The Diophantine equation $x(x + 1) \cdots (x + (m - 1)) + r = y^n$

by

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Dedicated to Robert Tijdeman on the occasion of his 60th birthday

1. Introduction. Erdős and Selfridge [7] proved that a product of consecutive integers can never be a perfect power. That is, the equation $x(x + 1) \cdots (x + (m - 1)) = y^n$ has no solutions in positive integers $x, y, m, n$ with $m, n > 1$. A natural problem is to study the equation

(1)  $x(x + 1)(x + 2) \cdots (x + (m - 1)) + r = y^n$

with a non-zero integral (or rational) parameter $r$. M. J. Cohen [6] proved that (1) has finitely many solutions with $m = n$, and Yuan Ping-zhi [13] used the classical theorems of Baker and Schinzel–Tijdeman to show that, with some obvious exceptions, there are at most finitely many solutions with a fixed $m$. (See Theorem 1.2 below.) Some special cases were completely solved by Abe [1] and Alemu [2].

In this paper we prove that (1) has finitely many solutions $(x, y, m, n)$ when $r$ is not a perfect power.

**Theorem 1.1.** Let $r$ be a non-zero rational number which is not a perfect power in $\mathbb{Q}$. Then (1) has at most finitely many solutions $(x, y, m, n)$ satisfying

(2)  $x, m, n \in \mathbb{Z}, \quad y \in \mathbb{Q}, \quad m, n > 1.$

Moreover, all the solutions can be explicitly determined.

We deduce Theorem 1.1 from three more particular results, one of which is the above-mentioned result of Yuan. First of all, let us display two infinite series of solutions which occur for two special values of $r$. For $r = 1/4$ we have the solutions

(3)  $x \in \mathbb{Z}, \quad y = \pm(x + 1/2), \quad m = n = 2.$

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For \( r = 1 \) we have infinitely many solutions
\[
(4) \quad x \in \mathbb{Z}, \quad y = \pm (x^2 + 3x + 1), \quad m = 4, \quad n = 2.
\]
In the following theorem \( m \) is fixed, and we solve (1) in \( x, y, n \).

**Theorem 1.2 (Yuan).** Let \( r \) be a non-zero rational number and \( m > 1 \) an integer.

1. Assume that \((m, r) \not\in \{(2, 1/4), (4, 1)\}\). Then (1) has at most finitely many solutions \((x, y, n)\) satisfying
\[
(5) \quad x, n \in \mathbb{Z}, \quad y \in \mathbb{Q}, \quad n > 1,
\]
and all the solutions can be explicitly determined.

2. Assume that \((m, r) = (2, 1/4)\) or \((m, r) = (4, 1)\). Then, besides the solutions from (3), respectively (4), equation (1) has at most finitely many solutions \((x, y, n)\) satisfying (5), and all these solutions can be explicitly determined.

Yuan formulates his result in a slightly different (and non-equivalent) form, and his proof is about three pages long. For the convenience of the reader, we give in Section 2 a concise proof of Theorem 1.2, following Yuan’s argument with some changes.

Theorem 1.2 implies that \( n \) is bounded in terms of \( m \) and \( r \). It turns out that, when \( r \neq \pm 1 \), it is bounded in terms of \( r \) only.

**Theorem 1.3.** Let \( r \) be a rational number distinct from 0 and \( \pm 1 \). Then there exists an effective constant \( C(r) \) with the following property. If \((x, y, m, n)\) is a solution of (1) satisfying (2) then \( n \leq C(r) \).

Now change the roles: \( n \) is fixed, \( m \) is variable.

**Theorem 1.4.** Let \( r \) be a non-zero rational number and \( n > 1 \) an integer. Assume that \( r \) is not an \( n \)th power in \( \mathbb{Q} \). Then (1) has at most finitely many solutions \((x, y, m)\) satisfying
\[
(6) \quad x, m \in \mathbb{Z}, \quad y \in \mathbb{Q}, \quad m > 1,
\]
and all the solutions can be explicitly determined.

In [8] this theorem is extended (non-effectively) to the equation \( x(x + 1) \ldots (x + (m - 1)) = g(y) \), where \( g(y) \) is an arbitrary irreducible polynomial.

Theorem 1.1 is an immediate consequence of Theorems 1.3 and 1.4. Indeed, assume that \( r \) is not a perfect power. Theorem 1.3 implies that \( n \) is effectively bounded in terms of \( r \). In particular, we have finitely many possible \( n \). Theorem 1.4 implies that for each \( n \) there are at most finitely many possibilities of \((x, y, m)\). This proves Theorem 1.1.

**Remark 1.5.** It is interesting to compare (1) with the classical equation of Catalan \( x^m - y^n = 1 \). This equation has been effectively solved by
Tijdeman [12], and recently Mihăilescu [9] (see also [5]) solved it completely. However, much less is known about the equation \( x^m - y^n = r \) for \( r \neq \pm 1 \). In contrast, for equation (1) the case \( r = \pm 1 \) seems to be the most difficult.

## 2. Proof of Theorem 1.2

In this section \( m > 1 \) is an integer and \( f_m(x) = x(x+1) \ldots (x+m-1) \).

**Proposition 2.1.** Let \( \lambda \) be a complex number. Then the polynomial \( f_m(x) - \lambda \) has at least two simple roots if

\[
(m, \lambda) \notin \left\{ \left(2, -\frac{1}{4}\right), \left(3, \frac{\pm 2}{3\sqrt{3}}\right), (4, -1) \right\}.
\]

It has at least three simple roots if \( m > 2 \) and

\[
(m, \lambda) \notin \left\{ \left(3, \frac{\pm 2}{3\sqrt{3}}\right), (4, -1), \left(4, \frac{9}{16}\right), \left(6, \frac{16(10 \pm 7\sqrt{7})}{27}\right) \right\}.
\]

**Proof.** By the theorem of Rolle, \( f_m'(x) \) has \( m-1 \) distinct real roots. Hence \( f_m(x) - \lambda \) may have roots of order at most 2. Beukers, Shorey and Tijdeman [4, Proposition 3.4] proved that for even \( m \) at most two double roots are possible, and for odd \( m \) only one double root may occur. It follows that for \( m \notin \{2, 3, 4, 6\} \) the polynomial \( f(x) - \lambda \) has at least three simple roots.

We are left with \( m \in \{2, 3, 4, 6\} \). Since the polynomial \( f(x) - \lambda \) has multiple roots if and only if \( \lambda \) is a stationary value of \( f(x) \) (that is, \( \lambda = f(\alpha) \) where \( \alpha \) is a root of \( f'(x) \)), it remains to determine the stationary values of each of the polynomials \( f_2, f_3, f_4, f_6 \) and count the simple roots of corresponding translates. The details are routine and we omit them.

**Corollary 2.2.** Let \( r \) be a non-zero rational number. The polynomial \( f_m(x) + r \) has at least two simple roots if \( (m, r) \notin \{(2, 1/4), (4, 1)\} \). It has at least three simple roots if \( m > 2 \) and \( (m, r) \notin \{(4, 1), (4, -9/16)\} \).

We shall use the classical results of Baker [3] and of Schinzel–Tijdeman [10] on the superelliptic equation

\[
f(x) = y^n.
\]

In Baker’s theorem \( n \in \mathbb{Z} \) is fixed.

**Theorem 2.3** (A. Baker). Assume that \( f(x) \in \mathbb{Q}[x] \) has at least three simple roots and \( n > 1 \), or \( f(x) \) has at least two simple roots and \( n > 2 \). Then (7) has only finitely many solutions in \( x \in \mathbb{Z} \) and \( y \in \mathbb{Q} \), and the solutions can be effectively computed.

(A non-effective version of this theorem goes back to Siegel [11].)

In the theorem of Schinzel and Tijdeman \( n \) becomes a variable.
Theorem 2.4 (Schinzel and Tijdeman). Let \( f(x) \in \mathbb{Q}[x] \) be a polynomial having at least two distinct roots. Then there exists an effective constant \( N(f) \) such that any solution of \((7)\) in \( x, n \in \mathbb{Z}, y \in \mathbb{Q} \) satisfies \( n \leq N(f) \).

Corollary 2.5. Let \( f(x) \in \mathbb{Q}[x] \) be a polynomial having at least three simple roots. Then \((7)\) has at most finitely many solutions in \( x, n \in \mathbb{Z}, y \in \mathbb{Q} \) satisfying \( n > 1 \). If \( f(x) \) has two simple roots then \((7)\) has only finitely many solutions with \( n > 2 \). In both cases the solutions can be explicitly determined.

Proof of Theorem 1.2. Corollaries 2.2 and 2.5 imply that the theorem is true if \( m > 2 \) and \((m, r) \notin \{(4, 1), (4, -9/16)\}\). It remains to consider the cases \( m = 2, (m, r) = (4, 1) \) and \((m, r) = (4, -9/16)\).

Case 1: \( m = 2, r \neq 1/4 \). In this case \( f_2(x) + r \) has two simple roots, and Corollary 2.5 implies that \( f_2(x) + r = y^n \) has at most finitely many solutions with \( n > 2 \) (and these solutions can be explicitly determined). We are left with the equation \( x(x+1) + r = y^2 \), which is equivalent to the equation \((x+1/2+y)(x+1/2-y) = 1/4 - r\), having finitely many solutions.

Case 2: \( m = 2, r = 1/4 \). In this case we have the equation \((x+1/2)^2 = y^n\). It has infinitely many solutions given by \((3)\) and no other solutions. Indeed, if \((x, y, n)\) is a solution with \( n > 2 \) then \( x+1/2 \) is a perfect power, which is impossible because its denominator is 2.

Case 3: \( m = 4, r = 1 \). In this case we have the equation \((x^2+3x+1)^2 = y^n\). It has infinitely many solutions given by \((4)\) and only finitely many other solutions, all of which can be explicitly determined.

Indeed, let \((x, y, n)\) be a solution with \( n > 2 \). If \( n \) is odd, then \( y \) is a perfect square: \( y = z^2 \) and \( x^2 + 3x + 1 = \pm z^n \). Since \( x^2 + 3x + 1 \) has two simple roots, the latter equation has, by Corollary 2.5, only finitely many solutions with \( n \geq 3 \).

If \( n = 2n_1 \) is even then \( x^2 + 3x + 1 = \pm y^{n_1} \), which has finitely many solutions with \( n_1 \geq 3 \). We are left with \( n = 4 \), in which case \( x^2 + 3x + 1 = \pm y^2 \). The equation \( x^2 + 3x + 1 = y^2 \) is equivalent to \((2x + 3 + 2y)(2x + 3 - 2y) = 5\), which has finitely many solutions. The equation \( x^2 + 3x + 1 = -y^2 \) is equivalent to \((2x + 3)^2 + 4y^2 = 5\), which has finitely many solutions as well.

Case 4: \( m = 4, r = -9/16 \). In this case we have the equation \((x+3/2)^2 \times (x^2 + 3x - 1/4) = y^n\). Since its left-hand side has two simple roots, this equation has, by Corollary 2.5, only finitely many solutions with \( n > 2 \). We are left with the equation \((x + 3/2)^2(x^2 + 3x - 1/4) = y^2\), which is equivalent to the equation \(16(x^2 + 3x + 1 - y)(x^2 + 3x + 1 + y) = 25\), having only finitely many solutions.

Theorem 1.2 is proved.
3. Proof of Theorems 1.3 and 1.4. Let \( \alpha \) be a non-zero rational number and \( p \) a prime number. Recall that \( \text{ord}_p(\alpha) \) is the integer \( t \) such that \( p^{-t} \alpha \) is a \( p \)-adic unit. The proofs of both theorems rely on the following simple observation.

**Proposition 3.1.** Let \( p \) be a prime number and \( t = \text{ord}_p(r) \). Then for any solution \( (x, y, m, n) \) of (1), satisfying (2), one has either \( m < (t + 1)p \) or \( n \mid t \).

**Proof.** Assume that \( m \geq (t + 1)p \). Then
\[
\text{ord}_p(x(x + 1)(x + 2) \cdots (x + (m - 1))) \geq t + 1.
\]
Hence
\[
\text{ord}_p(x(x + 1)(x + 2) \cdots (x + (m - 1)) + r) = t,
\]
that is, \( \text{ord}_p(y^n) = t \), which implies that \( n \mid t \). ■

**Proof of Theorem 1.3.** Since \( r \neq \pm 1 \), there exists a prime number \( p \) such that \( t = \text{ord}_p(r) \neq 0 \). Theorem 1.2 implies that for every \( m > 1 \) there exists an effective constant \( N(m) \) such that for any solution of (1) satisfying (2) we have \( n \leq N(m) \). Put \( C'(r) = \max\{N(m) : 2 \leq m < (t + 1)p \} \) if \( t > 0 \) and \( C'(r) = 0 \) if \( t < 0 \). Then \( n \leq C'(r) \) when \( m < (t + 1)p \), and \( n \leq |t| \) by Proposition 3.1 when \( m \geq (t + 1)p \). Thus, in any case \( n \leq C(r) := \max\{C'(r), |t|\} \).

**Proof of Theorem 1.4.** The proof splits into two cases.

**Case 1:** there is a prime \( p \) such that \( n \) does not divide \( t = \text{ord}_p(r) \). In this case Proposition 3.1 implies that \( m \leq (t + 1)p \). Also, \( (n, r) \notin \{(2, 1/4), (4, 1)\} \), because in both these cases \( r \) is an \( n \)-th power. Now Theorem 1.2 implies that we may have only finitely many solutions.

**Case 2:** \( n \) is even and \( r = -r_1^n \), where \( r_1 \in \mathbb{Q} \). Write \( z = (y/r_1)^{n/2} \). Let \( p \) be prime number congruent to 3 mod 4 and such that \( \text{ord}_p(r) = 0 \). If \( m \geq p \) then
\[
\text{ord}_p(1 + z^2) = \text{ord}_p(r^{-1}x(x + 1) \cdots (x + m - 1)) > 0,
\]
which implies that \(-1\) is a quadratic residue mod \( p \), a contradiction. Thus, \( m < p \) and Theorem 1.2 again implies that we may have only finitely many solutions. ■

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