Iwasawa $\lambda_3$-invariants of certain cubic fields

by

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1. Introduction. Let $l$ be a prime number and $k$ a finite extension of $\mathbb{Q}$. We denote by $\lambda_l(k)$ (resp. $\mu_l(k)$) the Iwasawa $\lambda$ (resp. $\mu$)-invariant of the cyclotomic $\mathbb{Z}_l$-extension $k_\infty/k$. If $k/\mathbb{Q}$ is an abelian extension, then it was shown by Ferrero and Washington [FW] that $\mu_l(k) = 0$ for any prime $l$. In [G1], Greenberg conjectured that $\lambda_l(k) = \mu_l(k) = 0$ for any totally real number field $k$. For a cyclic $l$-extension of $\mathbb{Q}$, one can deduce the following result from [I1] and [I3]:

**Theorem A.** Let $l$ be an odd prime number and $p$ a prime number which is congruent to 1 modulo $l$. Denote by $\mathbb{Q}(l)(p)$ the unique subfield of $\mathbb{Q}(\zeta_p)$ with $[\mathbb{Q}(l)(p) : \mathbb{Q}] = l$, where $\zeta_p$ is a primitive $p$th root of unity. If either

$$l^{(p-1)/l} \neq 1 \pmod{p} \quad \text{or} \quad p \neq 1 \pmod{l^2},$$

then $\lambda_l(\mathbb{Q}(l)(p)) = \mu_l(\mathbb{Q}(l)(p)) = 0$.

The authors of [FKOT] considered the case where $l = 3$ and both of the conditions of Theorem A are not satisfied. Put $k = \mathbb{Q}(3)(p)$. They have shown that $\lambda_3(k) = \mu_3(k) = 0$ if $(E_{\mathbb{Q}_1} : N_{k_1/\mathbb{Q}_1} E_{k_1}) = 9$, where $k_1$ (resp. $\mathbb{Q}_1$) is the first layer of the cyclotomic $\mathbb{Z}_3$-extension of $k$ (resp. $\mathbb{Q}$), $E_{k_1}$ (resp. $E_{\mathbb{Q}_1}$) is the unit group of $k_1$ (resp. $\mathbb{Q}_1$) and $N_{k_1/\mathbb{Q}_1}$ is the norm map from $k_1$ to $\mathbb{Q}_1$.

Recently, Komatsu investigated the field $k = \mathbb{Q}(3)(73)$ and proved that $\lambda_3(k) = \mu_3(k) = 0$ (see [K]). (Note that $3^{(73-1)/3} \equiv 1 \pmod{73}$, $73 \equiv 1 \pmod{3^2}$ and $(E_{\mathbb{Q}_1} : N_{k_1/\mathbb{Q}_1} E_{k_1}) = 3$.)

In the present paper, we give simple sufficient conditions on $p$ for $\lambda_3(\mathbb{Q}(3)(p)) = \mu_3(\mathbb{Q}(3)(p)) = 0$ and verify $\lambda_3 = \mu_3 = 0$ for many $\mathbb{Q}(3)(p)$'s including the case where $(E_{\mathbb{Q}_1} : N_{k_1/\mathbb{Q}_1} E_{k_1}) < 9$. Specifically, we show that $\lambda_3(\mathbb{Q}(3)(p)) = \mu_3(\mathbb{Q}(3)(p)) = 0$ for all $p < 10000$ with $p \equiv 1 \pmod{3}$ except for $p = 3907, 7219, 8011, 8677$.

2000 Mathematics Subject Classification: Primary 11R23; Secondary 11R18.

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2. Results. Our main criterion for \( \lambda_3(Q^{(3)}(p)) = \mu_3(Q^{(3)}(p)) = 0 \) is the following:

**Theorem 1.** Let \( k = Q^{(3)}(p) \) be a cubic field with conductor \( p \), where \( p \) is a prime number such that (a) \( 3^{(p-1)/3} \equiv 1 \pmod{p} \) and (b) \( p \equiv 1 \pmod{9} \). Put \( z = g^{(p-1)/9} \) for a primitive root \( g \) modulo \( p \). If

\[
\left( \frac{(z^2 - 1)(z^{-2} - 1)}{(z - 1)(z^{-1} - 1)} \right)^{(p-1)/3} \not\equiv 1 \pmod{p},
\]

then \( \lambda_3(k) = \mu_3(k) = 0 \).

**Proof.** First, we note that condition (a) holds if and only if 3 is decomposed in \( k \) and that condition (b) holds if and only if \( p \) is decomposed in \( Q_1 \). Let \( Q_1(p) \) be the mod \( p \) ray class field of \( Q_1 \) and \( Q_1(p)^{(3)} \) be the maximal subextension of \( Q_1(p)/Q_1 \) whose Galois group over \( Q_1 \) is an elementary abelian 3-group. Then \( k_1 \subseteq Q_1(p)^{(3)} \) and \( Q_1(p)^{(3)}/k_1 \) is unramified, because a prime of \( Q_1 \) lying over \( p \) ramifies in \( k_1 \) and its ramification index in \( Q_1(p)^{(3)}/Q_1 \) is 3. Denote by \( L^{ab}(k_1) \) the maximal unramified abelian 3-extension field of \( k_1 \) which is abelian over \( Q_1 \). Then \( Q_1(p)^{(3)} \subseteq L^{ab}(k_1) \).

Since \( Q_1 \) has class number prime to 3 and the ramification index of every ramified prime in \( L^{ab}(k_1)/Q_1 \) is 3, \( L^{ab}(k_1)/Q_1 \) has no cyclic subextension of degree 9. Hence \( L^{ab}(k_1)/Q_1 \) is an elementary abelian 3-extension of conductor \( p \). Therefore \( Q_1(p)^{(3)} = L^{ab}(k_1) \). We put \( F := Q_1(p)^{(3)} = L^{ab}(k_1) \) for simplicity. For a generator \( \sigma \) of \( \text{Gal}(k_1/Q_1) \), \( \text{Gal}(F/k_1) \simeq A(k_1)/A(k_1)^{3-1} \) by class field theory, where \( A(k_1) \) stands for the 3-Sylow subgroup of the ideal class group of \( k_1 \). Also by class field theory,

\[
\text{Gal}(F/Q_1) \simeq (I_p/S_p)/(I_p/S_p)^3,
\]

where \( I_p \) is the group of the fractional ideals of \( Q_1 \) which are prime to \( p \), and \( S_p = \{ \alpha \mathcal{O}_{Q_1} \mid \alpha \in Q_1^*, \alpha \equiv 1 \pmod{p} \} \subseteq I_p \), \( \mathcal{O}_{Q_1} \) being the integer ring of \( Q_1 \). Since the class number of \( Q_1 \) is prime to 3, we get the exact sequence

\[
E_{Q_1} \to (\mathcal{O}_{Q_1}/p)^x/((\mathcal{O}_{Q_1}/p)^x)^3 \to (I_p/S_p)/(I_p/S_p)^3 \to 0,
\]

where \( E_{Q_1} \) stands for the unit group of \( Q_1 \). Because

\[
(\mathcal{O}_{Q_1}/p)^x \simeq \bigoplus_{p \mid p} (\mathcal{O}_{Q_1}/p)^x \simeq (\mathbb{Z}/(p-1)\mathbb{Z})^\oplus 3,
\]

we get the isomorphism

\[
(\mathcal{O}_{Q_1}/p)^x/((\mathcal{O}_{Q_1}/p)^x)^3 \overset{(p-1)/3}{\approx} (\mathcal{O}_{Q_1}/p)^x \times (p-1)/3.
\]

Therefore it follows from (1)–(3) that

\[
\text{Gal}(F/Q_1) \simeq \text{Coker}(E_{Q_1} \overset{(p-1)/3}{\longrightarrow} (\mathcal{O}_{Q_1}/p)^x \times (p-1)/3).
\]
Since \((O_{Q_1}/p)^{(p-1)/3} \simeq \bigoplus_{p \mid p} (O_{Q_1}/p)^{(p-1)/3} \simeq (\mathbb{Z}/3\mathbb{Z})^3\) we obtain 3-rank\((A(k_1)/A(k_1)^{\sigma-1}) = 3\)-rank\((\text{Gal}(F/k_1)) \leq 2\) by (4).

Let \(\zeta = \zeta_9\) be a primitive 9th root of unity. We put \(\pi = (\zeta - 1)(\zeta^{-1} - 1)\). Then \(\pi \in Q_1\) and \(3O_{Q_1} = \pi^3O_{Q_1}\) in \(Q_1\). Now we choose \(\gamma \in \text{Gal}(Q_1/Q)\) such that \(\pi^\gamma = (\zeta^{-2} - 1)(\zeta^{-2} - 1)\). If we put

\[
C = \langle \pi^{\gamma - 1}, \pi^{\gamma(\gamma - 1)}, \pi^{\gamma^2(\gamma - 1)} \rangle \subseteq E_{Q_1},
\]

then \(C\) is a subgroup of the cyclotomic units of \(Q_1 = Q(\zeta_9)^+\) whose index is prime to 3. Since the class number of \(Q_1\) is prime to 3, we have \(3 \mid [E_{Q_1} : C]\). Therefore

\[(5) \quad \text{Gal}(F/Q_1) \simeq \text{Coker}(C) \rightarrow (O_{Q_1}/p)^{(p-1)/3} = \text{Coker}(\langle \eta, \eta^\gamma, \eta^{\gamma^2} \rangle) \rightarrow (O_{Q_1}/p)^{(p-1)/3},\]

by (4), where \(\eta = (\pi^{\gamma - 1})(p-1)/3\) and \(\varphi\) is the natural projection map.

From the above isomorphism, we find that 3-rank\((A(k_1)/A(k_1)^{\sigma-1}) \leq 1\) if and only if \(\text{Im}(\varphi) \neq 0\). Also, since \(\pi^{(p-1)/3} \cdot \pi^{(p-1)/3} \cdot \pi^{(p-1)/3} \cdot \pi^{(p-1)/3} = 3^{(p-1)/3} \equiv 1 \pmod{p}\) from assumption (a), we obtain \(\eta^\gamma = (\pi^{\gamma^2-\gamma})(p-1)/3 \equiv (\pi^{(p-1)/3} \cdot \pi^{(p-1)/3} \cdot \pi^{(p-1)/3} \cdot (\pi^{(p-1)/3} \cdot (\pi^{(p-1)/3}) - 2^{\gamma - 1} \equiv (\pi^{(p-1)/3} \cdot (\pi^{(p-1)/3}) \equiv \eta \pmod{p}\). Hence

\[(6) \quad \varphi(\eta) = \varphi(\eta^\gamma) = \varphi(\eta^{\gamma^2}).\]

We deduce from (5) and (6) that \(\text{Im}(\varphi) \neq 0\) is equivalent to \(\eta \equiv 1 \pmod{p}\), and that \(A(k_1)/A(k_1)^{\sigma-1} \neq 0\). For \(z \in \mathbb{Z}\) in the statement of the theorem, there exists a prime ideal \(\mathfrak{P}\) of \(Q(\zeta_9)\) lying above \(p\) such that \(\zeta_9 \equiv z \pmod{\mathfrak{P}}\). We denote by \(p\) the prime ideal of \(Q_1\) below \(\mathfrak{P}\). Then \(\eta \equiv 1 \pmod{p}\) if and only if condition (*) holds. If \(\eta \equiv 1 \pmod{p}\), then \(\eta \equiv \eta^{\gamma_i} \equiv 1 \pmod{p^{\gamma_i}}\) for \(i = 1, 2\) by (6), hence \(\eta \equiv 1 \pmod{p}\). Therefore \(\text{Im}(\varphi) \neq 0\) is equivalent to \(\eta \equiv 1 \pmod{p}\), which in turn is equivalent to condition (*). Consequently, the three statements: 3-rank\((A(k_1)/A(k_1)^{\sigma-1}) = 1\), \(\text{Im}(\varphi) = \langle \eta \pmod{p} \rangle \neq 0\), and condition (*), are equivalent.

Next we show that if (*) is satisfied, then the natural map \(D(k_1) \rightarrow A(k_1)/A(k_1)^{\sigma-1}\) is a non-zero map, where \(D(k_1)\) is the subgroup of \(A(k_1)\) consisting of the ideal classes which contain a product of prime ideals lying over 3. One can see that \(D(k_1) \rightarrow A(k_1)/A(k_1)^{\sigma-1}\) is a non-zero map if and only if \(\pi O_{Q_1}\) is not totally decomposed in \(F/Q_1\) by the canonical isomorphism \(\text{Gal}(F/k_1) \simeq A(k_1)/A(k_1)^{\sigma-1}\) and the fact that the prime \(\pi O_{Q_1}\) splits in \(k_1\). This is equivalent to

\[
\pi^{(p-1)/3} \pmod{p} \notin \text{Im}(\varphi),
\]

which in turn is equivalent to

\[
\pi^{(p-1)/3} \neq \eta^a \pmod{p} \quad \text{for any } a \in \mathbb{Z},
\]
because $\text{Im}(\varphi) = \langle \eta \mod p \rangle$ by (6). Now we assume that $\pi^{(p-1)/3} \equiv \eta^a \pmod{p}$ for some $a \in \mathbb{Z}$. Then $\pi^{(p-1)\gamma/3} \equiv \eta^a \gamma \equiv \pi^{(p-1)/3} \pmod{p}$ by (6), hence

$$\eta = \pi^{(p-1)(\gamma-1)/3} \equiv 1 \pmod{p}.$$  

But this contradicts the fact that $\text{Im}(\varphi) \neq 0$, which is equivalent to assumption (*). Hence assumption (*) implies that the map $D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}$ is non-zero. Also if condition (*) holds, then $A(k_1)/A(k_1)^{\sigma-1} \simeq \mathbb{Z}/3\mathbb{Z}$. Hence $D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}$ is surjective. Since $D(k_1)$ is a $\text{Gal}(k_1/\mathbb{Q})$-submodule of $A(k_1)$, the above surjection shows that $A(k_1) = D(k_1)$ by Nakayama’s lemma. $D(k_1)$ capitulates in $k_n$ for sufficiently large $n$, since Leopoldt’s conjecture is valid for an abelian number field $k$ (see [G1] and [B]). Therefore, $\lambda_{3}(k) = \mu_{3}(k) = 0$ by [O, Theorem], since $A(k_1)$ capitulates in $k_{\infty}$. ■

**Remark.** From the above proof, one can find that condition (*) holds if and only if $A(k_1) = D(k_1)$ under assumptions (a) and (b).

In the case where condition (*) does not hold, we give the following sufficient condition on $p$ for $\lambda_{3}(\mathbb{Q}(3)(p)) = \mu_{3}(\mathbb{Q}(3)(p)) = 0$:

**Theorem 2.** Let $k = \mathbb{Q}(3)(p)$ be a cyclic cubic field with conductor $p$, where $p$ is a prime number such that $p \equiv 1 \pmod{9}$ and $3^{(p-1)/3} \equiv 1 \pmod{p}$. Denote by $\chi$ a 3-adic Dirichlet character associated with $k$. Assume that

$$\left( \frac{(z^2 - 1)(z^{-2} - 1)}{(z - 1)(z^{-1} - 1)} \right)^{(p-1)/3} \equiv 1 \pmod{p},$$

and that $f(T, \chi)$ is irreducible in $\mathbb{Z}_3[\chi(\text{Gal}(k/\mathbb{Q}))][[T]]$, where $z$ is as in the statement of Theorem 1 and $f(T, \chi)$ is the Iwasawa power series associated with the 3-adic $L$-function $L_3(s, \chi)$, namely, $f(4^s - 1, \chi) = L_3(s, \chi)$ for $s \in \mathbb{Z}_3$. Then $\lambda_{3}(k) = \mu_{3}(k) = 0$.

**Proof.** Let $k_{\infty}/k$ be the cyclotomic $\mathbb{Z}_3$-extension, and $k_n$ the $n$th layer of $k_{\infty}/k$. Denote by $A(k_n)$ the 3-Sylow subgroup of the ideal class group of $k_n$. Put $G = \text{Gal}(k/\mathbb{Q})$, $A = \mathbb{Z}_3[[\text{Gal}(k_{\infty}/k)]]$, $X = \text{Gal}(L(k_{\infty})/k_{\infty})$ and $\mathfrak{X} = \text{Gal}(M(k_{\infty})/k_{\infty})$, where $L(k_{\infty})/k_{\infty}$ and $M(k_{\infty})/k_{\infty}$ are the maximal unramified pro-3 abelian extension and the maximal 3-ramified pro-3 abelian extension, respectively. Then $X$ and $\mathfrak{X}$ are finitely generated torsion $\Lambda[G]$-modules (see [I2]). Let $\gamma \in \text{Gal}(k_{\infty}(\zeta_3)/k(\zeta_3))$ be such that $\zeta^\gamma = \zeta^4$ for any 3-power-th root of unity $\zeta$, where $\zeta_3$ is a primitive 3rd root of unity, and put $\gamma = \tilde{\gamma}|_{k_{\infty}} \in \text{Gal}(k_{\infty}/k)$. In what follows we identify $\Lambda$ with $\mathbb{Z}_3[[T]]$ via the
correspondence $\gamma \leftrightarrow 1 + T$. For any $\mathbb{Z}_3[G]$-module $M$, we put

$$M_\chi = M \bigotimes_{\mathbb{Z}_3[G]} \mathbb{Z}_3[\chi(G)],$$

where $G$ acts on $\mathbb{Z}_3[\chi(G)]$ via $\chi$.

We now show that $X_\chi \simeq X$ and $X_\chi \simeq X$. Since the kernel of the map

$$\mathbb{Z}_3[G] \rightarrow \mathbb{Z}_3[\chi(G)], \quad \sum a_g g \mapsto \sum a_g \chi(g),$$

is $N_G \mathbb{Z}_3[G]$ ($N_G := \sum g \in G g$), we have $M_\chi = M/N_G M$ for any $\mathbb{Z}_3[G]$-module $M$. Because the class number of the $n$th layer of the cyclotomic $\mathbb{Z}_3$-extension $\mathbb{Q}_\infty/\mathbb{Q}$ is prime to 3, $N_G A(k_n) = 0$. Hence it follows from $X \simeq \lim \limits \rightarrow A(k_n)$ that $X_\chi \simeq X$ (as $\Lambda[G]$-modules), where the projective limit is taken with respect to the norm map. Next we show $X_\chi \simeq X$. It is enough to prove $X^G = 0$ since $N_G X \subseteq X^G$. Let $\sigma$ be a generator of $G$. Then $X/(\sigma - 1)X \simeq \text{Gal}(M(k_\infty)^{ab}/k_\infty)$, where $M(k_\infty)^{ab}$ is the maximal intermediate field of $M(k_\infty)/\mathbb{Q}_\infty$ which is abelian over $\mathbb{Q}_\infty$. Let $\mathfrak{p}$ be a prime of $\mathbb{Q}_\infty$ lying above $p$ and $I_{\mathfrak{p}}$ the inertia subgroup of $\text{Gal}(M(k_\infty)^{ab}/\mathbb{Q}_\infty)$ for $\mathfrak{p}$. Then $I_{\mathfrak{p}} \simeq \mathbb{Z}/3\mathbb{Z}$ and $\sum_{\mathfrak{p} | p} I_{\mathfrak{p}} = \text{Gal}(M(k_\infty)^{ab}/\mathbb{Q}_\infty)$ because $\mathbb{Q}_\infty$ has no proper 3-ramified pro-3 abelian extension. Since the number of primes of $\mathbb{Q}_\infty$ lying above $p$ is finite, $X/X^{\sigma - 1} \simeq \text{Gal}(M(k_\infty)^{ab}/k_\infty)$ is finite. From the exact sequence of $\Lambda$-modules

$$0 \rightarrow X^G \rightarrow X^{\sigma - 1} \xrightarrow{\sigma - 1} X \rightarrow X/X^{\sigma - 1} \rightarrow 0,$$

it follows that $\text{char}_{\Lambda}(X^G) = \text{char}_{\Lambda}(X/X^{\sigma - 1}) = \Lambda$, where $\text{char}_{\Lambda}(M)$ denotes the characteristic ideal of $M$ for any finitely generated torsion $\Lambda$-module $M$. Hence $X^G$ is finite. Because $X$ does not have non-trivial finite $\Lambda$-submodules ([G2]), we obtain $X^G = 0$. Thus $X_\chi \simeq X$.

From the Mazur–Wiles theorem ([MW, p. 214, Theorem])

$$\text{char}_{\Lambda[G]}(X_\chi) = f(4(1 + T)^{-1} - 1, \chi) \Lambda[G]_\chi$$

and the surjection $X_\chi \rightarrow X_\chi$, it follows that

$$\text{char}_{\Lambda[G]}(X_\chi) \supseteq f(4(1 + T)^{-1} - 1, \chi) \Lambda[G]_\chi.$$

Now assume that $X_\chi$ is infinite. Since $f(4(1 + T)^{-1} - 1, \chi)$ is irreducible in $\Lambda[G]_\chi = \mathbb{Z}_3[\chi(G)][[T]]$ by assumption, we see that

$$\text{char}_{\Lambda[G]}(X_\chi) = \text{char}_{\Lambda[G]_\chi}(X_\chi),$$

and $\text{Ker}(X_\chi \rightarrow X_\chi)$ is finite. Since $X_\chi \simeq X$ and $X$ does not have non-trivial finite $\Lambda$-submodules, $\text{Ker}(X_\chi \rightarrow X_\chi) = 0$, which implies $X \simeq X_\chi \simeq X_\chi \simeq X$. From the assumptions $(\frac{(z^2 - 1)(z^2 - 1)}{(z - 1)(z - 1)})(p - 1)/3 \equiv 1 \pmod{p}$ and $((z - 1)(z - 1)(p - 1)/3 \not\equiv 1 \pmod{p}$, we can see that $D(k_1) \neq 0$ as in the proof of Theorem 1, since $\text{Im}(\varphi) = 0$ from the first assumption, and $\pi(p - 1)/3 \neq 1 \pmod{p}$, i.e., $\pi(p - 1)/3 \pmod{p} \not\subseteq \text{Im}(\varphi) = 0$ from the second
(notations as in the proof of Theorem 1). We write $D(k_n)$ for the subgroup of $A(k_n)$ consisting of the ideal classes which contain a product of prime ideals of $k_n$ lying above 3. Because Leopoldt’s conjecture is valid for $k$ (see [B]), $\#D(k_n)$ is bounded (see [G1, Proposition 1]). Hence $\lim D(k_n)$ is a non-trivial finite $A$-submodule of $\lim A(k_n) \simeq X \simeq \mathfrak{x}$, because the norm map $D(k_m) \to D(k_n)$ is surjective for $m \geq n \geq 0$ and $D(k_1) \neq 0$. This contradicts the fact that $\mathfrak{x}$ does not have non-trivial finite $A$-submodules. Thus we have shown that $X \simeq X_\chi$ is finite, which is equivalent to $\lambda_3(k) = \mu_3(k) = 0$.

We obtain the following corollary to Theorem 2:

**COROLLARY 3.** Let $p$ be a prime number such that $p \equiv 1 \pmod{9}$ and $3^{(p-1)/3} \equiv 1 \pmod{p}$. Assume that
$$\left(\frac{z^2 - 1}{z - 1}\right)^{(p-1)/3} \equiv 1 \pmod{p},$$
$$((z - 1)(z^{-1} - 1))^{(p-1)/3} \not\equiv 1 \pmod{p},$$
where $z$ is as in the statement of Theorem 1. Denote by $\chi$ and $\omega$ a 3-adic Dirichlet character corresponding to $\mathbb{Q}(3)(p)$ and the Teichmüller character for the prime 3, respectively. If
$$B_{1,\chi \omega^{-1}} \neq 0 \pmod{3},$$
then $\lambda_3(\mathbb{Q}(3)(p)) = \mu_3(\mathbb{Q}(3)(p)) = 0$, where $B_{1,\chi \omega^{-1}}$ is the generalized Bernoulli number.

**Proof.** It is sufficient to show that $f(T, \chi) \in \mathbb{Z}_3[\zeta_3][[T]]$ is irreducible in $\mathbb{Z}_3[\zeta_3][[T]]$ by Theorem 2. Note that $-B_{1,\chi \omega^{-1}}$ is the constant term of $f(T, \chi)$, and that $g(T) \in \mathbb{Z}_3[\zeta_3][[T]]^\times$ if and only if $g(0) \in \mathbb{Z}_3[\zeta_3]$ is a unit. Hence we see immediately that $f(T, \chi)$ is irreducible in $\mathbb{Z}_3[\zeta_3][[T]]$.

One can easily check whether the conditions of Theorem 1 and Corollary 3 hold or not by computer. We give some examples below.

We consider the prime numbers $p \leq 10000$ congruent to 1 modulo 3. There exist 611 such $p$’s. By Theorem A, one can verify $\lambda_3(\mathbb{Q}(3)(p)) = \mu_3(\mathbb{Q}(3)(p)) = 0$ for 547 among them. The remaining 64 prime numbers are as follows:

73, 271, 307, 523, 577, 613, 757, 919, 991, 1009, 1117, 1531, 1549, 1621, 1783, 2179, 2251, 2269, 2287, 2341, 2971, 3079, 3187, 3529, 3853, 3889, 3907, 4177, 4339, 4483, 4933, 4951, 4969, 5059, 5077, 5113, 5527, 5851, 6067, 6211, 6247, 6301, 6481, 6553, 6967, 7219, 7507, 7561, 7669, 7687, 8011, 8191, 8461, 8677, 8803, 8893, 8929, 9001, 9109, 9181, 9343, 9613, 9901, 9973.

For 42 prime numbers $p$ among these, we can show that $\lambda_3(\mathbb{Q}(3)(p)) = \mu_3(\mathbb{Q}(3)(p)) = 0$ by using Theorem 1. The remaining 22 are:
991, 1117, 1549, 2251, 2269, 2341, 3907, 4483, 4933, 5527, 6247, 6481, 6967, 7219, 7669, 7687, 8011, 8677, 8803, 9001, 9181, 9901.

For 10 prime numbers $p$ among these, we can show that $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ by using Corollary 3. The remaining 12 are:

2269, 3907, 4933, 5527, 6247, 6481, 7219, 7687, 8011, 8677, 9001, 9901.

For 6 prime numbers $p$ among these, we can verify that $f(T, \chi)$ is irreducible and $((z - 1)(z^{-1} - 1))^{(p-1)/3} \equiv 1 \pmod{p}$ by computer, hence $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ by Theorem 2. The remaining 6 are:

2269, 3907, 6481, 7219, 8011, 8677.

For these, one can verify that $f(T, \chi)$ is reducible in the case $p = 7219$ and 8677. Also, one can verify that $((z - 1)(z^{-1} - 1))^{(p-1)/3} \equiv 1 \pmod{p}$ in the case $p = 3907$ and 8011. For $p = 2269$ and 6481, we do not know whether $f(T, \chi)$ is irreducible or not. In what follows we give another method to show $\lambda_3(\mathbb{Q}^{(3)}(p)) = \mu_3(\mathbb{Q}^{(3)}(p)) = 0$ for $p = 2269$ and 6481.

We would like to express our thanks to Prof. Masato Kurihara who communicated to us the following theorem. It gives an upper bound of the $\lambda_3$-invariant of $\mathbb{Q}^{(3)}(p)$:

**Theorem 4 (M. Kurihara).** Let $k = \mathbb{Q}^{(3)}(p)$ be a cyclic cubic field with conductor $p$, where $p$ is a prime number such that $p \equiv 1 \pmod{9}$ and $3^{(p-1)/3} \equiv 1 \pmod{p}$. If

$$((z - 1)(z^{-1} - 1))^{(p-1)/3} \not\equiv 1 \pmod{p},$$

then $\lambda_3(k) \leq 2$. ($\mu_3(k) = 0$ by the Ferrero–Washington theorem.)

**Proof.** If $((z - 1)(z^{-2} - 1))^{(p-1)/3} \not\equiv 1 \pmod{p}$, then $\lambda_3(k) = 0$ by Theorem 1. Hence we may assume that $((z - 1)(z^{-2} - 1))^{(p-1)/3} \equiv 1 \pmod{p}$.

Let $\sigma$ be a generator of $G = \text{Gal}(k_\infty/\mathbb{Q}_\infty)$. As in the proof of Theorem 1, we can see that

$$A(k_1)/A(k_1)^{\sigma-1} \simeq (\mathbb{Z}/3\mathbb{Z})^\oplus 2,$$

and

$$\text{Im}(D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}) \simeq \mathbb{Z}/3\mathbb{Z},$$

since $((z - 1)(z^{-2} - 1))^{(p-1)/3} \equiv 1 \pmod{p}$ and $((z - 1)(z^{-1} - 1))^{(p-1)/3} \not\equiv 1 \pmod{p}$. Let $A'(k_n)$ be the 3-Sylow subgroup of the 3-ideal class group of $k_n$, namely, $A'(k_n) = A(k_n)/D(k_n)$. Then

$$A'(k_1)/A'(k_1)^{\sigma-1} \simeq \text{Coker}(D(k_1) \to A(k_1)/A(k_1)^{\sigma-1}) \simeq \mathbb{Z}/3\mathbb{Z}$$

from (7) and (8).
We now also show that
\[
A'(k_2)/A'(k_2)^{\sigma-1} \simeq \mathbb{Z}/3\mathbb{Z}.
\]

Denote by \( \mathbb{Q}_2(p)^{(3)} \) the maximal abelian extension over \( \mathbb{Q}_2 \) (the second layer of the cyclotomic \( \mathbb{Z}_3 \)-extension over \( \mathbb{Q} \)) of conductor \( p \) whose Galois group over \( \mathbb{Q}_2 \) is an elementary abelian 3-group. Then, as in the proof of Theorem 1, we find that \( \mathbb{Q}_2(p)^{(3)} \) is the maximal unramified abelian 3-extension over \( k_2 \) which is abelian over \( \mathbb{Q}_2 \) and that \( \text{Gal}(\mathbb{Q}_2(p)^{(3)}/k_2) \simeq A(k_2)/A(k_2)^{\sigma-1} \), because the class number of \( \mathbb{Q}_2 \) is prime to 3.

Firstly, we consider the case \( p \not\equiv 1 \pmod{27} \). In this case, as in the proof of Theorem 1, we have \( \text{Gal}(\mathbb{Q}_2(p)^{(3)}/\mathbb{Q}_2) \simeq \text{Coker}(E_{\mathbb{Q}_2}(p^{3-1}/3) \rightarrow ((\mathcal{O}_{\mathbb{Q}_2}/p^3)^{\times}/(p^{3-1}/3)^{\times}) \) and 3-rank\(((\mathcal{O}_{\mathbb{Q}_2}/p)^{\times}/(p^{3-1}/3)^{\times}) = 3 \) because the prime \( p \) decomposes into three primes of degree three in \( \mathbb{Q}_2 \) by the assumption \( p \not\equiv 1 \pmod{27} \). Hence 3-rank\((A(k_2)/A(k_2)^{\sigma-1}) = 3\)-rank\(\left(\text{Gal}(\mathbb{Q}_2(p)^{(3)}/k_2)\right) \leq 2 \). Since the norm maps \( A(k_2) \rightarrow A(k_1) \) and \( D(k_2) \rightarrow D(k_1) \) are surjective, it follows from (7) and (8) that 3-rank\((A(k_2)/A(k_2)^{\sigma-1}) = 2 \) and \( \text{Im}(D(k_2) \rightarrow A(k_2)/A(k_2)^{\sigma-1}) \neq 0 \). Hence 3-rank\(\left(A'(k_2)/A'(k_2)^{\sigma-1}\right) \leq 1 \). Since the norm map \( A'(k_2)/A'(k_2)^{\sigma-1} \rightarrow A'(k_1)/A'(k_1)^{\sigma-1} \) is surjective, the group \( A'(k_2)/A'(k_2)^{\sigma-1} \) is non-trivial, so we have proved (10).

Next, we consider the case \( p \equiv 1 \pmod{27} \). As in the case \( p \not\equiv 1 \pmod{27} \), we deduce \( \text{Im}(D(k_2) \rightarrow A(k_2)/A(k_2)^{\sigma-1}) \neq 0 \) from (8). Since this image is generated by an ideal class containing a prime of \( k_2 \) lying above 3, we have
\[
\text{Im}(D(k_2) \rightarrow A(k_2)/A(k_2)^{\sigma-1}) \simeq \mathbb{Z}/3\mathbb{Z}.
\]

Let \( p_2 \) be a prime ideal of \( k_2 \) lying above \( p \) and \( p_1 \) the prime ideal of \( k_1 \) below \( p_2 \). We choose a primitive 9th root of unity \( \zeta_9 \) such that \((z-1)(z^{-1}-1) \equiv (\zeta_9-1)(\zeta_9^{-1}-1) \pmod{p_1} \). Further, we choose a primitive 27th root of unity \( \zeta_{27} \) such that \( N_{\mathbb{Q}_2/Q_1}((\zeta_{27}-1)(\zeta_{27}^{-1}-1)) = (\zeta_9-1)(\zeta_9^{-1}-1) \). Let \( \gamma \) be a topological generator of \( \Gamma = \text{Gal}(k_\infty/k) \) such that \((\zeta_9-1)(\zeta_9^{-1}-1)\gamma = (\zeta_9-1)(\zeta_9^{-2}-1) \), and \( \gamma_n \) the restriction of \( \gamma \) to \( k_n \). Then \( \gamma_n \) is a generator of \( \Gamma_n = \text{Gal}(k_n/k) \). For simplicity, we put \( \pi_i = (\zeta_{3i+1}-1)(\zeta_{3i+1}^{-1}-1) \) and \( \eta_i = (\pi_i^{n_i-1}/3) \) for \( i = 1, 2 \). Then it follows from the assumption \((z-1)(z^{-1}-1))^{(p-1)/3} \not\equiv 1 \pmod{p} \) that
\[
\pi_1^{(p-1)/3} \not\equiv 1 \pmod{p}.
\]

Also it follows from the assumption \((z^2-1)(z^{-2}-1))^{(p-1)/3} \equiv 1 \pmod{p} \) that
\[
\eta_1 \equiv 1 \pmod{p}.
\]

as in the proof of Theorem 1.

In this case, we have
\[
\text{Gal}(\mathbb{Q}_2^{(3)}/\mathbb{Q}_2) \simeq \text{Coker}(\eta_2^{Z_{[12]}} \rightarrow ((\mathcal{O}_{\mathbb{Q}_2}/p)^{\times}/(p^{3-1}/3)^{\times})
\]

as in the proof of Theorem 1 since $[E_{Q_2} : \eta_2^{Z[F_2]}]$ is prime to 3. Note that we have $((O_{Q_2}/p)^x)^{(p-1)/3} \simeq \bigoplus_{i=0}^8((O_{Q_2}/p^i)^x)^{(p-1)/3} \simeq (Z/3Z)^{\oplus 9}$. Let $I = \text{Ann}_{F_3[I_2]}(\eta_2 \mod p) \subseteq F_3[I_2]$ be the annihilator ideal of $\eta_2 \mod p \in ((O_{Q_2}/p)^x)^{(p-1)/3}$.

Then $\text{Im}(\eta_2^{Z[F_2]}) \to ((O_{Q_2}/p)^x)^{(p-1)/3} \simeq F_3[I_2]/I$. We claim that $I = (\gamma_2 - 1)^6 F_3[I_2] = (1 + \gamma_2 + \gamma_2^6) F_3[I_2]$. Since $\eta_2^{1+\gamma_2^2+\gamma_2^6} = \eta_1 \equiv 1 \pmod{p}$ by (13), we have $((1 + \gamma_2 + \gamma_2^6) F_3[I_2]) = (\gamma_2 - 1)^6 F_3[I_2] \subseteq I$. Now, assume that $(\gamma_2 - 1)^6 F_3[I_2] \not\subseteq I$. Since $F_3[I_2] \approx F_3[T]/(T^9 - 1) = F_3[T]/(T - 1)^9$ and $\text{F}_3[T]$ is a principal ideal domain, we must have $(\gamma_2 - 1)^5 \in I$. Then

$$1 \equiv (\gamma_2 - 1)^5 = ((\pi_2^{\gamma_2-1})^{(p-1)/3})^{(\gamma_2 - 1)^5} = (\pi_2^{\gamma_2 - 1})^6 \equiv (\pi_2^{1+\gamma_2^2+\gamma_2^6})^{(p-1)/3} \equiv \pi_1^{(p-1)/3} \pmod{p},$$

which contradicts (12). Thus we obtain $I = (\gamma_2 - 1)^6 F_3[I_2]$, hence

$$\text{Gal}(Q_3^3(p)/Q_3) \simeq \text{Coker}(\eta_2^{Z[F_2]}) \to ((O_{Q_2}/p)^x)^{(p-1)/3} \simeq F_3[I_2]/(\gamma_2 - 1)^6 \simeq (Z/3Z)^{\oplus 3}.$$

Therefore we have $A(k_2)/A(k_2)^{\sigma - 1} \simeq \text{Gal}(Q_3^3(p)/k_2) \simeq (Z/3Z)^{\oplus 2}$ and $A(k_2)/A(k_2)^{\sigma - 1} \simeq Z/3Z$ by (11). Thus we have proved (10).

Let $L'(k_{\infty})$ be the maximal unramified pro-3 abelian extension of $k_{\infty}$ in which every prime of $k_{\infty}$ lying above 3 splits completely, and $X' = \text{Gal}(L'(k_{\infty})/k_{\infty})$. Then $X' \simeq \text{lim}_{\rightarrow} A'(k_n)$, and it follows from $A'(k) = 0$ that $X'/\nu_n X' \simeq A'(k_n)$, where $\nu_n = 1 + \gamma + \gamma^2 + \ldots + \gamma^{3^n - 1}$ (see [I2]). Hence

$$\text{(14)} \quad (X'/X'^{\sigma - 1})^{\nu_n} X'/X'^{\sigma - 1} \simeq A'(k_n)/A'(k_n)^{\sigma - 1}$$

for all $n \geq 1$.

We need the following lemma:

**Lemma 5.** Let $\Lambda = Z_l[[\Gamma]]$, where $l$ is any prime number and $\Gamma$ is a pro-$l$ group isomorphic to $Z_l$. For a topological generator $\gamma$ of $\Gamma$, put $\nu_n = 1 + \gamma + \gamma^2 + \ldots + \gamma^{l^n - 1} \in \Lambda$. For a finitely generated $\Lambda$-module $M$ and some $n \geq 0$, if the identity map $M \simeq M$ induces the isomorphism

$$M/\nu_{n+1} M \simeq M/\nu_n M,$$

then $\nu_n M = 0$.

**Proof.** By assumption,

$$\text{Ker}(M/\nu_{n+1} M \to M/\nu_n M) \simeq \nu_n M/\nu_{n+1} M = \nu_n M/(\nu_{n+1}/\nu_n) \nu_n M = 0.$$

Since $\nu_{n+1}/\nu_n$ is contained in the maximal ideal of $\Lambda$, we have $\nu_n M = 0$ by Nakayama’s lemma. ■
We can apply Lemma 5 to \( M = X'/X'^{\sigma - 1} \) and \( n = 1 \) by (9), (10) and (14). Then we get \( \nu_1(X'/X'^{\sigma - 1}) = 0 \). Hence
\[
X'/X'^{\sigma - 1} \simeq X'/X'^{\sigma - 1}/\nu_1(X'/X'^{\sigma - 1}) \simeq A'(k_1)/A'(k_1)^{\sigma - 1} \simeq \mathbb{Z}/3\mathbb{Z}
\]
by (9) and (14). Therefore, there exists an \( x' \in X' \) such that \( X' = \mathbb{Z}_3[G]x' \) by Nakayama’s lemma. Since \( (1 + \sigma + \sigma^2)x' = 0 \), there is a surjection \( \mathbb{Z}_3^{\oplus 2} \simeq \mathbb{Z}_3[G]/N_G\mathbb{Z}_3[G] \to X' \). Hence \( \text{rank}_{\mathbb{Z}_3} X' \leq 2 \). Because \( \lim D(k_n) \) is finite, we have \( \lambda_3(k) = \text{rank}_{\mathbb{Z}_3}(\lim A(k_n)) = \text{rank}_{\mathbb{Z}_3}(\lim A(k_n)/\lim D(k_n)) = \text{rank}_{\mathbb{Z}_3} X' \leq 2 \). Thus we have proved Theorem 2. ☐

**Corollary 6.** Let \( k \) satisfy the assumptions of Theorem 4. If \( \lambda_3(k) \neq 0 \), then
\[
X' \simeq \mathbb{Z}_3^{\oplus 2} \quad \text{and} \quad \text{Tor}_{\mathbb{Z}_3}(X) = D,
\]
where \( X = \lim A(k_n) \) and \( D = \lim D(k_n) \).

**Proof.** We consider a surjective homomorphism
\[
f : \mathbb{Z}_3[\zeta_3] \simeq \mathbb{Z}_3[G]/N_G\mathbb{Z}_3[G] \to X', \quad \zeta_3 \mapsto \sigma \mapsto \sigma(x'),
\]
where \( x' \) is as in the proof of Theorem 3 \((X' = \mathbb{Z}_3[G]x')\). If \( \text{Ker}(f) \neq 0 \), then \( X' \) is finite because any non-zero ideal of \( \mathbb{Z}_3[\zeta_3] \) has finite index in \( \mathbb{Z}_3[\zeta_3] \). Hence \( \lambda_3(k) = \text{rank}_{\mathbb{Z}_3}(X') = 0 \). Therefore, under the assumption \( \lambda_3(k) \neq 0 \), \( f \) must be injective, hence an isomorphism. This shows that \( X' \simeq \mathbb{Z}_3^{\oplus 2} \). Further, from the exact sequence \( 0 \to D \to X \to X' \to 0 \) and the fact that the order of \( D \) is finite, we see immediately that \( \text{Tor}_{\mathbb{Z}_3}(X) = D \). ☐

Now, we investigate the case \( p = 2269 \). The number field computations in what follows where done by using KASH, version 2.2. The prime number \( p = 2269 \) satisfies the assumption of Theorem 4. Let \( M \) be the decomposition field of the polynomial \( x^3 + 124794x^2 + 5186218509x + 71770829079384 \) over \( \mathbb{Q} \). Then \( M \) is a cubic subfield of \( k_1 \) different from \( k \) and \( \mathbb{Q}_1 \), where \( k = \mathbb{Q}^{(3)}(2269) \). We find that \( A(M) \simeq \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \), where \( A(F) \) denotes the 3-Sylow subgroup of the ideal class group of \( F \) for any number field \( F \). Let \( \mathfrak{p} \) and \( \mathfrak{l} \) be the prime ideals of \( M \) lying above 2269 and 3, respectively. For a number field \( F \) and a fractional ideal \( \mathfrak{a} \) of \( F \), we denote by \( \pi_F(\mathfrak{a}) \) the projection of \( \mathfrak{a} \) to \( A(F) \). Then \( \pi_M(\mathfrak{p}) \neq 0 \) and \( \pi_M(\mathfrak{l}) = 0 \). Hence
\[
\pi_{k_1}(\mathfrak{p}_1) \notin D(k_1)
\]
for any prime ideal \( \mathfrak{p}_1 \) of \( k_1 \) lying above \( \mathfrak{p} \), because
\[
N_{k_1/M}(\pi_{k_1}(\mathfrak{p}_1)) = \pi_M(\mathfrak{p}) \notin N_{k_1/M}(D(k_1)) = (\pi_M(\mathfrak{l})) = 0.
\]

We shall apply the following lemma:

**Lemma 7.** Let \( l \geq 2 \) be a prime number and \( k/\mathbb{Q} \) a cyclic extension with \([k : \mathbb{Q}] = l\). Denote by \( F_\infty/F \) and \( F_n \) the cyclotomic \( \mathbb{Z}_l \)-extension and its \( n \)-th layer, respectively, for any number field \( F \). Let \( \mathfrak{p} \) be a prime ideal of \( k_n \) which
ramifies in $k_n/Q_{m}$. If $p$ splits completely in $k_m$ and $l$-rank $A'(k_m) < l^{m-n}$ for some $m \geq n$, then $\pi_{k_n}(p) \in A(k_n)$ capitulates in $k_\infty$, where $A'(k_n)$ is the $l$-Sylow subgroup of the $l$-ideal class group of $k_n$, and $\pi_{k_n}$ denotes the natural projection map from the ideal group of $k_n$ to the $l$-Sylow subgroup $A(k_n)$ of the ideal class group of $k_n$.

Proof. We write $I_K$, $A'(K)$ and $A(K)$ for the ideal group of $K$, the $l$-Sylow subgroup of the $l$-ideal class group of $K$, and the $l$-Sylow subgroup of the ideal class group of $K$, respectively, for any subfield $K$ of $\overline{Q}$. Also we denote by $\pi'_K$ and $\pi_K$ the natural projection maps from $I_K$ to $A'(K)$ and $A(K)$, respectively.

We write $\mathfrak{P}$ for a prime ideal of $k_m$ lying above $p$. Since $\mathfrak{P}^l \in I_{Q_m}$ and since $A'(Q_m) = 0$, we see that $\pi'_{k_m}(\mathfrak{P})^l = 0$. We consider the map $\psi : \mathbb{Z}/l\mathbb{Z}[\text{Gal}(k_m/k_n)] \to A'(k_m)[l]$, $f \mapsto f\pi'_{k_m}(\mathfrak{P})$, where $A'(k_m)[l]$ is the subgroup of $A'(k_m)$ consisting of the elements whose order divides $l$. It follows from the assumption that

$$\#(A'(k_m)[l]) < l^{m-n} = \#\mathbb{Z}/l\mathbb{Z}[\text{Gal}(k_m/k_n)].$$

Hence $\text{Ker}(\psi) \neq 0$, which implies $\text{Ker}(\psi)^{\text{Gal}(k_m/k_n)} \neq 0$. Because

$$\mathbb{Z}/l\mathbb{Z}[\text{Gal}(k_m/k_n)]^{\text{Gal}(k_m/k_n)} = \mathbb{Z}/l\mathbb{Z} \sum_{\gamma \in \text{Gal}(k_m/k_n)} \gamma,$$

we have $\sum_{\gamma \in \text{Gal}(k_m/k_n)} \gamma \in \text{Ker}(\psi)$. So $\pi_{k_m}(p) = \sum_{\gamma \in \text{Gal}(k_m/k_n)} \gamma \pi'_{k_m}(\mathfrak{P}) = 0$, namely, $\pi_{k_m}(p) \in D(k_m)$ since $A'(k_m) = A(k_m)/D(k_m)$, where $D(k_m)$ is the subgroup of $A(k_m)$ consisting of the ideal classes which contain a product of prime ideals of $k_m$ lying above $l$. Because $k/Q$ is abelian, $D(k_m)$ capitulates in $k_\infty$. Therefore $\pi_{k_n}(p)$ capitulates in $k_\infty$. \hfill \blacksquare

Assume that $\lambda_3(k) \neq 0$. Since $2269 \equiv 1 \pmod{27}$, a prime ideal $p_1$ of $k_1$ lying above $2269$ decomposes into three prime ideals in $k_2$. Then $\pi_{k_1}(p_1)$ capitulates in $k_\infty$ by Lemma 7, because $3$-rank($A'(k_2)$) \leq $3$-rank($X'$) = 2 by Corollary 6. Thus $\pi_{k_1}(p_1) \in \text{Ker}(A(k_1) \to A(k_\infty)) = \text{Im}($Tor$_{Z_3}X \to A(k_1)$) by [O, Proposition]. But this contradicts (15) since Tor$_{Z_3}X = D$ and Im($D \to A(k_1)$) = D($k_1$) by Corollary 6. Therefore, we have shown that $\lambda_3(k) = \mu_3(k) = 0$.

In the case $p = 6481$, one can verify $\lambda_3(Q^3(6481)) = \mu_3(Q^3(6481)) = 0$ in the same manner.

Consequently, we have the following result:

**Theorem 8.** $\lambda_3(Q^3(p)) = \mu_3(Q^3(p)) = 0$ for all prime numbers $p < 10000$ with $p \equiv 1 \pmod{3}$ but $p = 3907, 7219, 8011, 8677$.

**Acknowledgments.** We would like to express our thanks to Prof. Masato Kurihara for valuable discussions on the topic of this paper. We
would also like thank Prof. Ralph Greenberg for improving an earlier version of Lemma 7. We owe the proof of Lemma 7 given in this paper to him.

References


