Diophantine equations $E(x) = P(x)$
with $E$ exponential, $P$ polynomial

by

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Dedicated to Robert Tijdeman on his sixtieth birthday

1. Introduction. A theorem of Laurent [2] tells us that polynomial-exponential equations of a fairly general type have only finitely many solutions in integers. It would be desirable to have a version of this theorem with bounds on the number of solutions, which do not depend on the coefficients of the equation. This has been achieved for purely exponential equations [3], and for equations in one variable [4]. In the present paper we will indicate such bounds for certain solutions of the equation of the title.

More precisely, we will deal with equations

$$E(x) = P(x)$$

in $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, where $P$ is a polynomial and $E$ is exponential of the type

$$E(x) = E_1(x_1) + \cdots + E_n(x_n) + c,$$

where $c$ is a complex number, and

$$E_l(x) = a_{l1} x_1^{k_1} + \cdots + a_{l,k_l} x_{k_l} (l = 1, \ldots, n)$$

with $k_l > 0$ and $a_{li} \in \mathbb{C}, \alpha_{li} \in \mathbb{C}^\times$, where no $\alpha_{li}$ is a root of unity ($1 \leq l \leq n, 1 \leq i \leq k_l$). A solution of (1.1) will be called degenerate if

$$\sum_{l \in \lambda} E_l(x_l) = 0$$

for some nonempty subset $\lambda$ of $\{1, \ldots, n\}$. As will be pointed out in Section 2, it is an easy consequence of Laurent’s theorem that there are only finitely many nondegenerate solutions.

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The notation $A \ll B$ will mean that $A \leq c_0 B$ with an effective constant $c_0$ depending only on

$$N := \sum_{l=1}^{n} k_l \quad \text{and} \quad d := \text{total degree of } P.$$ 

Observe that $n \leq N$.

**Theorem.** Suppose $P$ has rational coefficients. Then all but $\ll 1$ solutions of (1.1) are degenerate.

On the other hand, it is easy to give examples of equations with infinitely many degenerate solutions.

A number $\alpha$ is a *radical* of $\beta$ if $\alpha^u = \beta$ for some $u \in \mathbb{N}$. When $P$ has rational coefficients, the equation (1.1) yields the relation

$$E(x) \in \mathbb{Q}.$$ 

In Theorem 1 of [5] it was shown that if no $\alpha_l$ is a radical of an algebraic number of degree $\leq N$, then all but $\ll 1$ solutions of (1.6) are degenerate, so that our present Theorem holds in this case. But observe that we now have the weaker hypothesis that no $\alpha_l$ is a root of unity. The proof of our Theorem will depend on [5], and on some assertions in [3], [4].

**Example.** Let $\alpha, \beta$ in $\mathbb{C}^\times$ be multiplicatively independent, and consider the equation

$$\alpha^{2x_1} - \alpha \cdot \alpha^{3x_2} + \beta^{x_3} - \beta^{5x_4} = x_2 + x_3 - x_1 - x_4.$$ 

The left hand side is as $E(x)$ in (1.2), (1.3), with $c = 0$, $n = 4$, and each $k_l = 1$. When $\lambda$ is a nonempty subset of $\{1, 2, 3, 4\}$, let $S(\lambda)$ be the set of solutions which have (1.4$\lambda$), but not (1.4$\lambda'$) for any nonempty set $\lambda' \subsetneq \lambda$. By the Theorem, all but $\ll 1$ solutions of (1.7) are in $S(\lambda)$ for some $\lambda$. When $\lambda = \{1, 2\}$, so that (1.4$\lambda$) becomes $\alpha^{2x_1} - \alpha \cdot \alpha^{3x_2} = 0$, we obtain $2x_1 = 1 + 3x_2$, therefore $x_1 = 3y + 2$, $x_2 = 2y + 1$ with $y \in \mathbb{Z}$. After insertion into (1.7) we have

$$\beta^{x_3} - \beta^{5x_4} = x_3 - x_4 - y - 1.$$ 

The Theorem does not apply to this last equation since the variable $y$ does not occur in the exponential function on the left hand side. As is easily seen, the only solutions are with $\beta^{x_3} - \beta^{5x_4} = 0$, unless $\beta$ is an algebraic integer. When $\beta \in \mathbb{Z}$ we obtain a 2-parameter family of solutions parametrized by $x_3, x_4$. On the other hand suppose $\beta$ is not a radical of a rational or a quadratic. Then all but $\ll 1$ solutions of (1.8) have $\beta^{x_3} - \beta^{5x_4} = 0$ by Theorem 1 of [5], so that $x_3 = 5x_4$ and $4x_4 - y - 1 = 0$, giving a 1-parameter family of solutions parametrized by $x_4$. As will be shown in Section 3, this conclusion holds under the weaker assumption that $\beta$ is not a radical of a rational, or a quadratic of norm 1. The assumption cannot be entirely
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dispensed with. For instance, if $\beta$ is a quadratic unit of norm $-1$ (so that it is a radical of a unit of norm $1$), the conjugate $\beta'$ of $\beta$ equals $-1/\beta$, and

$$\beta^{-5x_4} - \beta^{5x_4} = -\beta^{5x_4} = \beta^{5x_4} \in \mathbb{Z}$$

when $x_4$ is odd. We then have the family of solutions with $x_3 = -5x_4$, $x_4 = 2t + 1$ where $t \in \mathbb{Z}$.

Similar considerations apply when $\lambda = \{3, 4\}$. For all other nonempty sets $\lambda$ we claim that $|S(\lambda)| \ll 1$. For instance, take $\lambda = \{1, 2, 3\}$. According to [1] (see also the formulations in Section 2 of [5]), the solutions in $S(\lambda)$ fall into $\ll 1$ classes, and for solutions in a given class the triples $(\alpha^{2x_1}, -\alpha\cdot\alpha^{3x_2}, \beta^{x_3})$ are proportional to a given triple, i.e., will have $\alpha^{2x_1} = \gamma(-\alpha\cdot\alpha^{3x_2}) = \gamma'\beta^{x_3}$ for some $\gamma, \gamma'$. But these relations for fixed $\gamma, \gamma'$ have (by the multiplicative independence of $\alpha, \beta$) at most one solution in integers $x_1, x_2, x_3$. Or take $\lambda = \{1, 3\}$, which gives $\alpha^{2x_1} + \beta^{x_3} = 0$, hence $x_1 = x_3 = 0$ by the multiplicative independence of $\alpha, \beta$, and we obtain $-\alpha\cdot\alpha^{3x_2} - \beta^{5x_4} = x_2 - x_4$. By our Theorem, both sides vanish for all but $\ll 1$ solutions, and then $x_2 = x_4 = 0$.

2. Laurent’s theorem. Let polynomials $P_i(x) = P_i(x_1, \ldots, x_n)$ and exponential functions $\alpha_i^x = x_1^{a_{i1}} \ldots x_n^{a_{in}}$ ($1 \leq i \leq q$) with nonzero $a_{ij}$ be given. The symbol $\mathcal{P}$ will denote a partition of $\{1, \ldots, q\}$, also interpreted as a partition of the set of functions $P_i(x)\alpha_i^x$ ($i = 1, \ldots, q$). The notation $\Lambda \in \mathcal{P}$ will mean that $\Lambda$ is a subset determined by $\mathcal{P}$. Further $G(\mathcal{P})$ signifies the group of points $x \in \mathbb{Z}^n$ having $\alpha_i^x = \alpha_j^x$ for every pair $i, j$ of numbers lying in the same set $\Lambda \in \mathcal{P}$.

**Theorem 2.1** (M. Laurent [2]). Let $S(\mathcal{P})$ consist of solutions $x \in \mathbb{Z}^n$ of the system of equations

$$(2.1\mathcal{P}) \quad \sum_{i \in \Lambda} P_i(x)\alpha_i^x = 0 \quad (\Lambda \in \mathcal{P}),$$

which are not solutions of $(2.1\mathcal{P}')$ for any proper refinement $\mathcal{P}'$ of $\mathcal{P}$. Then $S(\mathcal{P})$ is finite if $G(\mathcal{P}) = \{0\}$.

We will derive the (qualitative) result that (1.1) has only finitely many nondegenerate solutions. This equation may be written as

$$(2.2) \quad \sum_{l, i} a_{li}\alpha_{li}^{x_l} - P(x)\alpha_0^x = 0$$

with $\alpha_0 = (1, \ldots, 1)$. It is of polynomial-exponential type with $q = N + 1$ summands. Each solution lies in a set $S(\mathcal{P})$ (not necessarily uniquely determined) where $\mathcal{P}$ is a partition of the set of summands. It will be enough to show that for any $\mathcal{P}$, either $S(\mathcal{P})$ is finite, or its elements are degenerate.
Let \( \mathcal{P} \) be given. Write 0 \( \sim \) 0, and for 1 \( \leq l \leq n \) write \( l \sim 0 \) (and also 0 \( \sim l \)) if both \(-P(x)\alpha_0^x\) and \(a_{il}\alpha_{li}^{xl}\) lie in \( \Lambda \) for some \( \Lambda \in \mathcal{P} \) and some \( i, 1 \leq i \leq k_l \). When 1 \( \leq l, m \leq n \), write \( l \sim m \) if both \(a_{il}\alpha_{li}^{xl}\) and \(a_{mj}\alpha_{mj}^{xm}\) lie in \( \Lambda \) for some \( \Lambda \in \mathcal{P} \) and some \( i, j \) with 1 \( \leq i \leq k_l, 1 \leq j \leq k_m \). On the other hand, for 0 \( \leq l, m \leq n \), write \( l \sim m \) if there are \( l_1, \ldots, l_\nu \) with \( l_1 = l, l_\nu = m \) and \( l_i \sim l_{i+1} \) (1 \( \leq t < \nu \)). Then \( \sim \) is an equivalence relation on the set \( \{0, 1, \ldots, n\} \).

**Case A:** There is just one equivalence class. We claim that \( G(\mathcal{P}) = \{0\} \), which by Laurent’s theorem implies the finiteness of \( \mathcal{S}(\mathcal{P}) \). We have \( l \sim 0 \) for some 1 \( \leq l \leq n \). Then \( x \in G(\mathcal{P}) \) has \( \alpha_{li}^{xl} = \alpha_0^x = 1 \) for some \( i \), therefore \( x_l = 0 \) since \( \alpha_{li} \) is not a root of unity. Say \( m \sim l \) with 1 \( \leq m \leq n \). Then \( \alpha_{mj}^{xm} = \alpha_{li}^{xl} = 1 \) for some \( i, j \), hence \( x_m = 0 \). Continuing in this way we see that 0 = \( x_l = x_m = \ldots \), so that indeed \( G(\mathcal{P}) = \{0\} \).

**Case B:** There is more than one equivalence class. Let \( \lambda = \{l_1, \ldots, l_\nu\} \) be an equivalence class not containing 0. All the \( a_{il}\alpha_{li}^{xl} \) with \( l \in \lambda, 1 \leq i \leq k_l \) belong to sets \( \Lambda \in \mathcal{P} \) which do not contain \(-P(x) = -P(x)\alpha_0^x\) or any \(a_{mj}\alpha_{mj}^{xm}\) with \( m \notin \lambda \). Let these sets be \( \Lambda_1, \ldots, \Lambda_s \). For \( x \in \mathcal{S}(\mathcal{P}) \), the sum of the \( a_{il}\alpha_{li}^{xl} \) with 1 \( \leq i \leq k_l \) and \( l \) belonging to some \( \Lambda_l \), is zero. The union of \( \Lambda_1, \ldots, \Lambda_s \) is the union of the \( a_{il}\alpha_{li}^{xl} \) with 1 \( \leq i \leq k_l \) and \( l \in \lambda \). Therefore (1.4\lambda) holds, and \( x \) is degenerate. ■

3. Rational values of \( \beta^x - \beta^y \). Suppose \( \beta \) is not a radical of a rational, or of a quadratic of norm 1. To prove a certain assertion made in the Introduction it will be enough to show that the set of integer pairs \((x, y)\) with \( x \neq y \) and \( \beta^x - \beta^y \) rational has cardinality \( \ll 1 \).

In view of Theorem 1 of [5] we may assume \( \beta \) to be algebraic. Say \( \beta \) is of degree \( D \), with conjugates \( \beta^{(1)} = \beta, \beta^{(2)}, \ldots, \beta^{(D)} \). Suppose at first that for some \( \sigma, 1 < \sigma \leq D \), the numbers \( \beta, \beta^{(\sigma)} \) are multiplicatively independent. The rationality of \( \beta^x - \beta^y \) implies the equation

\[
\beta^x - \beta^y - \beta^{(\sigma)x} + \beta^{(\sigma)y} = 0.
\]

When \( \mathcal{P} \) is a partition of the set of the four summands on the left hand side, define \( \mathcal{S}(\mathcal{P}) \) as in the preceding section. If \( \Lambda_0 = \{\beta^x, -\beta^y\} \) is a set of \( \mathcal{P} \), then \( \beta^x - \beta^y = 0 \), hence \( x = y \). We will show that for any partition \( \mathcal{P} \) not containing \( \Lambda_0 \), \( |\mathcal{S}(\mathcal{P})| \ll 1 \). When \( \mathcal{P} \) is no proper partition, so that for \((x, y) \in \mathcal{S}(\mathcal{P}) \) no proper subsum of (3.1) vanishes, then by [1], the solutions in \( \mathcal{S}(\mathcal{P}) \) fall into \( \ll 1 \) classes, with solutions in a given class having \( \beta^x = \gamma_1\beta^y = \gamma_2\beta^{(\sigma)x} = \gamma_3\beta^{(\sigma)y} \) with fixed \( \gamma_1, \gamma_2, \gamma_3 \). By the multiplicative independence of \( \beta, \beta^{(\sigma)} \), there can be at most one such pair \((x, y)\). On the other hand, if \( \mathcal{P} \) consists of \( \Lambda_1 = \{\beta^x, -\beta^{(\sigma)x}\} \) and \( \Lambda_2 = \{-\beta^y, \beta^{(\sigma)y}\} \), then again \( x = y = 0 \) for \((x, y) \in \mathcal{S}(\mathcal{P}) \); and the same holds if \( \Lambda_3 = \{\beta^x, \beta^{(\sigma)y}\} \in \mathcal{P} \).
We are left with the case when $\beta, \beta^{(\sigma)}$ are multiplicatively dependent for each $\sigma$. Say for some $\sigma$ we have $\beta^u = \beta^{(\sigma)v}$ with $(u, v) \neq (0, 0)$. Extend $\sigma$ to an element of the Galois group of the normal closure $N$ of $\mathbb{Q}(\beta)$. We obtain $\beta^{u^2} = (\beta^{(\sigma)^v} = \beta^{(\sigma^2)v^2}$, then $\beta^{u^3} = \beta^{(\sigma^3)v^3}, \ldots, \beta^{u^E} = \beta^{(\sigma^E)v^E} = \beta^{v^E}$, where $E = \deg N$. Since $\beta$ is not a root of unity this gives $u^E = v^E$, therefore $u = \pm v$. Introducing the equivalence relation $\sim$ on $\mathbb{C}^\times$ with $\sim \approx \sigma$ if $\sim \approx \sigma$ is a root of unity, we may conclude that for each $\sigma$, either $\beta \approx \beta^{(\sigma)}$ or $\beta \approx 1/\beta^{(\sigma)}$.

Suppose at first that $\beta \approx \beta^{(\sigma)}$ for each $\sigma$. Then $\beta^u = \beta^{(2)^u} = \ldots = \beta^{(D)^u}$ for some $u \in \mathbb{N}$, so that $\beta^u$ is a rational, and $\beta$ among its radicals. Otherwise, if $\beta \not\approx \beta^{(\sigma)}$, hence $\beta \approx 1/\beta^{(\sigma)}$ for some $\sigma$, it is easily seen that this holds for exactly half of the embeddings $\sigma$. So $D$ is even, and after suitable numbering, there is a $u \in \mathbb{N}$ with

$$\beta^u = \beta^{(2)^u} = \ldots = \beta^{(D/2)^u} = 1/\beta^{((D/2)+1)^u} = \ldots = 1/\beta^{(D)^u}. $$

Therefore $\beta^u$ is quadratic with conjugate $1/\beta^u$, so that its norm is 1. And $\beta$ is among its radicals. 

4. An auxiliary lemma. We now begin with the proof of our Theorem. When $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n$, define $\alpha^x$ as in Section 2. We will deal with functions

$$(4.1) \quad F(x) = \sum_{i=1}^m P_i(x) \alpha_i^x$$

with polynomials $P_i$ and distinct elements $\alpha_1, \ldots, \alpha_m$ of $(\mathbb{C}^\times)^n$. Say

$$P_i(x) = \sum_{j=1}^{c_i} c_{ij} M_{ij}(x) \quad (i = 1, \ldots, m)$$

where $M_{i1}, \ldots, M_{ie_i}$ are distinct monomials, and $c_{i1}, \ldots, c_{i, e_i}$ are nonzero.

We will write $F^* \prec F$ if $F^*$ is a function like $F$, with the same $\alpha_1, \ldots, \alpha_m$ and the same monomials $M_{ij}$, but arbitrary coefficients $c_{ij}^*$ ($1 \leq i \leq m$, $1 \leq j \leq e_i$), some of which may be zero.

For $\beta = (\beta_1, \ldots, \beta_q) \in \overline{\mathbb{Q}} \setminus \{0\}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$, write $h(\beta)$ for its absolute logarithmic height, as defined, e.g., in [3, §2]. Our former notation $h(\beta)$ then becomes $h(\beta, 1)$. When $\beta_i = (\beta_{i1}, \ldots, \beta_{iq_i})$ ($i = 1, \ldots, s$), set $h(\beta_1, \ldots, \beta_s) = h(\beta_{11}, \ldots, \beta_{1q_1}, \ldots, \beta_{s1}, \ldots, \beta_{sq_s})$. The following is similar to Lemma 3.3 in [3].

Lemma 4.1. Suppose $F(x)$ is as above, with the coefficients $c_{ij}$, and the components of each $\alpha_i$ in $\overline{\mathbb{Q}}^\times$. Set $c_i = (c_{i1}, \ldots, c_{i, e_i})$ ($i = 1, \ldots, m$) and $q =$
\(e_1 + \ldots + e_m\), and let \(d(F)\) be the maximal total degree of the monomials \(M_{ij}\). Let \(h_0\) be a positive real. Then solutions \(x \in \mathbb{Z}^n\) of

\[
F(x) = 0
\]

with \(x_1 \ldots x_n \neq 0\),

\[
h(\alpha_1^{x} c_1, \ldots, \alpha_m^{x} c_m) \geq h_0 |x|
\]

and maximum norm \(|x| \geq x_0(h_0, q, d(F))\) lie in \(\leq c(q)\) classes, and solutions in a given class \(C\) satisfy

\[
F_C^*(x) = 0
\]

where \(F_C^* < F\), but \(F_C^*\) is not a constant multiple of \(F\).

Proof. The equation (4.2) may be written as

\[
(c_{11} M_{11}(x) + \ldots + c_{1,e_1} M_{1,e_1}(x)) \alpha_1^x + \ldots
\]

\[
+ (c_{m1} M_{m1}(x) + \ldots + c_{m,e_m} M_{m,e_m}(x)) \alpha_m^x = 0.
\]

Introduce vectors \(X, Y\) with \(q\) components:

\[
X = (c_{11} \alpha_1^x, \ldots, c_{1,e_1} \alpha_1^x, \ldots, c_{m1} \alpha_m^x, \ldots, c_{m,e_m} \alpha_m^x),
\]

\[
Y = (M_{11}(x), \ldots, M_{1,e_1}(x), \ldots, M_{m1}(x), \ldots, M_{m,e_m}(x)).
\]

Set \(Z = X \ast Y := (X_1 Y_1, \ldots, X_q Y_q)\). Then (4.2) becomes

\[
Z_1 + \ldots + Z_q = 0.
\]

\(X\) lies in the multiplicative group \(F \subset (\mathbb{C}^\times)^q\) of rank \(\leq n + 1\) generated by the vectors \((\alpha_1^x, \ldots, \alpha_1^x, \ldots, \alpha_m^x, \ldots, \alpha_m^x)\) with \(x \in \mathbb{Z}^n\), and by \((c_{11}, \ldots, c_{1,e_1}, \ldots, c_{m1}, \ldots, c_{m,e_m})\). Now (4.3) becomes

\[
h(X) \geq h_0 |x|.
\]

On the other hand, \(Y \in \mathbb{Q}^q\), and since the \(x_i\) are nonzero, in fact \(Y \in (\mathbb{Q}^\times)^q\) with

\[
h(Y) \leq d(F) \log |x| + \log q.
\]

Therefore

\[
h(Y) \leq (1/4q^2)h(X)
\]

provided \(|x|\) is sufficiently large, say \(|x| \geq x_0(h_0, q, d(F))\). By the Corollary of Lemma 3.1 in [3], solutions \(x\) of (4.2) with (4.5) have \(Z = Z(x)\) in the union of at most \(c(q)\) proper subspaces of the \((q-1)\)-dimensional space given by (4.4). In such a subspace \(u_1 Z_1 + \ldots + u_q Z_q = 0\) where \((u_1, \ldots, u_q)\) is not proportional to \((1, \ldots, 1)\). A subspace corresponds to some \(F^* \prec F\) not proportional to \(F\), and any \(x\) with \(Z(x)\) in the subspace has \(F^*(x) = 0\).
5. A proposition which implies our Theorem. We will consider functions $G_r(x)$ in $x \in \mathbb{Z}^n$ given by

$$G_r(x) = \sum_{l=1}^{n} (g_{rl1}x_1 + \ldots + g_{rlk}x_k) + Q_r(x) \quad (r = 1, \ldots, p)$$

with polynomials $Q_r$, where all the data, i.e., the $g_{rli}$, $\alpha_{li}$ and the coefficients of the $Q_r$, are algebraic. We will suppose that each $\alpha_{li} \neq 0$, and that

$$h(\alpha_{l1}) \geq h > 0 \quad (l = 1, \ldots, n)$$

for some constant $h$. The coefficients $g_{rli}$ are not necessarily nonzero, but write $N$ for the number of those which are, and $d$ for the maximal total degree of $Q_1, \ldots, Q_p$.

**Proposition 5.1.** Suppose there is a partition of $\{1, \ldots, n\}$ into non-empty sets $S_1, \ldots, S_p$ such that

$$g_{rli} \neq 0 \quad \text{for } l \in S_r \quad (r = 1, \ldots, p).$$

Then the solutions $x \in \mathbb{Z}^n$ of the system of equations

$$G_r(x) = 0 \quad (r = 1, \ldots, p)$$

lie in the union of at most $c_1(h, N, d)$ hyperplanes of the type $x_l = \text{const}$, and $c_2(N, d)$ classes, with elements of a given class having

$$g_{rmj}\alpha_{mj}^{x_m} = \gamma g_{sli}\alpha_{li}^{x_l} \neq 0$$

for some pairs $(m, j) \neq (l, i)$, some $r, s$, and some constant $\gamma$.

Note that the coefficients of the polynomials $Q_r$ are not required to be rational. The proof of the proposition is postponed to the next section. Here we will deduce our Theorem from the case $p = 1$, the general case of the proposition being needed only for its proof.

In view of Theorem 1 of [5] we may assume the $\alpha_{li}$ $(1 \leq l \leq n, 1 \leq i \leq k_l)$ in the definition (1.2), (1.3) of $E(x)$ to be algebraic. It is not hard to see that we also may suppose the $a_{li}$ to be algebraic: this may be done by a specialization argument, or as follows.

Let $A = (a_{11}, \ldots, a_{1k_1}, \ldots, a_{nk_n}) \in \mathbb{Q}^N$ be the “coefficient vector” of $E$. We signify this by writing $E(x) = E(A; x)$. We may write

$$A = A_1 + \zeta_2 A_2 + \ldots + \zeta_r A_r$$

where each $A_i$ is in $\overline{\mathbb{Q}}^N$, and $1, \zeta_2, \ldots, \zeta_r$ are linearly independent over $\overline{\mathbb{Q}}$. Let $\xi$ be algebraic of degree $r$ over the number field generated by the entries of $A_1, \ldots, A_r$, and set

$$\tilde{A} = A_1 + \xi A_2 + \ldots + \xi^{r-1} A_r.$$
Since $P$ has coefficients in $\mathbb{Q} \subset \mathbb{Q}$, the equation (1.1), i.e., $E(A; x) = P(x)$, is equivalent to the system $E(A_1; x) = P(x), E(A_2; x) = \ldots = E(A_r; x) = 0$, which in turn is equivalent to $E(\tilde{A}; x) = P(x)$. Similarly, (1.4), i.e., \( \sum_{l \in \lambda} E_l(A; x_l) = 0 \), is equivalent to $\sum_{l \in \lambda} E_l(A; x_l) = 0$. Therefore it will suffice to prove the Theorem for $E(\tilde{A}; x)$. We may indeed assume the coefficients $a_{l_1}$ to be algebraic.

For a function of the type (1.2), (1.3), write $n = n(E)$, and $N = N(E)$ with $N$ given by (1.5), and set $d(P)$ for the total degree of a polynomial $P$. For $n \leq N$ let $R_d(N, n)$ be the maximal number of nondegenerate solutions of equation (1.1), over $E, P$ as in the Theorem, with $n(E) \leq n, N(E) \leq N, d(P) \leq d$, and with algebraic data. The Theorem will follow if we can show that $R_d(1, 1) \leq 1$, $R_d(N, 1) \ll R_d(N - 1, 1)$ when $N > 1$, and $R_d(N, n) \ll R_d(N - 1, n) + R_d(N, n - 1)$ when $n > 1$.

A function $E$ given by (1.2), (1.3) will be called proper if each $\alpha_{l_1}$ is algebraic, we have $a_{l_1} \neq 1$, and absolute logarithmic heights

\[ h(\alpha_{l_1}) \geq \text{Dob}(N) \quad (l = 1, \ldots, n) \]

where $\text{Dob}(N) = 1/(4N(\log^+ N)^3)$ with $\log^+ N = \max(1, \log N)$. By Theorem 2 of [5], there are maps $1T, \ldots, tT$ with $t \leq t_0(N)$, say $jT: \mathbb{Z}^{m_j} \to \mathbb{Z}^n$ with $0 \leq m_j \leq n$, such that every nondegenerate solution $x$ of (1.6), i.e., of $E(x) \in \mathbb{Q}$, is of the form

\[ x = jTy \quad (5.4) \]

for some $j$ and some $y \in \mathbb{Z}^{m_j}$. Furthermore, for each $j$ with $m_j > 0$ the function $j\hat{E}(y) := E(jTy)$ is again of the general type (1.2), (1.3), and is proper.

Observe that for $j$ with $m_j = 0$ there is just one $x$ coming from (5.4), and these together lead to at most $t_0(N) \ll 1$ solutions. We are therefore reduced to studying equations

\[ j\hat{E}(y) = P(jTy) \]

where $m_j > 0$. The maps $jT$ described in [5] are linear (not necessarily homogeneous) with integer coefficients, so that $P(jTy)$ again has rational coefficients. They further have the property that when $x = jTy$ is a nondegenerate solution of $E(x) \in \mathbb{Q}$, then $y$ is a nondegenerate solution of $j\hat{E}(y) \in \mathbb{Q}$. We thus may restrict ourselves to proper functions $E(x)$.

We now apply the proposition with $h = \text{Dob}(N), p = 1, G_1(x) = E(x) - P(x)$. Some of the solutions of (1.1), i.e., of $G_1(x) = 0$, lie in the union of $\ll 1$ hyperplanes $x_l = \text{const}$. When $n = 1$, these simply give $\ll 1$ solutions, and when $n > 1$, then $E_l(x_l)$ may be absorbed into the constant in (1.2), so that we get $\ll R_d(N, n - 1)$ nondegenerate solutions. The remaining solutions of (1.1) lie in $\ll 1$ classes, with elements of a given class.
having
\[(5.5) \quad a_{mj}x_m^{\alpha_{mj}} = \gamma a_{li}x_l^{\alpha_{li}} \]
for some \((l, i) \neq (m, j)\) and some \(\gamma\). There clearly can be no such class unless \(N > 1\).

When \(m = l\), the summands \(a_{li}x_l^{\alpha_{li}}\) and \(a_{lj}x_l^{\alpha_{lj}}\) in (1.3) can be combined to \((1 + \gamma)a_{li}x_l^{\alpha_{li}}\), so that \(k_l\) can be reduced, or we even have \(E_l(x_l) = 0\), so that \(x\) is degenerate. Thus the number of nondegenerate solutions in our class is at most \(R_d(N-1, n)\). Or, when \(n > 1\), we may also have \(m \neq l\) in (5.5). For \(x, x'\) in the same class, (5.5) yields \(x_m^{\alpha_{mj}} - x_m' = x_l^{\alpha_{li}} - x_l'\), and since \(\alpha_{mj}, \alpha_{li}\) are not roots of unity, this either determines \(x_l, x_m\) uniquely, or \(x_l = uz + x_l', x_m = wz + x_m'\) with fixed nonzero \(u, w, z\). Substitution into \(E(x) - P(x)\) gives a function in at most \(n - 1\) variables, so that the number of nondegenerate solutions in our class is \(\leq R_d(N, n-1)\).

6. Proof of Proposition 5.1. Order the monomials in \(x\) as \(M_1 = 1, M_2, M_3, \ldots\) such that the total degrees do not decrease. When \(Q\) is a nonzero polynomial, write \(\varrho(Q)\) for the maximum number \(\varrho\) such that \(M^{\varrho}\) occurs in \(Q\) with nonzero coefficient. Call \(Q\) normalized if this coefficient is 1. Set \(\varrho(Q) = 0\) when \(Q = 0\).

We will do downward induction from \(p = n\) to \(n - 1, n - 2, \ldots, 1\). Given a function
\[G(x) = \sum_{l=1}^{n} (g_{l1}x_l^{\alpha_{l1}} + \ldots + g_{lk}x_l^{\alpha_{lk}}) + Q(x)\]
with the \(\alpha_{li} \neq 0\) and \(Q\) a polynomial, write \(N(G)\) for the number of nonzero coefficients \(g_{li}\). Now set
\[\varrho = \sum_{r=1}^{p} \varrho(Q_r), \quad \mu = N + \varrho.\]
Given \(p\), Proposition 5.1 will be proved by induction on \(\mu\). Observe that \(n \leq N \leq \mu\).

CASE A: Some \(Q_r = 0\), say \(Q_1 = 0\). We will then deal with the equation \(G_1(x) = 0\) of purely exponential type. For a partition \(\mathcal{P}\) of the set of nonzero summands of \(G_1\) (this set is nonempty by the hypothesis), we have \(S(\mathcal{P}) = \emptyset\) if \(\mathcal{P}\) contains a singleton, i.e., a one-element set. We thus may suppose that for some set \(A \in \mathcal{P}\), two summands \(g_{l1i}x_l^{\alpha_{l1i}}\) and \(g_{1mj}x_m^{\alpha_{m1j}}\) with \((l, i) \neq (m, j)\) and nonzero \(g_{l1i}, g_{1mj}\) belong to \(A\). Invoking \([1]\) we see that solutions in \(S(A)\) fall into \(\ll 1\) classes, and \(g_{1mj}x_m^{\alpha_{m1j}} = \gamma g_{l1i}x_l^{\alpha_{l1i}}\) with fixed \(\gamma\) for solutions \(x\) in a given class.

CASE B: Each \(Q_r \neq 0\). After multiplying the \(G_r\)'s \((r = 1, \ldots, p)\) by suitable constants we may assume each \(Q_r\) to be normalized.
Suppose \( l \in S_r \), so that (5.2) holds. Since \( h(\alpha_{t1}) \geq h \) by (5.1), there is, e.g., by Lemma 6 of [5], an integer \( u_l \) such that
\[
h(g_{rt1}\alpha_{t1}^{x_{t1}-u_l}) \geq \frac{1}{4} h(\alpha_{t1})|x_{t1}| \geq \frac{1}{4} h|x_{t1}|
\]
for \( x_{t1} \in \mathbb{Z} \). Therefore \( h(g_{rt1}\alpha_{t1}^{x_{t1}}) \geq \frac{1}{4} h|x_{l} + u_l| = h_0|x_{l} + u_l| \) with
\[
h_0 = \frac{1}{4} h.
\]
Setting \( \hat{g}_{rti} = g_{rti}\alpha_{ti}^{-u_i}, \hat{x}_i = x_{l} + u_l \) we have \( g_{rti}\alpha_{ti}^{x_{ti}} = \hat{g}_{rti}\hat{x}_i \) \((i = 1, \ldots, k)\) and
\[
h(\hat{g}_{rt1}\alpha_{t1}^{\hat{x}_l}) \geq h_0|\hat{x}_l|
\]
for any \( x_{l} \in \mathbb{Z} \). We may express the functions \( G_1, \ldots, G_p \) in terms of \( \hat{x}_i \) instead of \( x_{l} \). We carry this out for each \( l \in S_r \), and then for each \( r, 1 \leq r \leq p \). These substitutions will not affect the numbers \( N(G_r), \varrho(Q_r) \), hence not \( N, \varrho \) or \( \mu \). Each \( Q_r \) will still be normalized. Also, the truth of the desired conclusion of the proposition will not be affected. We therefore may suppose after suitable substitutions that
\[
(6.1) \quad h(g_{rt1}\alpha_{t1}^{x_{t1}}) \geq h_0|x_{l}| \quad (1 \leq r \leq p, \ l \in S_r).
\]
When dealing with systems of equations (5.3) with given \( p \) and \( \mu \) which satisfy (6.1), and with normalized nonzero polynomials \( Q_r \), we will do induction on \( \sigma = \sum_{r=1}^{p} \sigma(Q_r) \), where \( \sigma(Q) \) denotes the number of nonzero coefficients of a polynomial \( Q \). We thus will have another layer of induction.

Without loss of generality we may restrict our attention to solutions \( x \) of (5.3) with
\[
|x| = |x_{1}|.
\]
But \( 1 \in S_r \) for some \( r \), and \( 1 \in S_1 \) without loss of generality. Now (6.1) yields \( h(g_{111}\alpha_{111}^{x_{111}}) \geq h_0|x_{1}| = h_0|x| \), which is \( h(g_{111}\alpha_{111}^{x_{111}}, 1) \geq h_0|x| \) in other notation. In view of this, and since \( Q_1 \), being normalized, has some coefficient 1, the vector whose components are the \( g_{11i}\alpha_{1i}^{x_{1i}} \) and the coefficients of \( Q_1 \), has height \( h_0|x| \). Thus (4.3) holds, and Lemma 4.1 applies. Some solutions of \( G_1(x) = 0 \) may lie on a hyperplane \( x_{l} = 0 \) for some \( l \). Next, there may be solutions with \( |x| < x_{0}(h_0, q, d(Q_1)) \). In the present situation \( q = N(G_1) + \sigma(Q_1) \) is bounded in terms of \( N, d, n \), where \( n \leq N \), so that such solutions certainly lie in not more than \( c_3(h, N, d) \) hyperplanes \( x_{1} = \text{const} \). In view of Lemma 4.1, the remaining solutions fall into at most \( c(q) \leq c_4(N, d) \) classes.

Solutions in a given class \( C \) have \( G_{1}^{*}(x) = 0 \), hence
\[
G_{1}(x) = G_{1}^{*}(x) = 0
\]
where $G^*_C < G_1$, but is not proportional to $G_1$. Say

$$G^*_C = \sum_{l=1}^{n} (g_{ll1}^* x_{l1} + \ldots + g_{lk}^* x_{lk}) + Q^*(x).$$

(An analogous notation will be used for functions $G^{**, G^o, G', G''}$ introduced below.) We will need the matrix $M$ with the $|S_1|$ columns

$$
\begin{pmatrix}
g_{l11}^* \\
g_{l1}^*
\end{pmatrix}
\quad (l \in S_1).
$$

**Subcase B1:** $M$ has rank 1. Then in the pencil of $G^*_1, G^*_C$ there is a nonzero $G^{**}$ with $g_{l1}^{**} = 0$ for each $l \in S_1$. Suppose first that $\varrho(Q^{**}) = \varrho(Q_1)$, so that $M_\varrho$ with $\varrho = \varrho(Q_1)$ occurs in $Q^{**}$ with a coefficient $\theta \neq 0$. Then

$$G^o = G_1 - \theta^{-1} G^{**}$$

has

$$g_{l1}^o = g_{l11} \neq 0 \quad (l \in S_1)$$

and $\varrho(Q^o) < \varrho(Q_1)$. We now replace $G_1, G_2, \ldots, G_p$ by $G^o, G_2, \ldots, G_p$, thus replacing $\varrho$ by a smaller number. Then also $\mu$ is diminished. Since (5.2) still holds with $g_{l11}^o$ in place of $g_{l11}$, induction on $\mu$ may be applied. Now suppose that $\varrho(Q^{**}) < \varrho(Q_1)$. Then after subtracting a suitable multiple of $G^{**}$ from $G_1$, we obtain a function $G^o$ which again has (6.2), where $M_\varrho$ with $\varrho = \varrho(Q_1)$ appears in $Q^o$ with coefficient 1, but where there are fewer summands, i.e., $N(G^o) < N(G_1)$ or $\sigma(Q^o) < \sigma(Q_1)$. Again we replace $G_1, G_2, \ldots, G_p$ by $G^o, G_2, \ldots, G_p$. When $N(G^o) < N(G_1)$, then $N$ and hence $\mu$ is diminished, and again induction on $\mu$ applies. When $N(G^o) = N(G_1)$, then $\mu$ remains unchanged. But $Q^o$ is normalized, and (6.1) is true with $g_{l11}^o$ in place of $g_{l11}$. Since $\sigma(Q^o) < \sigma(Q_1)$, induction on $\sigma$ finishes the argument.

**Subcase B2:** $M$ has rank 2. (This can only happen when $|S_1| \geq 2$, so that $p < n$.) In this case there is a $G^{**}$ in the pencil of $G_1, G^*_C$ with $g_{l11}^{**} = 0$, but $g_{l11}^{**} \neq 0$ for some $l \in S_1$. Set

$$S' = \{l \in S_1 \text{ with } g_{l11}^{**} = 0\},$$

$$S'' = S_1 \setminus S' = \{l \in S_1 \text{ with } g_{l11}^{**} \neq 0\}.$$

Then $S_1 = S' \cup S''$ is a partition into two nonempty sets. Setting $G' = G_1$, $G'' = G^{**}$ we have

$$g_{l1}' \neq 0 \quad \text{for } l \in S', \quad g_{l1}'' \neq 0 \quad \text{for } l \in S''.$$

Now $x$ is a common zero of the system

$$G'(x) = G''(x) = G_2(x) = \ldots = G_p(x) = 0.$$

Since $S' \cup S'' \cup S_2 \cup \ldots \cup S_p$ is a partition of $\{1, \ldots, n\}$, we may invoke the case $p + 1$ of the proposition. ■
References


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