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Some results on Oppenheim's "Factorisatio Numerorum" function

by

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1. Introduction. Let $f(n)$ denote the number of distinct unordered factorisations of the natural number n into factors larger than 1. For example, $f(28)=4$ as 28 has the following factorisations:

$$28,\ 2\cdot 14,\ 4\cdot 7,\ 2\cdot 2\cdot 7.$$

In this paper, we address three aspects of the function $f(n)$. For the first aspect, in [1], Canfield, Erdős and Pomerance mention without proof that the number of values of $f(n)$ that do not exceed x is $x^{o(1)}$ as $x\to\infty$. Our first theorem in this note makes this result explicit.

For a set $\mathcal{A}$ of positive integers we put $\mathcal{A}(x)=\{n\in\mathcal{A}: n\le x\}$.

THEOREM 1. *Let* $\mathcal{A}=\{f(m): m\in\mathbb{N}\}$. *Then*

$$\#\mathcal{A}(x)=x^{O(\log\log\log x/\log\log x)}.$$

Recall that Oppenheim [8] and independently Szekeres and Turán [11] considered the average value of $f(n)$ in the interval $(0,x]$ showing that

$$\frac{1}{x}\sum_{0<n\le x} f(n)=\frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}\left(1+O\left(\frac{1}{\sqrt{\log x}}\right)\right). \tag{1}$$

There is a large body of literature addressing average values of various arithmetic functions in short intervals. Our next result gives a lower bound for the average of $f(n)$ over a short interval $(x,x+y]$ which is of the same order as the average of $f(n)$ over the interval $(0,y]$.

THEOREM 2. *Uniformly for* $x>0$ *and* $y\ge 2$, *we have*

$$\frac{1}{y}\sum_{x<n\le x+y} f(n)\gg \frac{e^{2\sqrt{\log y}}}{(\log y)^{3/4}}.$$

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Finally, there are also several results addressing the behaviour of positive integers n which are multiples of some other arithmetic function of n. See, for example, [3], [5], [9] and [10] for problems related to counting positive integers n which are divisible by either $\omega(n)$, $\Omega(n)$ or $\tau(n)$, where these functions are the number of distinct prime factors of n, the number of total prime factors of n, and the number of divisors of n, respectively. Our next and last result gives upper and lower bounds on the counting function of the set of positive integers n which are multiples of $f(n)$.

Theorem 3. *Let $\mathcal{B} = \{n : f(n) \mid n\}$. Then*

$$\#\mathcal{B}(x) = \frac{x}{(\log x)^{1+o(1)}} \quad \text{as } x \to \infty.$$

2. Preliminaries and lemmas. The function $f(n)$ is related to various partition functions. For example, $f(2^n) = p(n)$, where $p(n)$ is the number of partitions of n. Furthermore, $f(p_1 \cdots p_k) = B_k$, where B_k is the kth Bell number which counts the number of partitions of a set with k elements into nonempty disjoint subsets. In general, $f(p_1^{\alpha_1} \cdots p_k^{\alpha_k})$ is the number of partitions of a multiset consisting of α_i copies of $\{i\}$ for each $i = 1, \ldots, k$. Throughout the paper, we write $\log x$ for the natural logarithm of x. We use p and q for prime numbers, O and o for the Landau symbols, and $\ll$ and $\gg$ for the Vinogradov symbols. The following asymptotic formula for the kth Bell number is due to de Bruijn [4].

Lemma 1.

$$\frac{\log B_k}{k} = \log k - \log\log k - 1 + \frac{\log\log k}{\log k} + \frac{1}{\log k} + O\left(\frac{(\log\log k)^2}{(\log k)^2}\right).$$

We also need the Stirling numbers of the second kind $S(k,l)$ which count the number of partitions of a k-element set into l nonempty disjoint subsets. Clearly,

$$B_k = \sum_{l=1}^{k} S(k,l). \tag{2}$$

We now formulate and prove a few lemmas about the function $f(n)$ which will come in handy later on.

The next lemma is an easy observation, so we state it without proof.

Lemma 2. *If $a \mid b$, then $f(a) \le f(b)$.*

We let p_n denote the nth prime number and $\alpha_1(n)$ denote the maximal exponent of a prime appearing in the prime factorisation of n. Let n be a positive integer with prime factorisation

$$n = q_1^{\alpha_1} \cdots q_k^{\alpha_k},$$

where $q_1, \ldots, q_k$ are distinct primes and $\alpha_1(n) := \alpha_1 \geq \cdots \geq \alpha_k$. We put $n_0 = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, and observe that $f(n) = f(n_0)$. This observation will play a crucial role in the proof of Theorem 1.

The following lemma gives upper bounds for $\alpha_1(n)$ and $\omega(n)$ when $f(n) \leq x$.

LEMMA 3. *Let* $n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$, *where* $\alpha_1 \geq \cdots \geq \alpha_k$ *and* $f(n) \leq x$. *Then*

(i) $\alpha_1 = O((\log x)^2)$;
(ii) $k = \omega(n) = O(\log x/\log\log x)$.

Proof. It follows from Lemma 2 that

$$f(n) \geq f(q_1^{\alpha_1}) = p(\alpha_1).$$

Using the asymptotic formula

$$p(n) = (1 + o(1)) \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}} \quad \text{as } n \to \infty, \tag{3}$$

due to Hardy and Ramanujan [6], we conclude that $\exp(c\sqrt{\alpha_1}) \leq x$ with some constant $c > 0$. Hence, (i) follows. In order to prove (ii), let $n_0' = p_1 \cdots p_k$. By Lemma 2, we have $f(n_0') \leq f(n) \leq x$. Furthermore, $f(n_0') = B_k$. It now follows from Lemma 1 that

$$\exp((1 + o(1))k \log k) = B_k \leq x$$

as $k \to \infty$, yielding

$$k = O\left(\frac{\log x}{\log\log x}\right),$$

which completes the proof of the lemma. ■

3. Proofs of the theorems

3.1. Proof of Theorem 1. For a positive integer n, we let again n_0 and $\alpha_1(n)$ be the functions defined earlier. We let $\mathcal{A}(x) = \{m_1, \ldots, m_t\}$ be such that $m_1 < \cdots < m_t$ and let $\mathcal{N} = \{n_1, \ldots, n_t\}$ be positive integers such that n_i is minimal among all positive integers n with $f(n) = m_i$ for all $i = 1, \ldots, t$. It is clear that if $n \in \mathcal{N}$, then n is of the form n_0. Since $\#\mathcal{A}(x) = t = \#\mathcal{N}$, it suffices to bound the cardinality of $\mathcal{N}$.

We partition this set as $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$, where

$$\mathcal{N}_1 = \{n \in \mathcal{N} : \alpha_1(n) \leq \log\log x\},$$
$$\mathcal{N}_2 = \left\{n \in \mathcal{N} : \omega(n) \leq \frac{\log x}{(\log\log x)^2}\right\}, \quad \mathcal{N}_3 = \mathcal{N} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2).$$

If $n \in \mathcal{N}_1$, then n has at most $O(\log x/\log\log x)$ prime factors (by Lemma 3), each appearing with an exponent of at most $\log\log x$.

Therefore,

$$\#\mathcal{N}_1 = (\log\log x)^{O(\log x/\log\log x)} = x^{O(\frac{\log\log\log x}{\log\log x})}. \tag{4}$$

Next, we observe that an integer in $\mathcal{N}_2$ has at most $\log x/(\log\log x)^2$ prime factors, each appearing with an exponent $O((\log x)^2)$ (by Lemma 3). Thus,

$$\#\mathcal{N}_2 \le (O(\log x)^2)^{\frac{\log x}{(\log\log x)^2}} = \exp\left(\frac{(2+o(1))\log x}{\log\log x}\right) = x^{o(\frac{\log\log\log x}{\log\log x})} \quad \text{as } x \to \infty. \tag{5}$$

Finally, let $n \in \mathcal{N}_3$, and write it as

$$n = p_1^{\alpha_1}\cdots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}}\cdots p_k^{\alpha_k},$$

where we put

$$i := \max\{j \le k : \alpha_j \ge y\} \quad \text{with} \quad y := \lfloor \log\log x/\log\log\log x\rfloor.$$

Observe that the divisors $p_{i+1}^{\alpha_{i+1}}\cdots p_t^{\alpha_t}$ of the numbers $n \in \mathcal{N}_3$ can be chosen in at most

$$y^k = y^{O(\frac{\log x}{\log\log x})} = \exp\left(O\left(\frac{\log x \log\log\log x}{\log\log x}\right)\right) \tag{6}$$

ways. Furthermore, by Lemma 3, the numbers $n' = p_1^{\alpha_1}\cdots p_i^{\alpha_i}$ can trivially be chosen in at most

$$(O((\log x)^2))^i = \exp(O(i\log\log x))$$

ways. Thus, writing $\mathcal{N}_4$ for the subset of $\mathcal{N}_3$ such that $i \le \log x/(\log\log x)^2$, we get

$$\#\mathcal{N}_4 \le \exp\left(O\left(\frac{\log x}{\log\log x}\right)\right). \tag{7}$$

From now on, we look at $n \in \mathcal{N}_5 = \mathcal{N}_3 \setminus \mathcal{N}_4$.

For each t, we let k_t be such that $S(t,k_t)$ is maximal among the numbers $S(t,k)$ for $k = 1,\ldots,t$. By formula (2), the definition of k_t, and Lemma 1,

$$S(t,k_t) \ge \frac{B_t}{t} = \frac{\exp((1+o(1))t\log t)}{t} = \exp((1+o(1))t\log t)$$

as $t \to \infty$. We now claim that

$$f(n) \ge f(n') \ge f((p_1\cdots p_i)^y) \ge \frac{S(i,k_i)^y}{(yk_i)!}.$$

The first two inequalities follow immediately from Lemma 2, so let us prove the last one.

Note that $S(i,k_i)$ counts the number of factorisations of $p_1\cdots p_i$ into precisely k_i factors. Therefore, $(S(i,k_i))^y$ counts the number of factorisations of $(p_1 \cdots p_i)^y$ into $k_i y$ square-free factors, where we count each such factorisation at most $(k_i y)!$ times. This establishes the claim.

Since i tends to infinity as $x \to \infty$ for all $n \in \mathcal{N}_5$, we get

$$S(i,k_i)^y \geq \exp((1+o(1))yi\log i)$$

as $x \to \infty$. Furthermore, we trivially have

$$(k_i y)! \leq (k_i y)^{k_i y} = \exp(k_i y \log(k_i y)).$$

Thus,

$$f(n) \geq \frac{S(i,k_i)^y}{(k_i y)!} \geq \exp\big((1+o(1))yi\log i - k_i y\log(k_i y)\big) \tag{8}$$

as $x \to \infty$. We next show that for our choices of y and i we have

$$k_i y \log(k_i y) = o(yi\log i) \quad \text{as } x \to \infty.$$

Indeed, using the fact

$$k_i = (1+o(1))\,\frac{i}{\log i} \quad \text{as } i \to \infty$$

(see, for example, [2]), we see that the above condition is equivalent to

$$\log y = o((\log i)^2),$$

which holds as $x \to \infty$ because

$$y = \log\log x/\log\log\log x \quad \text{and} \quad i > \log x/(\log\log x)^2.$$

Now the inequality $f(n) \leq x$ together with (8) and the fact that $\log i \geq (1+o(1))\log\log x$ implies that

$$i \leq (1+o(1))\,\frac{\log x}{y\log\log x} \quad \text{as } x \to \infty. \tag{9}$$

Thus, the numbers n' can be chosen in at most

$$(O((\log x)^2))^i \leq (O((\log x)^2))^{(1+o(1))\frac{\log x}{y\log\log x}} = x^{O(\frac{\log\log\log x}{\log\log x})} \tag{10}$$

ways. As we have already seen at (6), the complementary divisor $n/n' = p_{i+1}^{\alpha_{i+1}} \cdots p_t^{\alpha_t}$ of n can be chosen in at most

$$x^{O(\log\log\log x/\log\log x)} \tag{11}$$

ways also. Thus, the total number of choices for n in $\mathcal{N}_5$ is

$$\#\mathcal{N}_5 \leq x^{O(\log\log\log x/\log\log x)}. \tag{12}$$

Hence, from estimates (7) and (12), we get

$$\#\mathcal{N}_3 \leq \#\mathcal{N}_4 + \#\mathcal{N}_5 \leq x^{O(\log\log\log x/\log\log x)}. \tag{13}$$

From estimates (4), (5) and (13), we finally get

$$\#\mathcal{N} \leq \#\mathcal{N}_1 + \#\mathcal{N}_2 + \#\mathcal{N}_3 \leq x^{O(\log\log\log x/\log\log x)},$$

which completes the proof of the theorem.

3.2. Proof of Theorem 2. For ease of notation we put

$$S(x,y) := \sum_{x<n\le x+y} f(n).$$

Let z be some function of y tending to infinity with it such that $z\log z < o(\sqrt{\log y})$ as $y\to\infty$. Assume that $0 < x \le zy$. Write

$$S(x,y) = S(0,x+y) - S(0,x).$$

Observe that

$$\log(x+y) = \log y + O(\log z),$$

therefore

$$\begin{aligned}\exp(2\sqrt{\log(x+y)}) &= \exp(2\sqrt{\log y + O(\log z)})\\ &= \exp\left(2\sqrt{\log y} + O\left(\frac{\log z}{\sqrt{\log y}}\right)\right)\\ &= e^{2\sqrt{\log y}}\left(1 + O\left(\frac{\log z}{\sqrt{\log y}}\right)\right),\end{aligned}$$

and a similar estimate holds for $\exp(2\sqrt{\log x})$. Furthermore,

$$\frac{1}{(\log(x+y))^{3/4}} = \frac{1}{(\log y + O(\log z))^{3/4}} = \frac{1}{(\log y)^{3/4}}\left(1 + O\left(\frac{\log z}{\log y}\right)\right),$$

and again a similar estimate holds for $1/(\log x)^{3/4}$. Thus, using estimate (1), we see that in the range $0 < x \le zy$ the desired sum is

$$S(x,y) = S(0,x+y) - S(0,x) = \frac{ye^{2\sqrt{\log y}}}{2\sqrt{\pi}(\log y)^{3/4}}\left(1 + O\left(\frac{z\log z}{(\log y)^{1/2}}\right)\right).$$

This is even an asymptotic as $y\to\infty$ if we take $z := (\log y)^{1/2}(\log\log y)^{-2}$. We next assume that $x > yz$. For each integer $n \in (0,y]$, let $m(n)$ be the largest multiple of n in $(x, x+y]$ and write it as $m(n) = m_0(n)\cdot n$. Observe that $m_0(n) \ge x/n > x/y$. Thus, if $x \ge y^2$, then $x/n > y$. Let $\mathcal{M} = \{m(n) : n \in (0,y]\}$ and observe that in this range

$$\sum_{x<n\le x+y} f(n) \ge \sum_{m\in\mathcal{M}} f(m) \ge \sum_{0<n\le y} f(n),$$

where the last inequality follows by considering only factorisations of $m \in \mathcal{M}$ which are of the form

$$n_1\cdots n_k\cdot m_0(n)$$

for some $n \in (0,y]$, by remarking also that since $m_0(n) > y$, distinct factorisations of n will yield distinct factorisations of $m \in \mathcal{M}$. Thus, if $x > y^2$, the above argument yields

$$S(x,y) \ge S(0,y) = \frac{ye^{2\sqrt{\log y}}}{2\sqrt{\pi}(\log y)^{3/4}}\left(1 + O\left(\frac{1}{\sqrt{\log y}}\right)\right).$$

We now suppose that $yz \le x \le y^2$. We let

$$S(0,y) - S(0,y/2) = \sum_{y/2<n\le y} f(n) = S(0,y)\left(\frac{1}{2} + O\left(\frac{1}{\sqrt{\log y}}\right)\right).$$

To each factorisation $n_1 \cdots n_k$ of some $n \in \mathcal{I} := [y/2, y]$ we associate, as before, the factorisation $n_1 \cdots n_k \cdot m_0(n)$ of $m(n)$. Observe that $m_0(n) \in (x/n, x/n + y/n] \subset \mathcal{J} := (x/y, 2x/y + 2]$. Let $f_1(n)$ be the number of factorisations of n with two or more parts in $\mathcal{J}$. Note that $f_1(n) = 0$ unless $(x/y)^2 \le y$. Writing a factorisation counted by $f_1(n)$ as

$$a \cdot b \cdot m_1 \cdots m_s, \quad \text{where} \quad a, b \in \mathcal{J},$$

we get

$$\sum_{y/2\le n\le y} f_1(n) \le \sum_{\substack{a\le b\\ a,b\in\mathcal{J}}} \sum_{m\le y/ab} f(m) = \sum_{\substack{a\le b\\ a,b\in\mathcal{J}}} S(0, y/ab).$$

We split the above sum at $ab \le y/2$. In the low range, we use the fact that the function $u \mapsto e^{2\sqrt{\log u}}/(\log u)^{3/4}$ is increasing, to get

$$\sum_{\substack{a\le b\\ a,b\in\mathcal{J}\\ ab<y/2}} S(0,y/ab) \le \frac{ye^{2\sqrt{\log y}}}{2\sqrt{\pi}(\log y)^{3/4}} \left(\sum_{\substack{a\le b\\ a,b\in\mathcal{J}}} \frac{1}{ab}\right)\left(1 + O\left(\frac{1}{(\log y)^{1/2}}\right)\right).$$

Observe that

$$\begin{aligned}
\sum_{\substack{a\le b\\ a,b\in\mathcal{J}}} \frac{1}{ab} &\le \sum_{a\ge x/y} \frac{1}{a^2} + \frac{1}{2}\left(\sum_{a\in\mathcal{J}} \frac{1}{a}\right)^2 \\
&\le \left(\log\left(\frac{2x}{y} + 2\right) - \log\left(\frac{x}{y}\right) + O\left(\frac{1}{z}\right)\right)^2 + O\left(\frac{1}{z}\right) \\
&= \frac{1}{2}\left(\log 2 + O\left(\frac{1}{z}\right)\right)^2 + O\left(\frac{1}{z}\right) = \frac{(\log 2)^2}{2} + O\left(\frac{1}{z}\right).
\end{aligned}$$

In the larger range, we have $S(0, y/ab) = 1$. Thus, under the assumption that $(x/y)^2 \le y$,

$$\sum_{\substack{a\le b\\ a,b\in\mathcal{J}\\ ab>y/2}} S(0,y/ab) \le \sum_{a,b\in\mathcal{J}} 1 \ll (x/y)^2 \le y.$$

Putting everything together, we get

$$\sum_{y/2\le n\le y} f_1(n) \le S(0,y)\left(\frac{(\log 2)^2}{2} + O\left(\frac{(\log\log y)^2}{\sqrt{\log y}}\right)\right).$$

Therefore,

$$\sum_{y/2\le n\le y}(f(n)-f_1(n)) \ge S(0,y)\left(\frac{1}{2}-\frac{(\log 2)^2}{2}+O\left(\frac{(\log\log y)^2}{\sqrt{\log y}}\right)\right)$$
$$\gg S(0,y).$$

We now look only at the factorisations $m_1\cdots m_k m_0(n)$ of $m(n)$ for $n\in[y/2,y]$ arising from factorisations $m_1\cdots m_k$ of n counted by $f(n)-f_1(n)$. These might not be distinct but since the factorisation $m_1\cdots m_k$ of n has at most one part in $\mathcal{J}$, the interval containing $m_0(n)$ for all n under scrutiny, it follows that each such factorisation is counted at most twice. This shows

$$S(x,y)\ge\frac{1}{2}\sum_{y/2\le n\le y}(f(n)-f_1(n))\gg S(0,y),$$

which is what we wanted to prove.

3.3. Proof of Theorem 3. We observe that all primes are in $\mathcal{A}$ since $f(p)=1$ for all prime p. Thus,

$$\#\mathcal{A}(x)\gg\frac{x}{\log x}.$$

This completes the lower bound part of the theorem. To obtain the upper bound, we cover the set $\mathcal{A}(x)$ by three sets $\mathcal{A}_1,\ \mathcal{A}_2$ and $\mathcal{A}_3$ as follows:

$$\mathcal{A}_1=\{n\le x:\Omega(n)>10\log\log x\},$$
$$\mathcal{A}_2=\left\{n\le x:\omega(n)<\frac{\log\log x}{\log\log\log x}\right\},$$
$$\mathcal{A}_3=\{n\le x:n\equiv 0\pmod{f(n)},\ n\notin\mathcal{A}_1\cup\mathcal{A}_2\}.$$

We recall the bound

$$\#\{n\le x:\Omega(n)=k\}\ll\frac{kx\log x}{2^k}$$

valid uniformly in k (see, for example, Lemma 13 in [7]). Using the above estimate, we get

$$\#\mathcal{A}_1\le x\sum_{k>10\log\log x}\frac{k}{2^k}\ll\frac{x\log\log x}{2^{10\log\log x}}=o\left(\frac{x}{\log x}\right) \tag{14}$$

as $x\to\infty$. To find an upper bound for $\mathcal{A}_2$, we use the bounds (see page 200 of [12])

$$\#\{n\le x:\omega(n)=k\}\ll\frac{x}{(k-1)!}\,\frac{(\log\log x+c_1)^{k-1}}{\log x},$$

where $c_1>0$ is some constant. Using the elementary estimate $m!\ge(m/e)^m$ with $m=k-1$, we get

$$\#\{n\le x:\omega(n)=k\}\ll\frac{x}{\log x}\left(\frac{e\log\log x+c_2}{k-1}\right)^{k-1},$$

where $c_2 = ec_1$. The right hand side is an increasing function of k in our range for k versus x when x is large. Since $k < \log\log x/\log\log\log x$, we deduce that

$$(15)\qquad \#\mathcal{A}_2 \ll \frac{x}{\log x}\,(O(\log\log\log x))^{\frac{\log\log x}{\log\log\log x}} = \frac{x}{(\log x)^{1+o(1)}}$$

as $x \to \infty$.

Finally, we estimate $\mathcal{A}_3$. Each $n \in \mathcal{A}_3$ can be written as

$$n = q_1^{\alpha_1}\cdots q_k^{\alpha_k},$$

where $q_1,\dots,q_k$ are distinct primes, $\alpha_1 \ge \cdots \ge \alpha_k$, $\alpha_1 + \cdots + \alpha_k \le 10\log\log x$ and $k > K := \lfloor \log\log x/\log\log\log x\rfloor$. Let $\mathcal{T}$ be the set of all such tuples $(k,\alpha_1,\dots,\alpha_k)$. For each such n, we have

$$\begin{aligned} f(n) &\ge B_K \ge \exp((1+o(1))K\log K) \ge \exp((1+o(1))\log\log x)\\ &= (\log x)^{1+o(1)} \end{aligned}$$

as $x \to \infty$. The number of tuples $(k,\alpha_1,\dots,\alpha_k)$ satisfying the above conditions is at most

$$\#\mathcal{T} \ll \log\log x \sum_{n\le 10\log\log x} p(n),$$

where again $p(n)$ is the partition function of n. Using estimate (3), we conclude that

$$\#\mathcal{T} \ll (\log\log x)^2 \exp(O(\sqrt{\log\log x})) = (\log x)^{o(1)} \qquad \text{as } x \to \infty.$$

Thus,

$$(16)\qquad \#\mathcal{A}_3 \le \sum_{(k,\alpha_1,\dots,\alpha_k)\in\mathcal{T}} \frac{x}{f(p_1^{\alpha_1}\cdots p_k^{\alpha_k})} \le \frac{x\#\mathcal{T}}{B_K} = \frac{x}{(\log x)^{1+o(1)}}$$

as $x \to \infty$. Now inequalities (14), (15) and (16) yield the desired upper bound and complete the proof.

4. Comments. Quite likely, the results of Theorems 1 and 2 are not best possible. In this respect, we suggest the following questions:

QUESTION 1. *Is it true that* $\#\mathcal{A}(x) = \exp(O(\sqrt{\log x}))$*?*

QUESTION 2. *In the notations used in the proof of Theorem* 2*, is it true that*

$$S(x,y) \ge (1+o(1))S(0,y) \qquad \textit{as } y \to \infty\,?$$

Namely, is it true that the average value of $f(n)$ *in the interval* $(0,y]$ *is an asymptotic lower bound for the average value of* $f(n)$ *in any interval of length* y *as* $y \to \infty$*?*

Concerning Question 2 above, observe that our proof indicates that this is indeed the case except when $x \in [yz, y^2]$, where $z = (\log y)^{1/2}(\log\log y)^{-2}$.

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