A construction of pseudorandom binary sequences using both additive and multiplicative characters

by

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1. Introduction. In order to study the pseudorandomness of finite binary sequences, Mauduit and Sárközy introduced several definitions in [6]. For a given binary sequence

\[ E_N = \{e_1, \ldots, e_N\} \in \{-1, +1\}^N \]

the well-distribution measure of \( E_N \) is defined by

\[ W(E_N) = \max_{a,b,t} |U(E_N, t, a, b)| = \max_{a,b,t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|, \]

where the maximum is taken over all \( a, b, t \in \mathbb{N} \) such that \( 1 \leq a \leq a + (t - 1)b \leq N \), and the correlation measure of order \( l \) of \( E_N \) is defined as

\[ C_l(E_N) = \max_{M,D} |V(E_N, M, D)| = \max_{M,D} \left| \sum_{n=1}^{M} e_{n+d_1} \cdots e_{n+d_l} \right|, \]

where the maximum is taken over all \( D = (d_1, \ldots, d_l) \) and \( M \) such that \( 0 \leq d_1 < \cdots < d_l \leq N - M \).

The sequence \( E_N \) is considered to be a “good” pseudorandom sequence if both these measures \( W(E_N) \) and \( C_l(E_N) \) (at least for small \( l \)) are “small” in terms of \( N \) (in particular, both are \( o(N) \) as \( N \to \infty \)). This terminology is justified since for a truly random sequence \( E_N \) each of these measures is \( \ll \sqrt{N \log N} \). (For a more precise version of this result see [1].)

Using the Legendre symbol, Mauduit and Sárközy [6] showed an example of a “good” pseudorandom sequence. They defined a binary sequence by putting \( N = p - 1 \) where \( p \) is a prime number, and

\[ e_n = \left( \frac{n}{p} \right) \quad \text{for } n = 1, \ldots, p - 1. \]

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They proved that
\[ W(E_{p-1}) \ll p^{1/2} \log p, \quad C_l(E_{p-1}) \ll lp^{1/2} \log p. \]

Other large families of binary sequences with strong pseudorandom properties were studied in [4], [3], [5], [8], [7], [10].

In this paper a new construction of a large family of pseudorandom binary sequences is presented which uses both additive and multiplicative characters.

Let \( p \) be a prime, \( \psi \) an additive character, \( \chi \) a multiplicative character in \( \mathbb{F}_p \), \( \alpha \in \mathbb{C} \) with \( |\alpha| = 1 \), and \( f(x), g(x), q(x), r(x) \in \mathbb{F}_p[x] \). Let us define \( E_p \) by

\[
e_n = \begin{cases} 
+1 & \text{if } \Re \left( \alpha \psi \left( \frac{f(n)}{g(n)} \right) \chi \left( \frac{q(n)}{r(n)} \right) \right) \geq 0 \\
& \text{and } g(n), r(n), q(n) \neq 0, \\
-1 & \text{otherwise}.
\end{cases}
\]  

(2)

Note that this construction generalizes several earlier ones:

**CONSTRUCTION 1:** If \( \chi \) is the Legendre symbol, \( \psi \) is the trivial additive character, \( \alpha = 1 \), \( r(x) \) is a non-zero constant polynomial, we get an extended variant of (1), studied in [3]:

\[
e_n = \begin{cases} 
\left( \frac{q(n)}{p} \right) & \text{for } p \nmid q(n), \\
1 & \text{for } p \mid q(n),
\end{cases} \quad \text{for } n = 1, \ldots, p.
\]

**CONSTRUCTION 2:** If \( \chi \) is a general multiplicative character, \( \psi \) is the trivial additive character, \( \alpha = 1 \), \( r(x) \) is a non-zero constant polynomial, we get the construction studied in [8], [10], [9]:

\[
e_n = \begin{cases} 
+1 & \text{if } \Re(\chi(q(n))) \geq 0, \\
-1 & \text{otherwise},
\end{cases} \quad \text{for } n = 1, \ldots, p.
\]

**CONSTRUCTION 3:** If \( \psi \) is the additive character of the form \( \psi(n) = e(n/p) \) (where now \( e(\alpha) = e^{2\pi i \alpha} \)), \( \chi \) is the trivial multiplicative character, \( \alpha = i \), then we get a variant of pseudorandom sequences studied in [4], [5], [7]:

\[
e_n = \begin{cases} 
+1 & \text{if } r_p \left( \frac{f(n)}{g(n)} \right) < \frac{p}{2} \text{ for } p \nmid g(n), \\
-1 & \text{otherwise},
\end{cases} \quad \text{for } n = 1, \ldots, p,
\]

where \( r_p(n) \) denotes the least non-negative residue of \( n \) modulo \( p \).

Let us introduce the following notations: for a rational function \( F(x) = f(x)/g(x) \) let \( \deg F(x) = \deg f(x) - \deg g(x) \) and \( \deg^* F(x) = \deg f(x) + \deg g(x) \). Finally, let us denote the algebraic closure of \( \mathbb{F}_p \) by \( \overline{\mathbb{F}}_p \).
Theorem 1. Assume that $p$ is a prime number, $\chi$ is a non-principal multiplicative character modulo $p$ of order $d$, $\psi$ is a non-principal additive character modulo $p$, $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, $F(x) = f(x)/g(x)$, $Q(x) = q(x)/r(x) \in \mathbb{F}_p(x)$ are rational functions such that $(g(x), f(x)) = 1$ and $(q(x), r(x)) = 1$ and neither $q(x)$ nor $r(x)$ has a multiple zero in $\mathbb{F}_p$, and the binary sequence $E_p = \{e_1, \ldots, e_p\}$ is defined by (2). Then

$$W(E_p) \ll (\deg F + d \deg^* Q) p^{1/2}(\log p)^2.$$  

Theorem 2. Let $p, F(x), Q(x)$ and $E_p$ be as in Theorem 1. Assume also that $l \in \mathbb{N}$, $2 \leq l < p$ and one of the following conditions holds:

(a) $l = 2$;  
(b) $(4 \deg g)^l < p$, $(4 \deg^* Q)^l < p$;  
(c) $g(x) = (x+a_1) \ldots (x+a_k)$ (with $a_i \neq a_j$ for $i \neq j$) and $l \deg g < p/2$, $(4 \deg^* Q)^l < p$.

Then

$$C_l(E_p) \ll (l + 1)(\deg F + d \deg^* Q) p^{1/2}(\log p)^{l+1}.$$  

2. On hybrid character sums. The proofs of Theorems 1 and 2 will be based on hybrid character sum estimates. For rational functions $F(x), Q(x) \in \mathbb{F}_p(x)$ denote the union of the sets of poles of $F(x)$ and $Q(x)$ by $S$.

Definition 3. For $F(x), Q(x) \in \mathbb{F}_q(x)$ the character sum

$$\sum_{n \not\in s} \psi(F(n)) \chi(Q(n))$$

is degenerate if

$$F(x) = H(x)^p - H(x) + b$$

for some $b \in \mathbb{F}_q$ and $H(x) \in \mathbb{F}_q(x)$ and

$$Q(x) = b H(x)^d$$

for some $b \in \mathbb{F}_q$ and $H(x) \in \mathbb{F}_q(x)$.

If the character sum is degenerate, then all of the terms are constant, so one cannot give a non-trivial upper bound for the sum. For non-degenerate sums Perel’man gave a non-trivial upper bound in [11]:

Theorem 4. Let $\mathbb{F}_q$ be a finite field of characteristic $p$, $\chi$ be a non-principal multiplicative character of $\mathbb{F}_q$ of order $d$, and $\psi$ be a non-principal additive character of $\mathbb{F}_q$. Let $F(x) = f(x)/g(x), Q(x) = q(x)/r(x) \in \mathbb{F}_q(x)$.

Assume that the hybrid character sum is not degenerate and the following conditions hold:

1. If $F = f/g_{1}^{\lambda_1} \ldots g_{r}^{\lambda_r}$, where the polynomials $g_1, \ldots, g_r$ are non-constants and $(g_1, \ldots, g_r) = 1$ then $p \nmid \lambda_i$ when $\lambda_i > 0$ for $i = 1, \ldots, r$ and $p \nmid \deg F$ when $\deg F > 0$.
2. If $Q = q_{1}^{n_1} \ldots q_{u}^{n_u} / r_{1}^{m_1} \ldots r_{v}^{m_v}$ then $0 < n_i, m_i < d$ for all $i$.  

Then
\[
\left| \sum_{n \notin S} \psi(F(n)) \chi(Q(n)) \right| \leq (d_1 + d_2 - 2)q^{1/2} + d_1 + d_2 + 1
\]
with
\[
d_1 = \max\{\deg f, \deg g\} + s + \lambda, \quad d_2 = \deg q + \deg r + \mu,
\]
where \(s\) is the number of distinct zeros of \(g\), \(\lambda\) is 0 if \(\deg g \geq \deg f\) and 1 otherwise, \(\mu\) is 0 if \(d \mid \deg Q\) and 1 otherwise.

**Theorem 5.** Let \(p\) be a prime, let \(\psi\) be a non-principal additive character of \(\mathbb{F}_p\), and \(\chi\) a non-principal multiplicative character of \(\mathbb{F}_p\) of order \(d\). Furthermore, let \(F = f/g, \ Q = q/r\) be non-zero rational functions over \(\mathbb{F}_p\), and let \(s\) be the number of distinct zeros of \(g\) in \(\mathbb{F}_p\). Suppose that \(g(x) \nmid f(x)\) and \(Q(x)\) is not of the form \(bB(x)^d\) for any \(b \in \mathbb{F}_p\) and \(B(x) \in \mathbb{F}_p(x)\). If \(1 \leq N < p\) then

\[
\left| \sum_{\substack{0 \leq n < N \\atop n \notin S}} \psi(F(n)) \chi(Q(n)) \right| \leq 3(\max\{\deg f, \deg g\} + s + \deg q + \deg r)p^{1/2} \log p.
\]

**Proof.** We can assume that the degrees of all the polynomials are less than \(p\) since the result is trivial otherwise.

It follows from the basic properties of additive characters that

\[
\sum_{r=0}^{N-1} \frac{1}{p} \sum_{u=0}^{p-1} \psi(u(n-r)) = \begin{cases} 1 & \text{if } 0 \leq n < N, \\ 0 & \text{otherwise}. \end{cases}
\]

Let us denote the character sum in (6) by \(S_N\). We have

\[
S_N = \sum_{n \notin S} \psi(F(n)) \chi(Q(n)) \sum_{r=0}^{N-1} \frac{1}{p} \sum_{u=0}^{p-1} \psi(u(n-r))
\]

\[
= \frac{1}{p} \sum_{u=0}^{p-1} \left( \sum_{r=0}^{N-1} \psi(-ur) \right) \left( \sum_{n \notin S} \psi(F(n) + un) \chi(Q(n)) \right)
\]

\[
= \frac{1}{p} \sum_{u=1}^{p-1} \left( \sum_{r=0}^{N-1} \psi(-ur) \right) \left( \sum_{n \notin S} \psi(F(n) + un) \chi(Q(n)) \right)
\]

\[
+ \frac{N}{p} \sum_{n \notin S} \psi(F(n)) \chi(Q(n))
\]
and so

$$|S_N| \leq \frac{1}{p} \sum_{u=1}^{p-1} \left| \sum_{r=0}^{N-1} \psi(ur) \right| \left| \sum_{n \not\in S} \psi(F(n) + un)\chi(Q(n)) \right|$$

$$+ \frac{N}{p} \left| \sum_{n \not\in S} \psi(F(n))\chi(Q(n)) \right| .$$

For a fixed $u$ we consider the rational function

$$F_u(x) = F(x) + ux = \frac{f(x)}{g(x)} + ux .$$

To show that $F_u(x)$ satisfies the conditions of Theorem 4, it suffices to prove that $F_u(x)$ is not of the form $A(x)^p - A(x)$ with $A(x) \in \mathbb{F}_p(x)$. Suppose that

$$F_u(x) = \left( \frac{K(x)}{L(x)} \right)^p - \frac{K(x)}{L(x)}$$

with $K(x), L(x) \in \mathbb{F}_p[x]$ such that $(K(x), L(x)) = 1$. Then

$$L(x)^p(f(x) + uxg(x)) = (K(x)^{p-1} - L(x)^{p-1})K(x)g(x),$$

so $L(x)^p \mid g(x)$ as $(K(x), L(x)) = 1$. Since $\deg g(x) < p$, it follows that $L(x)$ is a nonzero constant polynomial. Thus we get

$$f(x) + uxg(x) = (\alpha K(x)^p + \beta K(x))g(x),$$

and hence

$$f(x) = (\alpha K(x)^p + \beta K(x) - ux)g(x),$$

for some $\alpha, \beta \in \mathbb{F}_p$ with $\alpha \beta \neq 0$.

Since $g(x) \nmid f(x)$ and either

$$\deg(\alpha K(x)^p + \beta K(x) - ux) > p$$

or

$$\deg(\alpha K(x)^p + \beta K(x) - ux) = 1$$

we see that (8) cannot hold.

Since $F(x) + ux$, $F(x)$ and $Q(x)$ satisfy the conditions of Theorem 4, we deduce from (7) that

$$|S_N| \leq \frac{1}{p} \left( \sum_{u=1}^{p-1} \left| \sum_{r=0}^{N-1} \psi(ur) \right| + N \right)$$

$$\cdot 2 \max\{\deg f, \deg g\} + s + \deg q + \deg r)p^{1/2}$$

and

$$\sum_{u=0}^{p-1} \left| \sum_{r=0}^{N-1} \psi(ur) \right| < \frac{4}{p} p \log p + 0.38p + 0.64,$$

by Theorem 1 in [2].
3. The well-distribution measure. To express the terms of $E_p$, we will need the generalization of Lemma 2 in [4].

**Lemma 6.** Let $m \in \mathbb{N}$, and let $\varepsilon$ be an $m$th root of unity. Then

$$\frac{1}{m} \sum_{-\lfloor m/2 \rfloor < a \leq \lfloor m/2 \rfloor} v_m(a) \varepsilon^a = \begin{cases} +1 & \text{if } -\pi/2 \leq \arg(\varepsilon) < \pi/2, \\
-1 & \text{otherwise,} \end{cases}$$

where $v_m(a)$ is a function of period $m$ such that $v_m(0) = 1$, and if $m$ is odd, then

$$v_m(a) = i^a \left( 1 + i \frac{(-1)^a - \cos(\pi a/m)}{\sin(\pi a/m)} \right) \quad \text{if } 1 \leq |a| < m/2,$$

while if $m$ is even, then

$$v_m(a) = \begin{cases} 0 & \text{if } a \text{ is even} \\
 i^a \left( 2 - 2i \frac{\cos(a\pi/m)}{\sin(a\pi/m)} \right) & \text{if } a \text{ is odd} \quad \text{if } 1 \leq |a| \leq m/2. \end{cases}$$

Furthermore, in both cases, $v_m(a) \ll m/a$ if $a \neq 0$.

**Proof.** For $m$ odd, the statement has been proved in [4]; for $m$ even the proof is similar. $\blacksquare$

**Proof of Theorem 1.** To prove the desired inequality, consider $a \in \mathbb{Z}$ and $b, t \in \mathbb{N}$ such that

$$(9) \quad 1 \leq a \leq a + (t - 1)b \leq p, \quad b < p.$$ 

Then by Lemma 6 we have

$$U(E_p, t, a, b) = \sum_{j=0}^{t-1} e_{a+jb}$$

$$= \frac{1}{dp} \sum_{-\lfloor dp/2 \rfloor < h \leq \lfloor dp/2 \rfloor} v_{dp}(h) \alpha^h$$

$$\cdot \left( \sum_{0 \leq j \leq t-1 \atop a+jb \not\in S} \psi(F(a+jb))^h \chi(Q(a+jb))^h + O \left( \sum_{0 \leq j \leq p \atop a+jb \not\in S} 1 \right) \right) + O \left( \deg f \right)$$

$$= \frac{1}{dp} \sum_{-dp/2 < h \leq dp/2} v_{dp}(h) \alpha^h \left( \sum_{0 \leq j \leq t-1 \atop a+jb \not\in S} \psi(F(a+jb))^h \chi(Q(a+jb))^{r_d(h)} \right)$$

$$+ O(|S|) + O(\deg f),$$

since $\chi(Q(n))^h = \chi(Q(n))^{r_d(h)}$ for $n \in \mathbb{F}_p$. 

If $0 < |h| \leq dp/2$ then $h \nmid p$ or $h \nmid d$ (and so $r_d(h) \nmid d$), thus the hybrid character sums are not degenerate. Furthermore,

$$\max\{\deg f, \deg g\} + s \leq 2(\deg f + \deg g)$$

and

$$\deg^* Q^{r_d(h)} = r_d(h) \deg^* Q \leq d \deg^* Q,$$

thus by Theorem 5 we have

$$|U(E_p, t, a, b)| = \left| \sum_{j=0}^{t-1} e_{a+jb} \right| \leq \frac{1}{dp} \sum_{-\lfloor dp/2 \rfloor < h \leq \lfloor dp/2 \rfloor, \ n \neq 0} |v_{dp}(h)| \sum_{0 \leq j \leq t-1, a+jb \notin S} \psi(F(a+jb))^h \chi(Q(a+jb))^{r_d(h)} + |v_{dp}(0)| + O(|S|) + O(\deg f)$$

$$\ll \frac{1}{dp} \sum_{-\lfloor dp/2 \rfloor < h \leq \lfloor dp/2 \rfloor, \ n \neq 0} |v_{dp}(h)| (\deg^* F + \deg^* Q^{r_d(h)}) p^{1/2} \log p + |v_{dp}(0)|$$

$$\ll (\deg^* F + \deg^* Q^{r_d(h)}) p^{1/2} (\log p)^2.$$  

4. The correlation measure

Proof of Theorem 2. Consider any $M < p$ and $D = (d_1, \ldots, d_l)$ such that $0 \leq d_1 < \cdots < d_l \leq p - M$. Then

$$V(E_p, M, D) = \sum_{n=1}^{M} e_{n+d_1} \cdots e_{n+d_l}$$

$$= \frac{1}{(dp)^l} \sum_{1 \leq n \leq M} \prod_{i=1}^{l} \sum_{n+d_1, \ldots, n+d_i \notin S} v_{dp}(h_i) \cdot \alpha^{h_i} (\psi(F(n+d_i))^h \chi(Q(n+d_i)))^{h_i} + O\left( \sum_{1 \leq n \leq M, n+d_1 \in S} 1 \right) + \cdots + O\left( \sum_{1 \leq n \leq M, n+d_l \in S} 1 \right) + O(l \deg f),$$

whence, separating the contribution of the term with $h_1 = \cdots = h_l = 0,$
\[ V(E_p, M, D) = \frac{1}{(dp)^l} (M + \mathcal{O}(|S|l)) \]

\[ + \frac{1}{(dp)^l} \sum_{-\lfloor dp/2 \rfloor < h_1 \leq \lfloor dp/2 \rfloor} \cdots \sum_{-\lfloor dp/2 \rfloor < h_l \leq \lfloor dp/2 \rfloor} v_{dp}(h_1) \ldots v_{dp}(h_l) \prod_{i=1}^{l} \alpha^{h_i} \]

\[ \cdot \sum_{1 \leq n \leq M \atop n + d_1, \ldots, n + d_l \notin S} \prod_{i=1}^{l} (\psi(F(n + d_i))\chi(Q(n + d_i))^h_i \]

\[ + \mathcal{O}(|S|l) + \mathcal{O}(l \deg f). \]

Now consider one of the innermost sums (where \((h_1, \ldots, h_l) \neq (0, \ldots, 0))\), and let \(h_{i_1} < \cdots < h_{i_r}\) be the non-zero \(h_i\)'s. Then

\[ \sum_{1 \leq n \leq M \atop n + d_1, \ldots, n + d_l \notin S} \prod_{i=1}^{l} (\psi(F(n + d_i))\chi(Q(n + d_i))^h_i \]

\[ = \sum_{1 \leq n \leq M \atop n + d_1, \ldots, n + d_l \notin S} \psi \left( \sum_{i=1}^{l} h_i F(n + d_i) \right) \chi \left( \prod_{i=1}^{l} Q(n + d_i)^{h_i} \right) \]

\[ = \sum_{1 \leq n \leq M \atop n + d_1, \ldots, n + d_r \notin S} \psi \left( \sum_{j=1}^{r} h_{i_j} F(n + d_{i_j}) \right) \chi \left( \prod_{j=1}^{r} Q(n + d_{i_j})^{r_d(h_{i_j})} \right) \]

\[ = \sum_{1 \leq n \leq M \atop n + d_1, \ldots, n + d_r \notin S} \psi \left( \frac{f_{h_1, \ldots, h_l}(n)}{g_{h_1, \ldots, h_l}(n)} \right) \chi \left( \frac{q_{h_1, \ldots, h_l}(n)}{r_{h_1, \ldots, h_l}(n)} \right) \]

with

\[ f_{h_1, \ldots, h_l}(x) = \sum_{t=1}^{r} h_{i_t} f(x + d_{i_t}) \prod_{1 \leq j \leq r \atop j \neq t} g(x + d_{i_j}) , \]

\[ g_{h_1, \ldots, h_l}(x) = \prod_{j=1}^{r} g(x + d_{i_j}) , \]

\[ q_{h_1, \ldots, h_l}(x) = \prod_{j=1}^{r} q(x + d_{i_j})^{r_d(h_{i_j})} , \]

\[ r_{h_1, \ldots, h_l}(x) = \prod_{j=1}^{r} r(x + d_{i_j})^{r_d(h_{i_j})} . \]
so that
\[
\deg f_{h_1,\ldots,h_l} \leq \deg f + (r - 1) \deg g \leq \deg f + (l - 1) \deg g,
\]
\[
\deg g_{h_1,\ldots,h_l} = r \deg g \leq l \deg g,
\]
\[
\deg^* \left( \frac{q_{h_1,\ldots,h_l}}{r_{h_1,\ldots,h_l}} \right) \leq \sum_{j=1}^{r} r_d(h_{i_j}) \deg^* Q \leq ld \deg^* Q.
\]

In order to give an upper bound for the character sum in (11), we have to show that this sum is not degenerate for every \((h_1,\ldots,h_l) \neq (0,\ldots,0)\).

First, suppose that \(p \nmid h_{i_j}\) for all \(j = 1,\ldots,r\). The following lemma (Lemmas 8 and 9 in [7]) shows that the character sum is not degenerate.

**Lemma 7.** If \(p, f(x), g(x)\) and \(l\) satisfy the conditions in Theorem 2 and \(p \nmid h_{i_j}\) for \(j = 1,\ldots,r\), then \(g_{h_1,\ldots,h_l}(x) \nmid f_{h_1,\ldots,h_l}(x)\).

By the lemma, from (11) we have
\[
\left| \sum_{1 \leq n \leq M \atop n+d_{i_1},\ldots,n+d_{i_r} \not\in S} \psi \left( \frac{f_{h_1,\ldots,h_l}(n)}{g_{h_1,\ldots,h_l}(n)} \right) \chi \left( \frac{q_{h_1,\ldots,h_l}(n)}{r_{h_1,\ldots,h_l}(n)} \right) \right| 
\leq 3 \left( \deg^* \left( \frac{f_{h_1,\ldots,h_l}}{g_{h_1,\ldots,h_l}} \right) + \deg^* \left( \frac{q_{h_1,\ldots,h_l}}{r_{h_1,\ldots,h_l}} \right) \right) p^{1/2} \log p 
\leq 3(l + 1)(\deg^* F + d \deg^* Q)p^{1/2} \log p,
\]

since
\[
\max \{ \deg f_{h_1,\ldots,h_l}, \deg g_{h_1,\ldots,h_l} \} + s_{h_1,\ldots,h_l} \leq \deg f + (l + 1) \deg g 
\leq (l + 1) \deg^* F
\]
where \(s_{h_1,\ldots,h_l}\) is the number of distinct zeros of \(g_{h_1,\ldots,h_l}\).

On the other hand, if there are some \(h_{i_j}\) such that \(p \mid h_{i_j}\), then \(d \mid h_{i_j}\) since \(0 < |h_{i_j}| \leq [dp/2]\). Let
\[
q'_{h_1,\ldots,h_l}(x) = \prod_{j=1}^{r} q(x + d_{i_j})^{r_d(h_{i_j})}, \quad r'_{h_1,\ldots,h_l}(x) = \prod_{j=1}^{r} r(x + d_{i_j})^{r_d(h_{i_j})}.
\]

From the assumption, none of these polynomials is constant. Thus it is enough to prove the following lemma:

**Lemma 8.** If \(p, q(x), r(x)\) and \(l\) satisfy the conditions in Theorem 2 and there exists an index \(j\) such that \(d \nmid h_{i_j}\), then
\[
\frac{q'_{h_1,\ldots,h_l}(x)}{r'_{h_1,\ldots,h_l}(x)} = bB(x)^d
\]
for no \(b \in \mathbb{F}_p\) and \(B(x) \in \mathbb{F}_p(x)\).
In order to prove this, we will need the following lemma from [5].

**Lemma 9.** Assume that $p$ is a prime number, $k,l \in \mathbb{N}$ and $k,l < p$. Assume also that one of the following conditions holds:

1. $l \leq 2$,
2. $(4k)^l < p$.

Then for all $A, B \subset \mathbb{Z}_p$ with $|A| = k$ and $|B| = l$, there is a $c \in \mathbb{Z}_p$ such that the equation

\[(a + b = c, a \in A, b \in B,)

has exactly one solution in $a, b$.

**Proof of Lemma 8.** We use the approach developed in [3]. We say that $\varrho(x), \sigma(x) \in \mathbb{F}_p[x]$ are equivalent, $\sigma \sim \varrho$, if there is an $a \in \mathbb{F}_p$ such that $\varrho(x + a) = \sigma(x)$. Clearly, this is an equivalence relation.

Write $q(x)$ and $r(x)$ as the product of irreducible polynomials over $\mathbb{F}_p$. It follows from our assumption on the polynomials that all of these irreducible factors are distinct. Let us divide these factors into groups of equivalent factors. A typical group has the following form: $\varrho(x + a_1), \ldots, \varrho(x + a_u)$ (where $u \leq \deg q$) belong to $q(x)$, and $\sigma(x + b_1), \ldots, \sigma(x + b_v)$ (where $v \leq \deg r$) belong to $r(x)$, where the constants $a_i, b_j$ are distinct by assumption.

By the definition of $q_{h_1, \ldots, h_l}$ and $r_{h_1, \ldots, h_l}$ the factors occurring in the polynomials for a given group have the following form: $\varrho(x + a_t + d_{i_j})$ for $t = 1, \ldots, u$ and $j = 1, \ldots, r$ and $\sigma(x + b_z + d_{i_j})$ resp. All these polynomials are equivalent, and no other irreducible factor belongs to this equivalence class.

Now set $A = \{a_1, \ldots, a_u, b_1, \ldots, b_v\}$, $B = \{d_{i_1}, \ldots, d_{i_r}\}$. It follows from assumption of Theorem 2 that either

$$|B| = r \leq l = 2$$

or

$$(4|A||B|)^l \leq (4(\deg q + \deg r))^l \leq (4\deg \ast Q)^l < p,$$

so that one of the assumptions (1) or (2) in Lemma 9 holds, and thus the lemma can be applied. Hence there is a $c \in \mathbb{F}_p$ that has exactly one representation (13). Thus either $\varrho(x + c) \not| q_{h_1, \ldots, h_l}(x)$ or $\varrho(x + c) \not| r_{h_1, \ldots, h_l}(x)$, so

$$\varrho(x + c) \not| q_{h_1, \ldots, h_l}(x)(r_{h_1, \ldots, h_l}(x))^{d-1}$$

but

$$(\varrho(x + c))^d \not| q_{h_1, \ldots, h_l}(x)(r_{h_1, \ldots, h_l}(x))^{d-1}. \blacksquare$$

By Lemma 8 the character sum in (12) is not degenerate, so the inequality also holds if there are some $h_{i_j}$ such that $p \mid h_{i_j}$. 

Thus (10) and (12) yield

\[
|V(E_p, M, D)| \ll \frac{1}{(dp)^l} \sum_{-[dp/2]<h_1\leq [dp/2]} \cdots \sum_{-[dp/2]<h_l\leq [dp/2]} v_{dp}(h_1) \dots v_{dp}(h_l) \\
\cdot \left| \sum_{n+d_1, \ldots, n+d_l \notin S} \psi \left( \prod_{i=1}^l h_i F(n + d_i) \right) \chi \left( \sum_{i=1}^l Q(n + d_i)^h_i \right) \right| \\
+ O(|S|l) + O(l \deg f)
\]

\[
\ll \frac{1}{(dp)^l} (l + 1)(\deg^* F + d \deg^* Q)p^{1/2} \log p \left( \sum_{|h|<dp/2} |v_{dp}(h)| \right)^l \\
+ O(|S|l) + O(l \deg f)
\]

\[
\ll \frac{1}{(dp)^l} (l + 1)(\deg^* F + d \deg^* Q)p^{1/2} \log p \left( 1 + \sum_{0<|h|<dp/2} \frac{dp}{h} \right)^l \\
+ O(|S|l) + O(l \deg^* Q)
\]

\[
\ll (l + 1)(\deg^* F + d \deg^* Q)p^{1/2}(\log p)^{l+1},
\]

which completes the proof of Theorem 2.

References


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