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Chebyshev bounds for Beurling numbers

by

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1. Introduction. Before the prime number theorem was proved for the rational integers, the true order of magnitude of the prime-counting function $\pi(x)$ was established for the first time by P. L. Chebyshev as

$$x/\log x \ll \pi(x) \ll x/\log x. \tag{1.1}$$

Here we shall study corresponding relations for Beurling generalized primes (henceforth, *g-primes*). Surveys of g-numbers are given in [BD1] and [MV]. As in the classical case, we call the analogues of (1.1) for g-primes lower and upper *Chebyshev bounds*.

Several conditions have been given for such g-prime bounds (e.g. [Di], [Zh]). It was conjectured by the first author that if the counting function $N(x)$ of integers of a g-number system $\mathcal{N}$ satisfies the integral condition

$$\int_1^\infty |N(x) - Ax|x^{-2}\,dx < \infty \tag{1.2}$$

for some positive number A, then Chebyshev bounds held, but this guess was disproved by an example of J.-P. Kahane ([Ka]).

J. Vindas ([Vn1]) showed that if one augmented (1.2) with the pointwise condition $N(x) - Ax = o(x/\log x)$, then Chebyshev bounds hold. We have found that Vindas' condition can be replaced by the weaker bound

$$N(x) - Ax = O(x/\log x) \tag{1.3}$$

(with a specific O-constant needed for the lower Chebyshev bound). Moreover, as we show in [DZ], the last condition is optimal in the class of pointwise bounds, because Chebyshev bounds can fail to hold if the right side of (1.3) is replaced by $O(f(x)x/\log x)$ for an (arbitrarily slowly growing) unbounded function $f(x)$.

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Added in proof. In a recent article [Vn2], J. Vindas has also shown how (1.2) and (1.3) yield Chebyshev upper bounds.

Here we show that Chebyshev bounds can also be established by using the slightly weaker mean-value conditions (1.4) and (1.5) (below) in place of (1.3) or Vindas' o-condition. As in the classical case, these bounds can be expressed equivalently in terms of the Chebyshev weighted prime-counting function

$$\psi(x) := \sum_{p^\alpha \le x} \log p,$$

namely, as

$$x \ll \psi(x) \ll x.$$

Main Theorem 1.1. *Suppose that the counting function $N(x)$ of the integers of a Beurling generalized number system $\mathcal{N}$ satisfies both* (1.2) *and*

$$\limsup_{x\to\infty} \Big(x^{-1} \int_1^x u^{-1} |N(u) - Au| \log u \, du \Big) < \infty \tag{1.4}$$

with some positive constant A. Then the Chebyshev function ψ of $\mathcal{N}$ satisfies $\psi(x) \ll x$. Moreover, if

$$\liminf_{x\to\infty} \Big(x^{-1} N(x) \log x - x^{-1} \int_1^x u^{-1} N(u) \log u \, du \Big) > 0 \tag{1.5}$$

also holds, then $\psi(x) \gg x$ for all sufficiently large x.

Remark. The inequality (1.4) is an average form of (1.3). Also, (1.5) has an equivalent form

$$\liminf_{x\to\infty} \Big(x^{-1}(N(x) - Ax) \log x - x^{-1} \int_1^x u^{-1} (N(u) - Au) \log u \, du \Big) > -A. \tag{1.6}$$

Theorem 1.1 has a direct consequence.

Corollary 1.2. *If* (1.2) *and*

$$\limsup_{x\to\infty} (x^{-1} |N(x) - Ax| \log x) < A/2 \tag{1.7}$$

are satisfied, or if (1.2) *and*

$$x^{-1}(N(x) - Ax) \log x = o(1)$$

hold, then $x \ll \psi(x) \ll x$ for sufficiently large x.

Hence Theorem 1.1 covers the results of [Di], [Vn1], [Vn2], and [Zh].

We use an analytic argument based on Bochner's proof of the Wiener–Ikehara theorem. Our key additional ingredients are a concrete version of Wiener's division theorem and uniform estimates of derivatives of the Fejér kernel on $\mathbb{R}$.

2. Set-up. As a preliminary, we note that an easy estimate based on (1.2) shows that $N(x) \ll x$, and hence

$$\psi(x) = \sum_{p^\alpha \le x} \log p \le N(x) \log x \ll x \log x,$$

since every power p^α of a g-prime p is a g-integer. This ensures the convergence of Mellin integrals involving ψ in the half-plane $\{s = \sigma + it : \sigma > 1\}$.

Our starting point is the Mellin formula

$$\int_0^\infty e^{-su}\psi(e^u)\,du = -\frac{\zeta'(s)}{s\zeta(s)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s}\frac{d}{ds}\log\{(s-1)\zeta(s)\},$$

valid for $\sigma > 1$, where $\zeta(s)$ is the zeta function associated with the g-number system $\mathcal{N}$. Let λ be a positive number to be chosen later. Following the Wiener–Ikehara method (see e.g. [BD1] or [MV]), multiply both sides of the last formula by

$$\Delta_\lambda(t) := \frac{1}{2}\left(1 - \frac{|t|}{2\lambda}\right)^+$$

and by e^{ity}. Then integrate over $-2\lambda < t < 2\lambda$ and exchange the order of integrations. We find

$$\begin{aligned}(2.1)\qquad & \int_0^\infty e^{-\sigma u}\psi(e^u)k_\lambda(y-u)\,du \\ &= \int_0^\infty e^{-(\sigma-1)u}k_\lambda(y-u)\,du - \int_0^\infty e^{-\sigma u}k_\lambda(y-u)\,du \\ &\quad - \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)\frac{e^{ity}}{s}\frac{d}{ds}\log\{(s-1)\zeta(s)\}\,dt,\end{aligned}$$

where

$$k_\lambda(x) := \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)e^{itx}\,dt = \lambda\left(\frac{\sin\lambda x}{\lambda x}\right)^2$$

is the Fejér kernel for $\mathbb{R}$ (see §6 for a discussion of its properties). Let

$$I_\sigma(y) := \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)\frac{e^{ity}}{s}\frac{d}{ds}\log\{(s-1)\zeta(s)\}\,dt, \qquad \sigma > 1.$$

Now let $\sigma \to 1+$. Since $k_\lambda > 0$, by the monotone convergence theorem, the integral on the left-hand side of (2.1) has a limit for each y. (But it is not guaranteed at the moment to be finite!) Also, from (6.1), $\int_0^\infty k_\lambda(y-u)\,du < \pi$, so the first two integrals on the right-hand side of (2.1) have finite limits.

Hence $I_\sigma(y)$ has a limit as well. It follows that

$$\int_0^\infty e^{-u}\psi(e^u)k_\lambda(y-u)\,du \\ = \int_0^\infty k_\lambda(y-u)\,du - \int_0^\infty e^{-u}k_\lambda(y-u)\,du - \lim_{\sigma\to 1+} I_\sigma(y). \tag{2.2}$$

It is easy to treat the first two integrals on the right side of (2.2) by making a change of variable and using familiar properties of the Fejér kernel. The first integral becomes

$$\int_{-\infty}^{y} k_\lambda(v)\,dv \to \pi$$

as $y\to\infty$. The second integral can be rewritten as

$$\int_{-\infty}^{y/2} e^{v-y}k_\lambda(v)\,dv + \int_{y/2}^{y} e^{v-y}k_\lambda(v)\,dv < \lambda\int_{-\infty}^{y/2} e^{v-y}\,dv + \int_{y/2}^{y} k_\lambda(v)\,dv \to 0$$

as $y\to\infty$. Thus we have

$$\int_0^\infty e^{-u}\psi(e^u)k_\lambda(y-u)\,du = \pi + o(1) - \lim_{\sigma\to 1+} I_\sigma(y), \tag{2.3}$$

where $o(1)$ denotes a function tending to 0 as $y\to\infty$.

We shall deduce Chebyshev bounds from (2.3) by the following steps. Since

$$\frac{1}{\pi}\int_{-\infty}^{\infty} k_\lambda(v)\,dv = 1,$$

the left side of (2.3) is an average of $\psi(e^u)/e^u$. Our main job will be to show that $\lim_{\sigma\to 1+}|I_\sigma(y)|$ is "sufficiently small" for all large values of y. This calculation is more delicate than that in the classical Wiener–Ikehara proof of the prime number theorem. The main reason is that, here, the function $(d/ds)\log\{(s-1)\zeta(s)\}$ does not have a continuous extension to the closed half-plane $\sigma\ge 1$. A version of Wiener's division theorem and derivatives of the Fejér kernel will play key roles in our argument.

3. A decomposition. We show first that $|(s-1)\zeta(s)-A|$ is small for s near 1. Let

$$E(x) := x^{-1}(N(x)-Ax) \quad\text{and}\quad g(s) := \frac{1}{A}\int_1^\infty x^{-s}E(x)\,dx, \qquad \sigma\ge 1.$$

Then, by (1.2),

$$H := \int_1^\infty x^{-1}|E(x)|\,dx = \int_0^\infty |E(e^u)|\,du < \infty \tag{3.1}$$

and, by (1.4),

$$x^{-1}\int_1^x |E(u)|\log u\,du \le B, \qquad 1 \le x < \infty, \tag{3.2}$$

for some constant B. Since

$$(s-1)\zeta(s) = As + (s-1)s\int_1^\infty x^{-s}E(x)\,dx = A\{s + (s-1)sg(s)\}, \tag{3.3}$$

we see that condition (1.2) guarantees a continuous extension of $(s-1)\zeta(s)$ to the closed half-plane $\{s : \sigma \ge 1\}$. It follows that

$$\frac{(s-1)\zeta(s)}{A} \to 1 \qquad \text{as } s \to 1,\ \sigma \ge 1,$$

and thus

$$\left|\frac{(s-1)\zeta(s)}{A} - 1\right| \le \frac{1}{2}, \qquad |s-1| \le \eta_1,\ \sigma \ge 1,$$

with some constant $\eta_1 > 0$. Letting

$$f(s) := 1 - \frac{(s-1)\zeta(s)}{A},$$

we now have

$$(s-1)\zeta(s) = A(1-f(s)), \quad |f(s)| \le 1/2, \qquad |s-1| \le \eta_1,\ \sigma \ge 1.$$

For s in this semidisc, we can write

$$\log\{(s-1)\zeta(s)\} = \log A + \log(1-f(s)).$$

It follows that

$$\frac{d}{ds}\log\{(s-1)\zeta(s)\} = -\frac{f'(s)}{1-f(s)}, \qquad |s-1| < \eta_1,\ \sigma > 1. \tag{3.4}$$

Again, from (3.3) and the definition of $f(s)$,

$$f(s) = -(s-1)\{1 + sg(s)\} \tag{3.5}$$

and hence

$$f'(s) = -\{1 + (2s-1)g(s) + s(s-1)g'(s)\}. \tag{3.6}$$

Substitution of (3.6) into (3.4) yields

$$\frac{d}{ds}\log\{(s-1)\zeta(s)\} = \frac{1+(2s-1)g(s)}{1-f(s)} + \frac{s(s-1)g'(s)}{1-f(s)}, \qquad |s-1| < \eta_1,\ \sigma > 1.$$

Therefore, for $0 < \lambda \le \eta_1/4$, $1 < \sigma \le 1 + \eta_1/2$,

$$(3.7)\qquad I_\sigma(y) = \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)\,\frac{e^{ity}}{s}\,\frac{1+(2s-1)g(s)}{1-f(s)}\,dt + \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)\,\frac{e^{ity}(s-1)g'(s)}{1-f(s)}\,dt$$
$$= I_{1,\sigma}(y) + I_{2,\sigma}(y),$$

say. The integrand of $I_{1,\sigma}(y)$ is continuous on the closed semidisc; it follows that

$$(3.8)\qquad I_1(y) := \lim_{\sigma\to 1+} I_{1,\sigma}(y) = \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)\frac{e^{ity}}{1+it}\,\frac{1+(1+2it)\,g(1+it)}{1-f(1+it)}\,dt$$

exists and is finite. The last integral tends to zero as $y \to \infty$, by the Riemann–Lebesgue lemma. Since $I_{1,\sigma}(y)$ has a limit as $\sigma \to 1+$ as does $I_\sigma(y)$, it follows that the limit of the third function in (3.7), $I_2(y) := \lim_{\sigma\to 1+} I_{2,\sigma}(y)$, also exists; it remains to study this function.

4. Further analysis of $I_{2,\sigma}(y)$. Using (3.5), write part of the integrand of $I_{2,\sigma}(y)$ as

$$\frac{s-1}{1-f(s)} = \frac{s-1}{1+(s-1)\{1+sg(s)\}}\,.$$

We note that as $s = 1+\epsilon+it \to 1+it$,

$$\frac{s-1}{1+(s-1)\{1+sg(s)\}} - \frac{it}{(1+it)\{1+itg(s)\}} \to 0.$$

After some algebra, we find that the difference satisfies

$$(4.1)\qquad \frac{s-1}{1+(s-1)\{1+sg(s)\}} - \frac{it}{(1+it)\{1+itg(s)\}}$$
$$= \frac{(\sigma-1)\,\{1-it(s-1)\,g(s)\}}{s(1+it)(1+it\,g(s))\{1+(s-1)g(s)\}} := (\sigma-1)R(s).$$

By the definition of $g(s)$ and (3.1) we have $|g(s)| \le H/A$, so

$$|itg(s)| \le |(s-1)g(s)| \le 1/2, \qquad |s-1| \le \eta_2,\ \sigma \ge 1,$$

for some constant $\eta_2 > 0$. *Without loss of generality, we henceforth assume also that* $0 < \eta_2 \le \min\{\eta_1, 1\}$ *and write* $D = \{s : \sigma \ge 1,\ |s-1| \le \eta_2\}$. The denominator of $R(s)$ is bounded away from 0 on D, so the function is continuous there.

We now insert the relation (4.1) into the integrand of $I_{2,\sigma}(y)$. For $0 < \lambda \le \eta_2/4$ and $1 < \sigma \le 1 + \eta_2/2$, we find

$$(4.2)\qquad I_{2,\sigma}(y) = \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)\frac{e^{ity}g'(s)it\,dt}{(1+it)(1+itg(s))} + \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)e^{ity}g'(s)(\sigma-1)R(s)\,dt =: I_{3,\sigma}(y) + I_{4,\sigma}(y),$$

say. Since $R(s)$ is continuous on the compact semidisc D, it is bounded there by some constant R, say. Therefore,

$$|I_{4,\sigma}(y)| \le R\int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)(\sigma-1)|g'(s)|\,dt.$$

We have

$$g'(s) = -\frac{1}{A}\int_1^\infty x^{-s}E(x)\log x\,dx$$

and hence

$$(\sigma-1)|g'(s)| \le \frac{1}{A}\int_1^\infty x^{-\sigma}(\sigma-1)(\log x)|E(x)|\,dx.$$

Note that

$$x^{-\sigma}(\sigma-1)(\log x)|E(x)| \le x^{-1}|E(x)|$$

and

$$x^{-(\sigma-1)}(\sigma-1)\log x \to 0 \qquad \text{as } \sigma\to 1+$$

for each point $x \ge 1$. By the dominated convergence theorem,

$$\int_1^\infty x^{-\sigma}(\sigma-1)(\log x)|E(x)|\,dx \to 0 \qquad \text{as } \sigma\to 1+.$$

Therefore

$$(\sigma-1)|g'(s)| \to 0 \qquad \text{as } \sigma\to 1+$$

uniformly for all $t\in\mathbb{R}$. It follows that

$$(4.3)\qquad I_4(y) := \lim_{\sigma\to 1+} I_{4,\sigma}(y) = 0.$$

It remains to study $I_3(y) := \lim_{\sigma\to 1+} I_{3,\sigma}(y)$. This exists since $I_{2,\sigma}(y)$ and $I_{4,\sigma}(y)$ have limits as $\sigma\to 1+$. Since $|itg(s)| \le 1/2$ on D, we can write

$$\frac{it}{(1+it)(1+itg(s))} = \frac{it}{1+it}\sum_{\nu\ge 0}(-1)^\nu (it)^\nu g(s)^\nu.$$

Thus

$$I_{3,\sigma}(y) = \sum_{\nu\ge 0}(-1)^\nu J_{\nu,\sigma}(y),$$

where

$$J_{\nu,\sigma}(y) = \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t) e^{ity} g'(s) \frac{(it)^{\nu+1} g(s)^\nu}{1+it}\, dt. \tag{4.4}$$

5. $J_{\nu,\sigma}$ as a convolution. We represent $J_{\nu,\sigma}$ as an additive convolution of L^1 functions by using two familiar Fourier relations. Suppose $f \in L^1[0,\infty)$, $h \in L^1(-\infty,\infty)$,

$$\hat{f}(x) := \int_0^\infty f(t) e^{-ixt}\, dt, \qquad \check{f}(x) := \hat{f}(-x),$$

and $f \star h(x) = \int_0^\infty h(x-t) f(t)\, dt$. We have (by changing the integration order)

$$\hat{f}(t)\hat{h}(t) = (f \star h)\hat{\ }(t) \tag{5.1}$$

with $f \star h \in L^1$ and also

$$\int_{-\infty}^{\infty} h(t) e^{ity} \hat{f}(t)\, dt = (\check{h} \star f)(y). \tag{5.2}$$

Recall (4.4), which we rewrite as

$$J_{\nu,\sigma}(y) = \int_{-\infty}^{\infty} h(t) e^{ity} F_\sigma(t)\, dt, \qquad \sigma > 1,\ \nu = 0, 1, 2, \ldots,$$

with

$$h(t) = h_\nu(t) := \Delta_\lambda(t)(it)^{\nu+1} \quad \text{and} \quad F_\sigma(t) = F_{\nu,\sigma}(t) := g'(s) g(s)^\nu / (1+it).$$

We shall express $J_{\nu,\sigma}$ as a convolution as in (5.2). Note first that

$$\check{h}(y) = \int_{-2\lambda}^{2\lambda} \Delta_\lambda(t)(it)^{\nu+1} e^{ity}\, dt = k_\lambda^{(\nu+1)}(y). \tag{5.3}$$

It remains to show that $F_\sigma(t)$ can be expressed as the Fourier transform of an L^1 function f_σ. For $\sigma > 1$ and $u > 0$, set

$$G_\sigma(u) := A^{-1} e^{-(\sigma-1)u} E(e^u), \qquad G_\sigma^d(u) := -u G_\sigma(u), \qquad Z(u) := e^{-u}.$$

The factors of $F_\sigma(t)$ have the Fourier representations

$$g(s) = \frac{1}{A}\int_0^\infty e^{-itu}e^{-(\sigma-1)u}E(e^u)\,du =: (G_\sigma)\hat{}(t),$$

$$g'(s) = -\frac{1}{A}\int_0^\infty e^{-itu}e^{-(\sigma-1)u}uE(e^u)\,du =: (G_\sigma^d)\hat{}(t),$$

$$\frac{1}{1+it} = \int_0^\infty e^{-itu}e^{-u}\,du =: \hat{Z}(t).$$

It follows from (5.1) and the preceding formulas that $F_\sigma(t) = \hat{f}_\sigma(t)$, with

$$f_\sigma(u) = (G_\sigma^d \star G_\sigma^{\star\nu} \star Z)(u),$$

the convolution of $\nu+2$ functions, each in $L^1[0,\infty)$. (In order to have $G_\sigma^d(u)$ in $L^1[0,\infty)$, we have assumed $\sigma > 1$.)

We combine the formulas of the last two paragraphs with (5.2) to get

$$J_{\nu,\sigma}(y) = \int_{-\infty}^{\infty} h(t)e^{ity}\hat{f}_\sigma(t)\,dt = (k_\lambda^{(\nu+1)} \star G_\sigma^d \star G_\sigma^{\star\nu} \star Z)(y). \tag{5.4}$$

6. Derivatives of the Fejér kernel. The last expression contains derivatives of the Fejér kernel. Recall that the Fejér kernel is defined on $\mathbb{R}$, for each positive real number λ, by

$$k_\lambda(x) := \frac{1}{2}\int_{-2\lambda}^{2\lambda}\left(1 - \frac{|t|}{2\lambda}\right)e^{ixt}\,dt.$$

Integration shows that

$$k_\lambda(x) = \lambda\left(\frac{\sin\lambda x}{\lambda x}\right)^2,$$

and we have the familiar relations

$$\int_{-\infty}^{\infty} k_\lambda(u)\,du = \pi, \tag{6.1}$$

$$\int_{|u|>\delta} k_\lambda(u)\,du \le 2/(\lambda\delta), \tag{6.2}$$

the latter for any $\delta > 0$.

Here we establish L^1 estimates for derivatives of k_λ on $\mathbb{R}$.

Lemma 6.1. *Let $0 < \lambda \le 1/2$. Then for $\nu = 1, 2, \ldots,$*

$$\int_{-\infty}^{\infty} |k_\lambda^{(\nu)}(x)|\,dx \le \frac{8(2\lambda)^\nu}{\nu+1}. \tag{6.3}$$

Proof. We begin with an absolute bound for $k_\lambda^{(\nu)}(x)$, to be used for $|x|$ small. Start with (5.3) and get the simple inequality

$$|k_\lambda^{(\nu)}(x)| \le \int_0^{2\lambda} \left(1 - \frac{t}{2\lambda}\right) t^\nu \, dt = \frac{(2\lambda)^{\nu+1}}{(\nu+1)(\nu+2)}, \qquad x \in \mathbb{R}. \tag{6.4}$$

For application to larger $|x|$, we shall show that

$$|k_\lambda^{(\nu)}(x)| \le \frac{8(2\lambda)^{\nu-1}}{3x^2}. \tag{6.5}$$

Starting from the relation

$$k_\lambda(x) = \int_0^{2\lambda} \left(1 - \frac{t}{2\lambda}\right) \cos xt \, dt$$

and making ν differentiations, we get

$$k_\lambda^{(\nu)}(x) = \int_0^{2\lambda} \left(1 - \frac{t}{2\lambda}\right) t^\nu T(xt) \, dt,$$

where $T = \pm \sin$ or $\pm \cos$, depending on ν (mod 4). Integrate by parts twice, with $T_1 := \int T$ and $T_2 := \int T_1$. We find

$$k_\lambda^{(\nu)}(x) = \frac{T_2(2\lambda x)}{x^2}(2\lambda)^{\nu-1} + \int_0^{2\lambda} \frac{T_2(xt)}{x^2} \nu t^{\nu-2} \left\{\nu - 1 - \frac{\nu+1}{2\lambda} t\right\} dt,$$

so

$$|k_\lambda^{(\nu)}(x)| \le \frac{(2\lambda)^{\nu-1}}{x^2} + \int_0^{2\lambda} \frac{\nu t^{\nu-2}}{x^2} \left|\nu - 1 - \frac{\nu+1}{2\lambda} t\right| dt.$$

To treat the last integral, set $t^* = 2\lambda(\nu-1)/(\nu+1)$ and note that

$$\nu - 1 - \frac{(\nu+1)t}{2\lambda} \begin{cases} > 0, & 0 \le t < t^*, \\ < 0, & t^* < t \le 2\lambda. \end{cases}$$

Thus

$$\begin{aligned} |k_\lambda^{(\nu)}(x)| &\le \frac{(2\lambda)^{\nu-1}}{x^2} + \frac{1}{x^2} \int_0^{t^*} \nu t^{\nu-2} \left\{\nu - 1 - \frac{\nu+1}{2\lambda} t\right\} dt \\ &\qquad - \frac{1}{x^2} \int_{t^*}^{2\lambda} \nu t^{\nu-2} \left\{\nu - 1 - \frac{\nu+1}{2\lambda} t\right\} dt \\ &= 2(2\lambda)^{\nu-1} x^{-2} \left\{1 + \left(\frac{\nu-1}{\nu+1}\right)^{\nu-1}\right\} \qquad (= 2/x^2 \text{ if } \nu = 1). \end{aligned}$$

Now

$$\left(\frac{\nu-1}{\nu+1}\right)^{\nu-1} \le \frac{1}{3}, \qquad \nu = 2, 3, \ldots,$$

because

$$\frac{d}{d\nu}\log\left\{\left(\frac{\nu+1}{\nu-1}\right)^{\nu-1}\right\} < 0.$$

Thus (6.5) holds for all non-zero x and positive integers ν.

The estimates of (6.4) and (6.5) change relative size at

$$X = \frac{\sqrt{8(\nu+1)(\nu+2)/3}}{2\lambda}.$$

Using the symmetry of $|k_\lambda^{(\nu)}(x)|$ and estimate (6.4) on $(0, X)$ and (6.5) on (X, ∞), we see that

$$\int_{-\infty}^{\infty} |k_\lambda^{(\nu)}(x)|\,dx \le \frac{2(2\lambda)^{\nu+1}X}{(\nu+1)(\nu+2)} + \frac{16(2\lambda)^{\nu-1}}{3X} = 8(2\lambda)^\nu\left\{\frac{2/3}{(\nu+1)(\nu+2)}\right\}^{1/2}.$$

With a trivial estimate of the square root, we get (6.3). ∎

Remark. By using (6.4) for $|x| < 3$ and (6.5) for $|x| \ge 3$, we find the pointwise bound

$$|k_\lambda^{(\nu)}(x)| < \frac{3(2\lambda)^{\nu-1}}{1+x^2} \tag{6.6}$$

for all $x \in \mathbb{R}$, $0 < \lambda \le 1/2$ and $\nu = 1, 2, \ldots$.

7. Two inequalities for $I_3(y)$. Here we combine the convolution identities of (5.1) and (5.2) with estimates for $E(e^u)$ and derivatives of the Fejér kernel. What we find will first justify letting $\sigma \to 1+$ in $J_{\nu,\sigma}$ and then give inequalities (7.4) and (7.8) for $I_3(y)$, which are key for proving the Chebyshev bounds.

Let

$$m_\sigma(u) := (Z \star G_\sigma^d)(u) = \frac{-1}{A}\int_0^u e^{-(u-v)}e^{-(\sigma-1)v}vE(e^v)\,dv.$$

Changing the variables in (3.2) yields

$$|m_\sigma(u)| \le \frac{1}{A}\int_0^u e^{-(u-v)}v|E(e^v)|\,dv \le \frac{B}{A} \quad \text{for } 0 \le u < \infty. \tag{7.1}$$

By the dominated convergence theorem,

$$m(u) := \lim_{\sigma\to 1+} m_\sigma(u)$$

exists for each $u > 0$ and also satisfies $|m(u)| \le B/A$.

Arguing inductively on ν, using the relation $\int_0^\infty |G_1(u)|\,du = H/A$ from (3.1), we see that

$$(G_\sigma^{\star\nu} \star Z \star G_\sigma^d)(u) = (G_\sigma^{\star\nu} \star m_\sigma)(u) \to (G_1^{\star\nu} \star m)(u)$$

as $\sigma \to 1+$, again by dominated convergence, and

$$|(G_\sigma^{\star\nu} \star Z \star G_\sigma^d)(u)| < \frac{B\,H^\nu}{A^{\nu+1}}, \qquad |(G_1^{\star\nu} \star m)(u)| < \frac{BH^\nu}{A^{\nu+1}}.$$

Combining the preceding bounds for $G_\sigma^{\star\nu} \star Z \star G_\sigma^d$ with the L^1 estimate (6.3) for Fejér derivatives, we see that

$$J_{\nu,\sigma}(y) = \int_{-\infty}^{y} \big(G_\sigma^d \star G_\sigma^{\star\nu} \star Z\big)(y-t) k_\lambda^{(\nu+1)}(t)\,dt$$

is absolutely integrable. By one last application of the dominated convergence theorem, we conclude that $J_\nu(y) = \lim_{\sigma\to 1+} J_{\nu,\sigma}(y)$ exists; moreover,

$$\big|J_\nu(y)\big| < \frac{16\lambda B}{(\nu+2)A}\,(2\lambda H/A)^\nu \le (4B/A)\,(2\lambda H/A)^\nu, \qquad \nu \ge 0 \tag{7.2}$$

(with the usual proviso that $\lambda \le 1/2$).

It now follows that $I_3(y)$, the limit of $I_{3,\sigma}(y)$, satisfies

$$I_3(y) = \lim_{\sigma\to 1+} \sum_{\nu\ge 0} (-1)^\nu J_{\nu,\sigma}(y) = \sum_{\nu \ge 0} (-1)^\nu J_\nu(y), \tag{7.3}$$

if we further assume that λ is sufficiently small that $2\lambda H/A < 1$. In this case, $\sum_\nu |J_\nu(y)| < \infty$ and the last equation is justified by the Weierstrass M-test. Therefore, we deduce from (7.3) and (7.2) that

$$|I_3(y)| < \frac{4B/A}{1 - 2\lambda H/A} \le \frac{8B}{A}, \tag{7.4}$$

uniformly in y, provided (1.2) and (1.4) hold and $0 < \lambda \le \min(\eta_2, A/H)/4$.

Now suppose that (1.5) also is satisfied. From the equivalent form (1.6), we see that

$$uE(e^u) - \int_0^u e^{-(u-v)} vE(e^v)\,dv \ge -A + \epsilon, \qquad u \ge u_0, \tag{7.5}$$

with some positive number ϵ and sufficiently large u_0. We apply this inequality for another estimate of $\lim_{\sigma\to 1+} J_{0,\sigma}(y)$ to prove a lower Chebyshev bound.

We start this calculation by noting that

$$\frac{itg'(s)}{1+it} = \Big(1 - \frac{1}{1+it}\Big) g'(s) = -\frac{1}{A}\int_0^\infty e^{-itu}\{e^{-(\sigma-1)u} uE(e^u) - Am_\sigma(u)\}\,du.$$

Hence

$$J_{0,\sigma}(y) = -\frac{1}{A}\int_0^\infty k_\lambda(y-u)\{e^{-(\sigma-1)u}uE(e^u) - Am_\sigma(u)\}\,du \tag{7.6}$$

$$= -\frac{1}{A}\int_0^\infty k_\lambda(y-u)e^{-(\sigma-1)u}\Big\{uE(e^u) - \int_0^u e^{-(u-v)}vE(e^v)\,dv\Big\}\,du$$

$$- \frac{1}{A}\int_0^\infty k_\lambda(y-u)P(\sigma,u)\,du,$$

where

$$P(\sigma,u) := \int_0^u \{e^{-(\sigma-1)u} - e^{-(\sigma-1)v}\}e^{-(u-v)}vE(e^v)\,dv.$$

By (7.5), the right-hand side of (7.6) is at most

$$\frac{A-\epsilon}{A}\int_{u_0}^\infty k_\lambda(y-u)e^{-(\sigma-1)u}\,du$$

$$- \frac{1}{A}\int_0^{u_0} k_\lambda(y-u)e^{-(\sigma-1)u}\Big(uE(e^u) - \int_0^u e^{-(u-v)}vE(e^v)\,dv\Big)\,du$$

$$- \frac{1}{A}\int_0^\infty k_\lambda(y-u)P(\sigma,u)\,du.$$

Let us consider the preceding three integrals. By (7.1), $|P(\sigma,u)| \le B$ for all $\sigma \ge 1$ and $u > 0$; also $P(\sigma,u) \to 0$ as $\sigma \to 1+$. Thus, the last integral tends to 0 as $\sigma \to 1+$, by the dominated convergence theorem. The integrand of the second integral is bounded for $0 < u < u_0$, so it too has a limit as $\sigma \to 1+$. Moreover, if

$$S := \sup_{0<u<u_0}\Big|uE(e^u) - \int_0^u e^{-(u-v)}vE(e^v)\,dv\Big|,$$

then the second integral has absolute value at most

$$\frac{S}{A}\int_0^{u_0} k_\lambda(y-u)\,du = \frac{S}{A}\int_{y-u_0}^{y} k_\lambda(v)\,dv \to 0$$

as $y \to \infty$ by the Cauchy condition for convergent integrals. The monotone convergence theorem applies to the first integral, and we find

$$\lim_{\sigma\to1+}\int_{u_0}^\infty k_\lambda(y-u)e^{-(\sigma-1)u}\,du = \int_{-\infty}^{y-u_0} k_\lambda(v)\,dv = \pi + o(1)$$

as $y \to \infty$. It follows from these observations that

$$\lim_{\sigma\to1+} J_{0,\sigma}(y) \le \Big(1 - \frac{\epsilon}{A}\Big)\pi + o(1). \tag{7.7}$$

The last formula, together with (7.2) for $J_\nu(y)$ with $\nu \ge 1$, gives

$$I_3(y) \le \left(1 - \frac{\epsilon}{A}\right)\pi + \frac{16BH\lambda}{A^2} + o(1) \tag{7.8}$$

as $y \to \infty$, assuming as before that $0 < \lambda \le \min(\eta_2, A/H)/4$.

8. Proof of Theorem 1. Suppose first that conditions (1.2) and (1.4) are satisfied and that we have $0 < \lambda \le \min(\eta_2, A/H)/4$. Starting from the basic relation (2.2) and using the decompositions (3.7) and (4.2), we showed that $I_1(y) \to 0$ as $y \to \infty$, $I_4(y) = 0$, and $|I_3(y)| < 8B/A$. Together, these results give

$$\int_0^\infty e^{-u}\psi(e^u)k_\lambda(y-u)\,du \le \pi + \frac{8B}{A} + o(1). \tag{8.1}$$

By the monotonicity of $\psi(u)$ and e^u and the Fejér kernel estimates (6.1) and (6.2), the left-hand side of (8.1) is at least

$$e^{-y-\delta}\psi(e^{y-\delta}) \int_{y-\delta}^{y+\delta} k_\lambda(y-u)\,du \ge e^{-y-\delta}\psi(e^{y-\delta})\left(\pi - \frac{2}{\lambda\delta}\right)$$

for $0 < \delta < y$. Fixing λ to satisfy the preceding conditions and choosing a constant $\delta > 0$ sufficiently large that $\lambda\delta > 4$, inequality (8.1) gives

$$\limsup_{y\to\infty} e^{-y-\delta}\psi(e^{y-\delta}) \le \frac{2}{\pi}\left(\pi + \frac{8B}{A}\right) =: C$$

i.e.,

$$\limsup_{x\to\infty} e^{-x}\psi(e^x) \le Ce^{2\delta}.$$

This proves the Chebyshev upper bound. For use below, note that

$$e^{-x}\psi(e^x) \le M \tag{8.2}$$

for all $x \ge 0$ with some constant M.

Finally, suppose that (1.5) is also satisfied. By the set of relations used for the upper bound, but this time with $I_3(y)$ estimated by (7.8), we get

$$\int_0^\infty e^{-u}\psi(e^u)k_\lambda(y-u)\,du \ge \pi - \left(1 - \frac{\epsilon}{A}\right)\pi - \frac{16BH\lambda}{A^2} + o(1). \tag{8.3}$$

Using monotonicity again, along with the bound (8.2) and estimates of $\int k_\lambda$ and its tail, we see that the left-hand side of (8.3) is bounded above by

$$\int_{y-\delta}^{y+\delta} e^{-u}\psi(e^u)k_\lambda(y-u)\,du + M \int_{|u-y|\ge\delta} k_\lambda(y-u)\,du \le \pi e^{-y+\delta}\psi(e^{y+\delta}) + \frac{2M}{\lambda\delta}$$

for $0 < \delta < y$. Choose λ satisfying $0 < \lambda < \min\{\eta_2, A/H, \epsilon A\pi/(8BH)\}/4$. Then the inequality (8.3) yields

$$\pi \liminf_{y\to\infty} e^{-y+\delta}\psi(e^{y+\delta}) + \frac{2M}{\lambda\delta} \ge \pi - \left(1 - \frac{\epsilon}{A}\right)\pi - \frac{16BH\lambda}{A^2} \ge \frac{\epsilon\pi}{2A}.$$

Fixing λ and choosing a constant δ large enough that $\lambda\delta > 8AM/(\epsilon\pi)$, we see that

$$\liminf_{x\to\infty} e^{-x}\psi(e^x) \ge e^{-2\delta}\epsilon/(4A) > 0.$$

This proves the Chebyshev lower bound.

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