi|n|y) q^n.$$

Here $\Gamma(a, y) = \int_y^\infty e^{-t} t^{a-1} dt$ denotes the incomplete Gamma function.

The Maass raising and lowering operators R_k and L_k on functions $f : \mathbb{H} \to \mathbb{C}$ are defined by

$$R_k = 2i \frac{\partial}{\partial \tau} + k y^{-1} = i\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) + k y^{-1},$$
$$L_k = -2iy^2 \frac{\partial}{\partial \bar{\tau}} = -iy^2 \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

We also recall the differential operator

$$D := \frac{1}{2\pi i} \frac{d}{d\tau}.$$

For $\epsilon > 0$ we denote by $\mathcal{F}(\epsilon)$ the truncated fundamental domain

$$\mathcal{F}(\epsilon) = \{\tau \in \mathbb{H} \,|\, |x| \le 1/2,\ |\tau| \ge 1,\ y \le 1/\epsilon\}$$

for $\mathrm{SL}_2(\mathbb{Z})$, and we define the truncated fundamental domain for $\Gamma_0(N)$ by

$$\mathcal{F}_N(\epsilon) = \bigcup_{\gamma \in \Gamma_0(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \gamma \mathcal{F}(\epsilon).$$

We denote by $(\cdot, \cdot)$ the Petersson inner product. Now we introduce the regularized inner product $(f, h)^{\mathrm{reg}}$ for $f \in M_k^!(\Gamma_0(N))$ and $h \in M_k(\Gamma_0(N))$. It is defined to be the constant term in the Laurent expansion at $s = 0$ of the meromorphic continuation in s of the function

$$\lim_{\epsilon \to 0^+} \int_{\mathcal{F}_N(\epsilon)} f(\tau) \overline{h(\tau)} y^{k-s} \frac{dx\,dy}{y^2}.$$

As in [8, Remark 8], if $f \in M_k^!(\Gamma_0(N))$ has vanishing constant term at every cusp of $\Gamma_0(N)$, then

$$(f, h)^{\mathrm{reg}} = \lim_{\epsilon \to 0^+} \int_{\mathcal{F}_N(\epsilon)} f(\tau) \overline{h(\tau)} y^k \frac{dx\,dy}{y^2}.$$

For $k \ge 2$ there is an antilinear differential operator $\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to S_k(\Gamma_0(N))$ defined by

$$\xi_{2-k}(f)(\tau) := y^k\, \overline{L_{2-k} f(\tau)}.$$

Finally, we can define the regularized bilinear pairing $\{f, g\}$ with $f \in M_k^!(\Gamma_0(N))$ and $g \in H_{2-k}(\Gamma_0(N))$ as

$$\{f, g\} := (f, \xi_{2-k}(g))^{\mathrm{reg}}.$$

3. Proof of Theorem 1.1. Let C_N denote the set of all cusps for $\Gamma_0(N)$, and h_s the width for $s \in C_N$. For each cusp $s \in C_N$, take $\sigma_s \in \mathrm{SL}_2(\mathbb{Z})$ so that $\sigma_s(s) = \infty$. Then, at every cusp $s \in C_N$, $f \in M_k^!(\Gamma_0(N))$ and $g \in H_{2-k}(\Gamma_0(N))$ have Fourier expansions

$$(f|_k\sigma_s^{-1})(\tau) = \sum_{n \gg -\infty} c_f(s,n) q_{h_s}^n,$$

$$(g|_{2-k}\sigma_s^{-1})(\tau) = \sum_{n \gg -\infty} c_g^+(s,n) q_{h_s}^n + \sum_{n<0} c_g^-(s,n) \Gamma(k-1, 4\pi|n|y/h_s) q_{h_s}^n,$$

where $q_{h_s} = e^{2\pi i\tau/h_s}$. We denote by $g_s^+(\tau)$ (respectively, $g_s^-(\tau)$) the holomorphic (respectively, nonholomorphic) part of $(g|_{2-k}\sigma_s^{-1})(\tau)$.

THEOREM 3.1. *Let $f \in M_k^!(\Gamma_0(N))$ and $g \in H_{2-k}(\Gamma_0(N))$. Suppose that f has vanishing constant terms at all cusps of $\Gamma_0(N)$. Then, with the notation as above,*

$$\{f,g\} = \sum_{s \in C_N} h_s \Big(\sum_{m+n=0} c_f(s,m) c_g^+(s,n) \Big).$$

Proof. Note that

$$\begin{aligned} d(f(\tau)g(\tau)d\tau) &= \bar{\partial}(f(\tau)g(\tau)\,d\tau) = f(\tau)\left(\frac{\partial}{\partial\bar{\tau}} g(\tau)\right) d\bar{\tau} \wedge d\tau \\ &= -f(\tau)(L_{2-k}g)\frac{dxdy}{y^2} = -f(\tau)\overline{\xi_{2-k}(g)}y^k \,\frac{dxdy}{y^2}. \end{aligned}$$

We put

$$\gamma(s,\epsilon) = \{\tau \in \mathcal{F}_N(\epsilon) \,|\, \mathrm{Im}(\sigma_s\tau) = 1/\epsilon\}.$$

Then

$$\begin{aligned} \{f,g\} &= -\lim_{\epsilon\to 0^+} \int_{\mathcal{F}_N(\epsilon)} d(f(\tau)g(\tau)\,d\tau) = -\lim_{\epsilon\to 0^+} \oint_{\partial\mathcal{F}_N(\epsilon)} f(\tau)g(\tau)\,d\tau \\ &= -\lim_{\epsilon\to 0^+} \sum_{s\in C_N} \int_{\gamma(s,\epsilon)} f(\tau)g(\tau)\,d\tau \\ &= -\lim_{\epsilon\to 0^+} \sum_{s\in C_N} \int_{\sigma_s(\gamma(s,\epsilon))} (f|_k\sigma_s^{-1})(w)(g|_{2-k}\sigma_s^{-1})(w)\,dw \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{with } w = \sigma_s(\tau) \\ &= \lim_{\epsilon\to 0^+} \sum_{s\in C_N} \int_{-h_s/2+i/\epsilon}^{h_s/2+i/\epsilon} (f|_k\sigma_s^{-1})(w)(g|_{2-k}\sigma_s^{-1})(w)\,dw \\ &= \lim_{\epsilon\to 0^+} \sum_{s\in C_N} \int_{-h_s/2}^{h_s/2} (f|_k\sigma_s^{-1})(x+i/\epsilon)(g|_{2-k}\sigma_s^{-1})(x+i/\epsilon)\,dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{with } w = x + i/\epsilon \end{aligned}$$

$$= \lim_{\epsilon\to 0^+} \sum_{s\in C_N} \Big(\int\limits_{-h_s/2}^{h_s/2} (f|_k\sigma_s^{-1})(x+i/\epsilon)\cdot g_s^-(x+i/\epsilon)\,dx$$
$$+ \int\limits_{-h_s/2}^{h_s/2} (f|_k\sigma_s^{-1})(x+i/\epsilon)\cdot g_s^+(x+i/\epsilon)\,dx\Big).$$

If we decompose $(f|_k\sigma_s^{-1})(\tau) = P_{f,s}(\tau) + R_{f,s}(\tau)$ with the principal part

$$P_{f,s}(\tau) = \sum_{-\infty\ll m<0} c_f(s,m)q_{h_s}^m$$

of $(f|_k\sigma_s^{-1})(\tau)$, then

$$\int\limits_{-h_s/2}^{h_s/2} P_{f,s}(x+i/\epsilon)\cdot g_s^-(x+i/\epsilon)\,dx = \int\limits_{-h_s/2}^{h_s/2} \Big(\sum_{-\infty\ll m<0} c_f(s,m)e^{\frac{2\pi i m}{h_s}(x+i/\epsilon)}\Big)$$
$$\times \Big(\sum_{n<0} c_g^-(s,n)\Gamma\Big(k-1, 4\pi|n|\frac{1}{\epsilon h_s}\Big)e^{\frac{2\pi i m}{h_s}(x+i/\epsilon)}\Big)\,dx$$
$$= h_s \sum_{\substack{-\infty\ll m<0,\, n<0\\ m+n=0}} c_f(s,m)c_g^-(s,n)\Gamma\Big(k-1, 4\pi|n|\frac{1}{\epsilon h_s}\Big) = 0.$$

Furthermore,

$$\lim_{\epsilon\to 0^+} \int\limits_{-h_s/2}^{h_s/2} R_{f,s}(x+i/\epsilon)\cdot g_s^-(x+i/\epsilon)\,dx = 0$$

because $g_s^-(x+i/\epsilon)$ decays to zero due to the behavior of the incomplete Gamma function $\Gamma(k-1, 4\pi|n|y) \sim e^{-4\pi|n|y}$ as $y\to\infty$.

Therefore,

$$\{f,g\} = \lim_{\epsilon\to 0^+} \sum_{s\in C_N} \int\limits_{-h_s/2}^{h_s/2} (f|_k\sigma_s^{-1})(x+i/\epsilon)\cdot g_s^+(x+i/\epsilon)\,dx$$
$$= \lim_{\epsilon\to 0^+} \sum_{s\in C_N} \int\limits_{-h_s/2}^{h_s/2} \Big(\sum_{m\gg-\infty} c_f(s,m)e^{\frac{2\pi i m}{h_s}(x+i/\epsilon)}\Big)$$
$$\times \Big(\sum_{n\gg-\infty} c_g^+(s,n)e^{\frac{2\pi i n}{h_s}(x+i/\epsilon)}\Big)\,dx$$
$$= \lim_{\epsilon\to 0^+} \sum_{s\in C_N} h_s\Big(\sum_{m+n=0} c_f(s,m)c_g^+(s,n)\Big)$$
$$= \sum_{s\in C_N} h_s\Big(\sum_{m+n=0} c_f(s,m)c_g^+(s,n)\Big). \blacksquare$$

Hereafter we assume that $k \geq 2$.

LEMMA 3.2. *Let m be a positive integer and $g_m \in H^{\infty}_{2-k}(\Gamma)$ be such that $g_m = g_m^- + g_m^+$ and $g_m^+ = q^{-m} + O(1)$.*

(i) *If $k > 2$, then the g_m is unique.*
(ii) *If $k = 2$, then the g_m is unique up to constants.*

Proof. Suppose that there are two elements satisfying the definition of g_m and let $h \in H^{\infty}_{2-k}(\Gamma)$ be their difference. By Theorem 3.1 we obtain $0 = \{\phi, h\} = (\phi, \xi_{2-k}(h))$ for every $\phi \in S_k(\Gamma)$. It then follows that $\xi_{2-k}(h) = 0$, i.e. $h \in M^!_{2-k}(\Gamma) \cap H^{\infty}_{2-k}(\Gamma)$. Furthermore, $h \in M_{2-k}(\Gamma)$ since h is holomorphic at all cusps. Therefore $h = 0$ if $k > 2$, and h should be constant if $k = 2$. ∎

The uniqueness of the grid in Theorem 1.1 follows immediately from Lemma 3.2. For $m, N \geq 1$ and $k \geq 2$, let $Q(-m, k, N; \tau)$ be the harmonic weak Maass form of weight $2-k$ for $\Gamma_0(N)$ defined as in [8]. Let $\Gamma = \Gamma_0^+(p)$. For $m, n \geq 1$ and $k \geq 2$ we take

$$\begin{aligned} g_m(\tau) &= Q(-m, k, p; \tau) + Q(-m, k, p; \tau)|_{2-k} W_p, \\ f_n(\tau) &= (-n)^{1-k} D^{k-1} g_n(\tau) \end{aligned}$$

and for $n = 0$ and $k > 2$,

$$f_0(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma.$$

Indeed $f_0(\tau)$ is the classical Eisenstein series. For details, we refer the reader to [18, Theorems 2.6.6 and 2.6.9].

The following proposition comes from [21, Proposition 3.3].

PROPOSITION 3.3. *If $k \geq 2$ and $m, N \geq 1$, then $Q(-m, k, N; \tau) \in H^{\infty}_{2-k}(\Gamma_0(N))$ and $Q^+(-m, k, N; \tau) = q^{-m} + O(1)$.*

Proof. For the convenience of the reader, we prove that the principal part of $Q(-m, k, N; \tau)$ at all cusps other than ∞ is constant. It follows from [8, Section 6.2] that

$$D^{k-1} Q(-m, k, N; \tau) = -m^{k-1} P(-m, k, N; \tau), \tag{3.1}$$

where $P(-m, k, N; \tau)$ is the classical Poincaré series defined as

$$P(-m, k, N; \tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e^{-2\pi i m \tau}|_k \gamma.$$

Let s be a cusp which is not $\Gamma_0(N)$-equivalent to ∞, and $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ with

$\sigma(\infty) = s$. Then

$$\begin{aligned} D^{k-1}&(Q(-m,k,N;\tau)|_{2-k}\sigma) \\ &= (D^{k-1}Q(-m,k,N;\tau))|_k\sigma \quad \text{by Bol's identity [2]} \\ &= -m^{k-1}P(-m,k,N;\tau)|_k\sigma \quad \text{by (3.1)} \\ &= -m^{k-1}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} e^{-2\pi i m\tau}|_k\gamma\sigma. \end{aligned}$$

Since $\gamma\sigma(\infty)\neq\infty$ for all $\gamma\in\Gamma_0(N)$, we see that $\gamma\sigma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ with $c\neq 0$. Thus

$$|e^{-2\pi i m\tau}|_k\gamma\sigma| = \frac{e^{2\pi m y/|c\tau+d|^2}}{|c\tau+d|^k} \le \frac{e^{2\pi m/y}}{|c\tau+d|^k}.$$

This implies that if $k>2$, then $P(-m,k,N;\tau)|_k\sigma$ is bounded as $y\to\infty$, whence $Q(-m,k,N;\tau)$ has a constant principal part at the cusp s.

Now we assume $k=2$. In this case,

$$\begin{aligned} \varphi_m^*(\tau) &= M_{0,1/2}(4\pi m y)e^{-2\pi i m x} \\ &= 2\pi\sqrt{m}y^{1/2}I_{1/2}(2\pi m y)e^{-2\pi i m x} \quad \text{by [1, (13.6.3)] or [7, Section 3.1].} \end{aligned}$$

This implies that our $Q(-m,2,N;\tau)$ equals $2\pi\sqrt{m}\,\mathcal{F}_m(\tau,1)$, where $\mathcal{F}_m(\tau,s)$ is the Niebur-Poincaré series. Since $\mathcal{F}_m(\tau,1)$ has a constant principal part at all cusps other than ∞ (see [19, 17] and [10, Section 2]), we have the desired assertion. ∎

We remark that g_m and f_n have Fourier expansions

$$\begin{aligned} g_m(\tau) &= g_m^-(\tau) + q^{-m} + \sum_{n\ge 0} c_{g_m}^+(n)q^n \in H_{2-k}(\Gamma_0^+(p)), \\ f_n(\tau) &= q^{-n} + \sum_{m>0} c_{f_n}(m)q^m \in M_k^!(\Gamma_0^+(p)). \end{aligned}$$

Theorem 3.4. *Let $\Gamma=\Gamma_0^+(p)$. With the same notation as above:*

(i) *If $k>2$, then*

$$c_{g_m}^+(n) = -c_{f_n}(m) \quad \textit{for all } m\ge 1 \textit{ and } n\ge 0.$$

(ii) *If $k=2$, then the relation in* (i) *holds for all $m,n\ge 1$.*

Proof. If $n=0$ and $k>2$, then $\{f_0,g_m\} = (f_0,\xi_{2-k}(g_m)) = 0$ since f_0 is an Eisenstein series and $\xi_{2-k}(g_m)$ is a cusp form (see [18, Theorem 2.6.10]). For $n\ge 1$, we can infer

$$\{f_n,g_m\} = (f_n,\xi_{2-k}(g_m))^{\mathrm{reg}} = (-n)^{1-k}(D^{k-1}g_n,\xi_{2-k}(g_m))^{\mathrm{reg}} = 0$$

from [8, Corollary 4.3] or [3, Theorem 1.2]. By Theorem 3.1 we have

$$
\begin{aligned}
0 = \{f_n, g_m\} &= \sum_{s\in C_p} h_s\Big(\sum_{m'+n'=0} c_{f_n}(s,m')c^+_{g_m}(s,n')\Big)\\
&= \sum_{m'+n'=0} c_{f_n}(\infty,m')c^+_{g_m}(\infty,n') + p\sum_{m'+n'=0} c_{f_n}(0,m')c^+_{g_m}(0,n')\\
&= (1+p)\sum_{m'+n'=0} c_{f_n}(\infty,m')c^+_{g_m}(\infty,n')\\
&= (1+p)(c_{f_n}(\infty,m) + c^+_{g_m}(\infty,n)). \ \blacksquare
\end{aligned}
$$

Let $\Gamma = \Gamma_0(N)$. For $m, n \geq 1$ and $k \geq 2$ we take

$$g_m(\tau) = Q(-m,k,N;\tau), \qquad f_n(\tau) = (-n)^{1-k}D^{k-1}g_n(\tau)$$

and for $n = 0$ and $k > 2$,

$$f_0(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} 1|_k\gamma.$$

We observe from [21, Proposition 3.3] that $g_m \in H^\infty_{2-k}(\Gamma_0(N))$ and $f_n \in M^\infty_k(\Gamma_0(N))$. Now we write the Fourier expansions of g_m and f_n as

$$g_m(\tau) = g^-_m(\tau) + q^{-m} + \sum_{n\geq 0} c^+_{g_m}(n)q^n, \qquad f_n(\tau) = q^{-n} + \sum_{m>0} c_{f_n}(m)q^m.$$

Theorem 3.5. *Let $\Gamma = \Gamma_0(N)$. With the same notation as above:*

(i) *If $k > 2$, then*

$$c^+_{g_m}(n) = -c_{f_n}(m) \quad \textit{for all } m \geq 1 \textit{ and } n \geq 0.$$

(ii) *If $k = 2$, then the relation in* (i) *holds for all $m, n \geq 1$.*

Proof. We proceed as in the proof of Theorem 3.4. First, $\{f_n, g_m\} = 0$ for the same reason, and hence

$$
\begin{aligned}
0 &= \{f_n, g_m\} = \sum_{s\in C_N} h_s\Big(\sum_{m'+n'=0} c_{f_n}(s,m')c^+_{g_m}(s,n')\Big)\\
&= \sum_{m'+n'=0} c_{f_n}(\infty,m')c^+_{g_m}(\infty,n') + \sum_{s\in C_N-\{\infty\}} h_s\Big(\sum_{m'+n'=0} c_{f_n}(s,m')c^+_{g_m}(s,n')\Big)\\
&= c_{f_n}(\infty,m) + c^+_{g_m}(\infty,n) + \sum_{s\in C_N-\{\infty\}} h_s\Big(\sum_{m'+n'=0} c_{f_n}(s,m')c^+_{g_m}(s,n')\Big).
\end{aligned}
$$

The Maass–Poincaré series g_m has the Fourier expansion at $s \in C_N - \{\infty\}$ of the form

$$g_m|_{2-k}\sigma_s^{-1} = g^-_{m,s} + \sum_{l\geq 0} c^+_{g_m}(s,l)q^l_{h_s},$$

and also f_n $(n \geq 1)$ has the Fourier expansion

$$f_n|_k\sigma_s^{-1} = (-n)^{1-k}D^{k-1}(g_n|_{2-k}\sigma_s^{-1}) = \sum_{l>0} c_{f_n}(s,l)q^l_{h_s}$$

with

$$c_{f_n}(s,l) = (-n)^{1-k} c^+_{g_n}(s,l)(l/h_s)^{k-1}.$$

We also note that the Eisenstein series f_0 vanishes at all cusps $s \in C_N - \{\infty\}$. Therefore,

$$\sum_{s \in C_N - \{\infty\}} h_s \Big(\sum_{m'+n'=0} c_{f_n}(s,m') c^+_{g_m}(s,n') \Big) = 0. \ \blacksquare$$

In fact, we can prove Theorem 3.5 by dealing with their Fourier coefficients explicitly as follows.

LEMMA 3.6. *For any $m, n \in \mathbb{Z}$, let $K(m,n,c)$ be the Kloosterman sum*

$$K(m,n,c) = \sum_{v(c)^\times} e\Big(\frac{m\bar{v} + nv}{c}\Big).$$

Then

$$K(-m,n,c) = K(-n,m,c).$$

Proof. By direct computation,

$$\begin{aligned} K(-m,n,c) &= \sum_{v(c)^\times} e\Big(\frac{-m\bar{v} + nv}{c}\Big) \\ &= \sum_{v(c)^\times} e\Big(\frac{-m(-v) + n(-\bar{v})}{c}\Big) = K(-n,m,c). \ \blacksquare \end{aligned}$$

LEMMA 3.7. *With the notation as above, for all $m, n \geq 1$,*

$$c^+_{g_n}(m) m^{k-1} = c^+_{g_m}(n) n^{k-1}.$$

Proof. We recall from Remark 1 that the Fourier coefficient of q^n in g^+_m is

$$c^+_{g_m}(n) = -2\pi(-1)^{k/2} \sum_{\substack{c>0 \\ c \equiv 0\,(N)}} \Big(\frac{m}{n}\Big)^{(k-1)/2} \frac{K(-m,n,c)}{c} I_{k-1}\Big(\frac{4\pi\sqrt{|mn|}}{c}\Big).$$

By Lemma 3.6,

$$\begin{aligned} &c^+_{g_m}(n) \\ &= \Big(\frac{m}{n}\Big)^{k-1} \Big(-2\pi(-1)^{k/2} \sum_{\substack{c>0 \\ c \equiv 0\,(N)}} \Big(\frac{n}{m}\Big)^{(k-1)/2} \frac{K(-n,m,c)}{c} I_{k-1}\Big(\frac{4\pi\sqrt{|nm|}}{c}\Big)\Big) \\ &= \Big(\frac{m}{n}\Big)^{k-1} c^+_{g_n}(m). \ \blacksquare \end{aligned}$$

Another proof of Theorem 3.5. When $m, n \geq 1$, Theorem 3.5 follows immediately from Lemma 3.7:

$$c_{f_n}(m) = (-n)^{1-k} c^+_{g_n}(m) m^{k-1} = (-n)^{1-k} c^+_{g_m}(n) n^{k-1} = -c^+_{g_m}(n).$$

Finally when $n = 0$, we find from Lemma 3.6 and the formulas for the Fourier coefficients $c_{f_0}(m)$ and $c^+_{g_m}(0)$ given in Remark 1 that

$$c_{f_0}(m) = -c^+_{g_m}(0). \quad \blacksquare$$

4. Proof of Theorem 1.2. The Hecke operators T_n act on harmonic weak Maass forms and on weakly holomorphic modular forms in the usual way, and the formula for the action on the Fourier coefficients is the same. Since $g_1|T_m$ belongs to the space $H^\infty_{2-k}(\Gamma)$ and has principal part $m^{1-k}q^{-m}$, it follows from Lemma 3.2 that $g_m = m^{k-1}g_1|T_m$. We then have

$$\begin{aligned}
-f_n &= D^{k-1}(n^{1-k}g_n) = D^{k-1}(g_1|T_n) = D^{k-1}\Big(n^{-k/2}\sum_{\substack{ad=n\\0\le b<d}} g_1|_{2-k}\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)\Big)\\
&= n^{-k/2}\sum_{\substack{ad=n\\0\le b<d}} (D^{k-1}g_1)|_k\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right) = n^{1-k}\sum_{\substack{ad=n\\0\le b<d}} n^{k/2-1}(D^{k-1}g_1)|_k\left(\begin{smallmatrix} a & b\\ 0 & d\end{smallmatrix}\right)\\
&= n^{1-k}(D^{k-1}g_1)|T_n = -n^{1-k}f_1|T_n.
\end{aligned}$$

Now we take $\psi = f_1 - f_{k,1}$ where $f_{k,1}$ is the weakly holomorphic modular form in the space $M^\infty_k(\Gamma)$ defined in Remark 1(ii). Then $\psi \in M_k(\Gamma)$. Since $f_n = n^{1-k}f_1|T_n$, we obtain $f_n = n^{1-k}\psi|T_n + n^{1-k}f_{k,1}|T_n$. Now the assertion follows since $\psi|T_n \in M_k(\Gamma)$ and $n^{1-k}f_{k,1}|T_n \in n^{1-k}\mathbb{Z}((q^{-1}, q))$.

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References

[1] M. Abramowitz and I. Stegun, *Pocketbook of Mathematical Functions*, Verlag Harri Deutsch, 1984.

[2] G. Bol, *Invarianten linearer Differentialgleichungen*, Abh. Math. Sem. Univ. Hamburg 16 (1949), 1–28.

[3] K. Bringmann, B. Kane and R. C. Rhoades, *Duality and differential operators for harmonic Maass forms*, Dev. Math. 28 (2013), 85–106.

[4] J. H. Bruinier, *Borcherds Products on $O(2, l)$ and Chern Classes of Heegner Divisors*, Lecture Notes in Math. 1780, Springer, 2002.

[5] J. H. Bruinier and M. Bundschuh, *On Borcherds products associated with lattices of prime discriminant*, Ramanujan J. 7 (2003), 49–61.

[6] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. 125 (2004), 45–90.

[7] J. H. Bruinier and K. Ono, *Algebraic formula for the coefficients of half-integral weight harmonic weak Maass forms*, Adv. Math., to appear.

[8] J. H. Bruinier, K. Ono, and R. C. Rhoades, *Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues*, Math. Ann. 342 (2008), 673–693.

[9] B. Cho and Y. Choie, *Zagier duality for harmonic weak Maass forms of integral weight*, Proc. Amer. Math. Soc. 139 (2011), 787–797.

[10] D. Choi, D. Jeon, S.-Y. Kang, and C. H. Kim, *Exact formulas for traces of singular moduli of higher level modular functions*, J. Number Theory 128 (2008), 700–707.

[11] S. Choi, *Arithmetic of weakly holomorphic modular forms for Hecke groups*, preprint.

[12] S. Choi, *The space of cusp forms and Poincaré series*, preprint.

[13] S. Choi and C. H. Kim, *Congruences for Hecke eigenvalues in higher level cases*, J. Number Theory 131 (2011), 2023–2036.

[14] S. Choi and C. H. Kim, *Basis for the space of weakly holomorphic modular forms in higher level cases*, J. Number Theory 133 (2013), 1300–1311.

[15] W. Duke and P. Jenkins, *On the zeros and coefficients of certain weakly holomorphic modular forms*, Pure Appl. Math. Quart. 4 (2008), 1327–1340.

[16] P. Guerzhoy, *On weak harmonic Maass-modular grids of even integral weights*, Math. Res. Lett. 16 (2009), 59–65.

[17] S.-Y. Kang and C. H. Kim, *Arithmetic properties of traces of singular moduli on congruence subgroups*, Int. J. Number Theory 6 (2010), 1755–1768.

[18] T. Miyake, *Modular Forms*, Springer, Berlin, 1989.

[19] D. Niebur, *A class of nonanalytic automorphic functions*, Nagoya Math. J. 52 (1973), 133–145.

[20] K. Ono, *Algebraicity of harmonic Maass forms*, Ramanujan J. 20 (2009), 297–309.

[21] R. C. Rhoades, *Linear relations among Poincaré series via harmonic weak Maass forms*, Ramanujan J. 29 (2012), 311–320.

[22] D. Zagier, *Traces of singular moduli*, in: Motives, Polylogarithms and Hodge Theory, Part I (Irvine, CA, 1998), Int. Press Lecture Ser. 3, Int. Press, Somerville, MA, 2002, 211–244.

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