Some results on Oppenheim’s “Factorisatio Numerorum” function

by

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1. Introduction. Let \( f(n) \) denote the number of distinct unordered factorisations of the natural number \( n \) into factors larger than 1. For example, \( f(28) = 4 \) as 28 has the following factorisations:

\[
28, \ 2 \cdot 14, \ 4 \cdot 7, \ 2 \cdot 2 \cdot 7.
\]

In this paper, we address three aspects of the function \( f(n) \). For the first aspect, in [1], Canfield, Erdős and Pomerance mention without proof that the number of values of \( f(n) \) that do not exceed \( x \) is \( x^{o(1)} \) as \( x \to \infty \). Our first theorem in this note makes this result explicit.

For a set \( A \) of positive integers we put \( A(x) = \{n \in A : n \leq x\} \).

**Theorem 1.** Let \( A = \{f(m) : m \in \mathbb{N}\} \). Then

\[
\#A(x) = x^{O(\log \log \log x/\log \log x)}.
\]

Recall that Oppenheim [8] and independently Szekeres and Turán [11] considered the average value of \( f(n) \) in the interval \((0, x]\) showing that

\[
\frac{1}{x} \sum_{0<n\leq x} f(n) = \frac{e^{2\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}} \left( 1 + O\left( \frac{1}{\sqrt{\log x}} \right) \right).
\]

(1)

There is a large body of literature addressing average values of various arithmetic functions in short intervals. Our next result gives a lower bound for the average of \( f(n) \) over a short interval \((x, x+y]\) which is of the same order as the average of \( f(n) \) over the interval \((0, y]\).

**Theorem 2.** Uniformly for \( x > 0 \) and \( y \geq 2 \), we have

\[
\frac{1}{y} \sum_{x<n\leq x+y} f(n) \gg \frac{e^{2\sqrt{\log y}}}{(\log y)^{3/4}}.
\]
Finally, there are also several results addressing the behaviour of positive integers $n$ which are multiples of some other arithmetic function of $n$. See, for example, [3], [5], [9] and [10] for problems related to counting positive integers $n$ which are divisible by either $\omega(n)$, $\Omega(n)$ or $\tau(n)$, where these functions are the number of distinct prime factors of $n$, the number of total prime factors of $n$, and the number of divisors of $n$, respectively. Our next and last result gives upper and lower bounds on the counting function of the set of positive integers $n$ which are multiples of $f(n)$.

THEOREM 3. Let $\mathcal{B} = \{ n : f(n) \mid n \}$. Then

$$\#\mathcal{B}(x) = \frac{x}{(\log x)^{1+o(1)}} \quad \text{as} \quad x \to \infty.$$ 

2. Preliminaries and lemmas. The function $f(n)$ is related to various partition functions. For example, $f(2^n) = p(n)$, where $p(n)$ is the number of partitions of $n$. Furthermore, $f(p_1 \cdots p_k) = B_k$, where $B_k$ is the $k$th Bell number which counts the number of partitions of a set with $k$ elements into nonempty disjoint subsets. In general, $f(p_1^{\alpha_1} \cdots p_k^{\alpha_k})$ is the number of partitions of a multiset consisting of $\alpha_i$ copies of $\{i\}$ for each $i = 1, \ldots, k$.

Throughout the paper, we write $\log x$ for the natural logarithm of $x$. We use $p$ and $q$ for prime numbers, $O$ and $o$ for the Landau symbols, and $\ll$ and $\gg$ for the Vinogradov symbols. The following asymptotic formula for the $k$th Bell number is due to de Bruijn [4].

**Lemma 1.**

$$\frac{\log B_k}{k} = \log k - \log \log k - 1 + \frac{\log \log k}{\log k} + \frac{1}{\log k} + O \left( \frac{(\log \log k)^2}{(\log k)^2} \right).$$

We also need the Stirling numbers of the second kind $S(k, l)$ which count the number of partitions of a $k$-element set into $l$ nonempty disjoint subsets. Clearly,

$$B_k = \sum_{l=1}^{k} S(k, l).$$

We now formulate and prove a few lemmas about the function $f(n)$ which will come in handy later on.

The next lemma is an easy observation, so we state it without proof.

**Lemma 2.** If $a \mid b$, then $f(a) \leq f(b)$.

We let $p_n$ denote the $n$th prime number and $\alpha_1(n)$ denote the maximal exponent of a prime appearing in the prime factorisation of $n$. Let $n$ be a positive integer with prime factorisation

$$n = q_1^{\alpha_1} \cdots q_k^{\alpha_k},$$
where \( q_1, \ldots, q_k \) are distinct primes and \( \alpha_1(n) := \alpha_1 \geq \cdots \geq \alpha_k \). We put \( n_0 = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \), and observe that \( f(n) = f(n_0) \). This observation will play a crucial role in the proof of Theorem 1.

The following lemma gives upper bounds for \( \alpha_1(n) \) and \( \omega(n) \) when \( f(n) \leq x \).

**Lemma 3.** Let \( n = q_1^{\alpha_1} \cdots q_k^{\alpha_k} \), where \( \alpha_1 \geq \cdots \geq \alpha_k \) and \( f(n) \leq x \). Then

(i) \( \alpha_1 = O((\log x)^2) \);

(ii) \( k = \omega(n) = O(\log x / \log \log x) \).

**Proof.** It follows from Lemma 2 that \( f(n) \geq f(q_1^{\alpha_1}) = p(\alpha_1) \).

Using the asymptotic formula

\[
p(n) = (1 + o(1)) \frac{\exp(\pi \sqrt{2n/3})}{4n\sqrt{3}} \quad \text{as } n \to \infty,
\]
due to Hardy and Ramanujan [6], we conclude that \( \exp(c\sqrt{\alpha_1}) \leq x \) with some constant \( c > 0 \). Hence, (i) follows. In order to prove (ii), let \( n'_0 = p_1 \cdots p_k \). By Lemma 2 we have \( f(n'_0) \leq f(n) \leq x \). Furthermore, \( f(n'_0) = B_k \).

It now follows from Lemma 1 that

\[
\exp((1 + o(1))k \log k) = B_k \leq x
\]
as \( k \to \infty \), yielding

\[
k = O\left( \frac{\log x}{\log \log x} \right),
\]
which completes the proof of the lemma.

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**3. Proofs of the theorems**

**3.1. Proof of Theorem 1.** For a positive integer \( n \), we let again \( n_0 \) and \( \alpha_1(n) \) be the functions defined earlier. We let \( \mathcal{A}(x) = \{m_1, \ldots, m_t\} \) be such that \( m_1 < \cdots < m_t \) and let \( \mathcal{N} = \{n_1, \ldots, n_t\} \) be positive integers such that \( n_i \) is minimal among all positive integers \( n \) with \( f(n) = m_i \) for all \( i = 1, \ldots, t \). It is clear that if \( n \in \mathcal{N} \), then \( n \) is of the form \( n_0 \). Since \( \#\mathcal{A}(x) = t = \#\mathcal{N} \), it suffices to bound the cardinality of \( \mathcal{N} \).

We partition this set as \( \mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3 \), where

\[
\mathcal{N}_1 = \{n \in \mathcal{N} : \alpha_1(n) \leq \log \log x\},
\]
\[
\mathcal{N}_2 = \left\{n \in \mathcal{N} : \omega(n) \leq \frac{\log x}{(\log \log x)^2}\right\},
\]
\[
\mathcal{N}_3 = \mathcal{N} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2).
\]

If \( n \in \mathcal{N}_1 \), then \( n \) has at most \( O(\log x / \log \log x) \) prime factors (by Lemma 3), each appearing with an exponent of at most \( \log \log x \).
Therefore,

\( \#N_1 = (\log \log x)^O(\log x/\log \log x) = x^{O(\log \log \log x/\log \log x)} \).

Next, we observe that an integer in \( N_2 \) has at most \( \log x/(\log \log x)^2 \) prime factors, each appearing with an exponent \( O(\log x) \) (by Lemma 3). Thus,

\[
\#N_2 \leq (O(\log x)^2)^{\log x/(\log \log x)^2} = \exp\left(\frac{(2 + o(1)) \log x}{\log \log x}\right) = x^{o(\log \log \log x/\log \log x)} \quad \text{as } x \to \infty.
\]

Finally, let \( n \in N_3 \), and write it as

\[
n = p_1^{\alpha_1} \cdots p_i^{\alpha_i} p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k},
\]

where we put

\[
i := \max\{j \leq k : \alpha_j \geq y\} \quad \text{with} \quad y := \lfloor \log \log x/\log \log \log x \rfloor.
\]

Observe that the divisors \( p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k} \) of the numbers \( n \in N_3 \) can be chosen in at most

\[
y^k = y^{O(\log x/\log \log x)} = \exp\left(O\left(\frac{\log x \log \log x}{\log \log x}\right)\right)
\]

ways. Furthermore, by Lemma 3 the numbers \( n' = p_1^{\alpha_1} \cdots p_i^{\alpha_i} \) can trivially be chosen in at most

\[
(O((\log x)^2))^i = \exp(O(i \log \log x))
\]

ways. Thus, writing \( N_4 \) for the subset of \( N_3 \) such that \( i \leq \log x/(\log \log x)^2 \), we get

\[
\#N_4 \leq \exp\left(O\left(\frac{\log x}{\log \log x}\right)\right).
\]

From now on, we look at \( n \in N_5 = N_3 \setminus N_4 \).

For each \( t \), we let \( k_t \) be such that \( S(t, k_t) \) is maximal among the numbers \( S(t, k) \) for \( k = 1, \ldots, t \). By formula (2), the definition of \( k_t \), and Lemma 1

\[
S(t, k_t) \geq B_t/\gamma = \exp((1 + o(1))t \log t) = \exp((1 + o(1))t \log t)
\]

as \( t \to \infty \). We now claim that

\[
f(n) \geq f(n') \geq f((p_1 \cdots p_i)^y) \geq \frac{S(i, k_i)^y}{(yk_i)!}.
\]

The first two inequalities follow immediately from Lemma 2 so let us prove the last one.

Note that \( S(i, k_i) \) counts the number of factorisations of \( p_1 \cdots p_i \) into precisely \( k_i \) factors. Therefore, \( (S(i, k_i))^y \) counts the number of factorisa-
tions of \((p_1 \cdots p_i)^y\) into \(k_i y\) square-free factors, where we count each such factorisation at most \((k_i y)!\) times. This establishes the claim.

Since \(i\) tends to infinity as \(x \to \infty\) for all \(n \in \mathcal{N}_5\), we get
\[
S(i, k_i)^y \geq \exp((1 + o(1))yi \log i)
\]
as \(x \to \infty\). Furthermore, we trivially have
\[
(k_i y)! \leq (k_i y)^{k_i y} = \exp(k_i y \log(k_i y)).
\]
Thus,
\[
(8) \quad f(n) \geq \frac{S(i, k_i)^y}{(k_i y)!} \geq \exp((1 + o(1))yi \log i - k_i y \log(k_i y))
\]
as \(x \to \infty\). We next show that for our choices of \(y\) and \(i\) we have
\[
k_i y \log(k_i y) = o(yi \log i) \quad \text{as} \quad x \to \infty.
\]
Indeed, using the fact
\[
k_i = (1 + o(1)) \frac{i}{\log i} \quad \text{as} \quad i \to \infty
\]
(see, for example, [2]), we see that the above condition is equivalent to
\[
\log y = o((\log i)^2),
\]
which holds as \(x \to \infty\) because
\[
y = \log \log x / \log \log \log x \quad \text{and} \quad i > \log x / (\log \log x)^2.
\]
Now the inequality \(f(n) \leq x\) together with (8) and the fact that \(\log i \geq (1 + o(1)) \log \log x\) implies that
\[
(9) \quad i \leq (1 + o(1)) \frac{\log x}{y \log \log x} \quad \text{as} \quad x \to \infty.
\]
Thus, the numbers \(n'\) can be chosen in at most
\[
(10) \quad (O((\log x)^2))^i \leq (O((\log x)^2))^{(1+o(1)) \frac{\log x}{y \log \log x}} = x^{O(\frac{\log \log \log x}{\log \log x})}
\]
ways. As we have already seen at (6), the complementary divisor \(n/n' = p_{i+1}^{\alpha_{i+1}} \cdots p_t^{\alpha_t}\) of \(n\) can be chosen in at most
\[
(11) \quad x^{O(\log \log \log x / \log \log x)}
\]
ways also. Thus, the total number of choices for \(n\) in \(\mathcal{N}_5\) is
\[
(12) \quad \#\mathcal{N}_5 \leq x^{O(\log \log \log x / \log \log x)}.
\]
Hence, from estimates (7) and (12), we get
\[
(13) \quad \#\mathcal{N}_3 \leq \#\mathcal{N}_4 + \#\mathcal{N}_5 \leq x^{O(\log \log \log x / \log \log x)}.
\]
From estimates (4), (5) and (13), we finally get
\[
\#\mathcal{N} \leq \#\mathcal{N}_1 + \#\mathcal{N}_2 + \#\mathcal{N}_3 \leq x^{O(\log \log \log x / \log \log x)},
\]
which completes the proof of the theorem.
3.2. Proof of Theorem 2. For ease of notation we put

\[ S(x, y) := \sum_{x<n\le x+y} f(n). \]

Let \( z \) be some function of \( y \) tending to infinity with it such that \( z \log z < o(\sqrt{\log y}) \) as \( y \to \infty \). Assume that \( 0 < x \leq zy \). Write

\[ S(x, y) = S(0, x+y) - S(0, x). \]

Observe that

\[ \log(x + y) = \log y + O(\log z), \]

therefore

\[
\exp(2\sqrt{\log(x+y)}) = \exp(2\sqrt{\log y + O(\log z)}) \\
= \exp\left(2\sqrt{\log y} + O\left(\frac{\log z}{\sqrt{\log y}}\right)\right) \\
= e^{2\sqrt{\log y} \left(1 + O\left(\frac{\log z}{\sqrt{\log y}}\right)\right)},
\]

and a similar estimate holds for \( \exp(2\sqrt{\log x}) \). Furthermore,

\[
\frac{1}{(\log(x+y))^{3/4}} = \frac{1}{(\log y + O(\log z))^{3/4}} = \frac{1}{(\log y)^{3/4}} \left(1 + O\left(\frac{\log z}{\log y}\right)\right),
\]

and again a similar estimate holds for \( 1/(\log x)^{3/4} \). Thus, using estimate (1), we see that in the range \( 0 < x \leq zy \) the desired sum is

\[
S(x, y) = S(0, x+y) - S(0, x) = \frac{ye^{2\sqrt{\log y}}}{2\sqrt{\pi}(\log y)^{3/4}} \left(1 + O\left(\frac{\log z}{(\log y)^{1/2}}\right)\right).
\]

This is even an asymptotic as \( y \to \infty \) if we take \( z := (\log y)^{1/2}(\log log y)^{-2} \).

We next assume that \( x > yz \). For each integer \( n \in (0, y] \), let \( m(n) \) be the largest multiple of \( n \) in \( (x, x+y] \) and write it as \( m(n) = m_0(n) \cdot n \).

Observe that \( m_0(n) \geq x/n > x/y \). Thus, if \( x \geq y^2 \), then \( x/n > y \). Let \( \mathcal{M} = \{m(n) : n \in (0, y]\} \) and observe that in this range

\[
\sum_{x<n\le x+y} f(n) \geq \sum_{m \in \mathcal{M}} f(m) \geq \sum_{0<n\le y} f(n),
\]

where the last inequality follows by considering only factorisations of \( m \in \mathcal{M} \) which are of the form

\[ n_1 \cdots n_k \cdot m_0(n) \]

for some \( n \in (0, y] \), by remarking also that since \( m_0(n) > y \), distinct factorisations of \( n \) will yield distinct factorisations of \( m \in \mathcal{M} \). Thus, if \( x > y^2 \), the above argument yields

\[
S(x, y) \geq S(0, y) = \frac{ye^{2\sqrt{\log y}}}{2\sqrt{\pi}(\log y)^{3/4}} \left(1 + O\left(\frac{1}{\sqrt{\log y}}\right)\right).
\]
We now suppose that $yz \leq x \leq y^2$. We let
\[
S(0, y) - S(0, y/2) = \sum_{y/2 < n \leq y} f(n) = S(0, y) \left( \frac{1}{2} + O \left( \frac{1}{\sqrt{\log y}} \right) \right).
\]

To each factorisation $n_1 \cdots n_k$ of some $n \in \mathcal{I} := [y/2, y]$ we associate, as before, the factorisation $n_1 \cdots n_k \cdot m_0(n)$ of $m(n)$. Observe that $m_0(n) \in (x/n, x/y + y/n] \subset \mathcal{J} := (x/y, 2x/y + 2]$. Let $f_1(n)$ be the number of factorisations of $n$ with two or more parts in $\mathcal{J}$. Note that $f_1(n) = 0$ unless $(x/y)^2 \leq y$. Writing a factorisation counted by $f_1(n)$ as
\[
a \cdot b \cdot m_1 \cdots m_s, \quad \text{where} \quad a, b \in \mathcal{J},
\]
we get
\[
\sum_{y/2 \leq n \leq y} f_1(n) \leq \sum_{a \leq b} \sum_{m \leq y/ab} f(m) = \sum_{a \leq b} S(0, y/ab).
\]

We split the above sum at $ab \leq y/2$. In the low range, we use the fact that the function $u \mapsto e^{2\sqrt{\log u}/(\log u)^{3/4}}$ is increasing, to get
\[
\sum_{a \leq b} S(0, y/ab) \leq \frac{ye^{2\sqrt{\log y}}}{2\sqrt{\pi}(\log y)^{3/4}} \left( \sum_{a \leq b} \frac{1}{ab} \right) \left( 1 + O \left( \frac{1}{(\log y)^{1/2}} \right) \right).
\]

Observe that
\[
\sum_{a \leq b} \frac{1}{ab} \leq \sum_{a \geq x/y} \frac{1}{a^2} + \frac{1}{2} \left( \sum_{a \in \mathcal{J}} \frac{1}{a} \right)^2
\]
\[
\leq \left( \log \left( \frac{2x}{y} + 2 \right) - \log \left( \frac{x}{y} \right) + O \left( \frac{1}{z} \right) \right)^2 + O \left( \frac{1}{z} \right)
\]
\[
= \frac{1}{2} \left( \log 2 + O \left( \frac{1}{z} \right) \right)^2 + O \left( \frac{1}{z} \right) = \frac{(\log 2)^2}{2} + O \left( \frac{1}{z} \right).
\]

In the larger range, we have $S(0, y/ab) = 1$. Thus, under the assumption that $(x/y)^2 \leq y$,
\[
\sum_{a \leq b} S(0, y/ab) \leq \sum_{a \in \mathcal{J}} 1 \ll (x/y)^2 \leq y.
\]

Putting everything together, we get
\[
\sum_{y/2 \leq n \leq y} f_1(n) \leq S(0, y) \left( \frac{(\log 2)^2}{2} + O \left( \frac{\log \log y)^2}{\sqrt{\log y}} \right) \right).
\]
Therefore,
\[
\sum_{y/2 \leq n \leq y} (f(n) - f_1(n)) \geq S(0, y) \left( \frac{1}{2} - \frac{(\log 2)^2}{2} + O \left( \frac{(\log \log y)^2}{\sqrt{\log y}} \right) \right)
\]
\[\gg S(0, y).\]

We now look only at the factorisations \(m_1 \cdots m_km_0(n)\) of \(m(n)\) for \(n \in [y/2, y]\) arising from factorisations \(m_1 \cdots m_k\) of \(n\) counted by \(f(n) - f_1(n)\). These might not be distinct but since the factorisation \(m_1 \cdots m_k\) of \(n\) has at most one part in \(J\), the interval containing \(m_0(n)\) for all \(n\) under scrutiny, it follows that each such factorisation is counted at most twice. This shows
\[
S(x, y) \geq \frac{1}{2} \sum_{y/2 \leq n \leq y} (f(n) - f_1(n)) \gg S(0, y),
\]
which is what we wanted to prove.

3.3. Proof of Theorem 3. We observe that all primes are in \(A\) since \(f(p) = 1\) for all prime \(p\). Thus,
\[
\#A(x) \gg \frac{x}{\log x}.
\]
This completes the lower bound part of the theorem. To obtain the upper bound, we cover the set \(A(x)\) by three sets \(A_1\), \(A_2\) and \(A_3\) as follows:
\[
A_1 = \{n \leq x : \Omega(n) > 10 \log \log x\},
\]
\[
A_2 = \{n \leq x : \omega(n) < \frac{\log \log x}{\log \log \log x}\},
\]
\[
A_3 = \{n \leq x : n \equiv 0 (\text{mod } f(n)), n \notin A_1 \cup A_2\}.
\]

We recall the bound
\[
\#\{n \leq x : \Omega(n) = k\} \ll \frac{kx \log x}{2^k}
\]
valid uniformly in \(k\) (see, for example, Lemma 13 in [7]). Using the above estimate, we get
\[
\#A_1 \leq x \sum_{k > 10 \log \log x} \frac{k}{2^k} \ll \frac{x \log \log x}{2^{10 \log \log x}} = o \left( \frac{x}{\log x} \right)
\]
as \(x \to \infty\). To find an upper bound for \(A_2\), we use the bounds (see page 200 of [12])
\[
\#\{n \leq x : \omega(n) = k\} \ll \frac{x}{(k-1)!} \frac{(\log \log x + c_1)^{k-1}}{\log x},
\]
where \(c_1 > 0\) is some constant. Using the elementary estimate \(m! \geq (m/e)^m\) with \(m = k - 1\), we get
\[
\#\{n \leq x : \omega(n) = k\} \ll \frac{x}{\log x} \left( \frac{e \log \log x + c_2}{k-1} \right)^{k-1},
\]
where \( c_2 = ec_1 \). The right hand side is an increasing function of \( k \) in our range for \( k \) versus \( x \) when \( x \) is large. Since \( k < \log \log x / \log \log \log x \), we deduce that

\[
\#A_2 \ll \frac{x}{\log x} \left( O(\log \log x) \right)^{\log \log x} = \frac{x}{(\log x)^{1+o(1)}}
\]
as \( x \to \infty \).

Finally, we estimate \( A_3 \). Each \( n \in A_3 \) can be written as

\[ n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}, \]

where \( q_1, \ldots, q_k \) are distinct primes, \( \alpha_1 \geq \cdots \geq \alpha_k \), \( \alpha_1 + \cdots + \alpha_k \leq 10 \log \log x \) and \( k > K := \lfloor \log \log x / \log \log \log x \rfloor \). Let \( T \) be the set of all such tuples \((k, \alpha_1, \ldots, \alpha_k)\). For each such \( n \), we have

\[
f(n) \geq B_K \geq \exp((1 + o(1))K \log K) \geq \exp((1 + o(1)) \log \log x)
\]
as \( x \to \infty \). The number of tuples \((k, \alpha_1, \ldots, \alpha_k)\) satisfying the above conditions is at most

\[
\#T \ll \log \log x \sum_{n \leq 10 \log \log x} p(n),
\]

where again \( p(n) \) is the partition function of \( n \). Using estimate (3), we conclude that

\[
\#T \ll (\log \log x)^2 \exp(O(\sqrt{\log \log x})) = (\log x)^{o(1)} \quad \text{as } x \to \infty.
\]

Thus,

\[
\#A_3 \leq \sum_{(k, \alpha_1, \ldots, \alpha_k) \in T} \frac{x}{f(p_1^{\alpha_1} \cdots p_k^{\alpha_k})} \leq \frac{x \#T}{B_K} = \frac{x}{(\log x)^{1+o(1)}}
\]
as \( x \to \infty \). Now inequalities (14), (15) and (16) yield the desired upper bound and complete the proof.

4. Comments. Quite likely, the results of Theorems 1 and 2 are not best possible. In this respect, we suggest the following questions:

**Question 1.** Is it true that \( \#A(x) = \exp(O(\sqrt{\log x})) \)?

**Question 2.** In the notations used in the proof of Theorem 2, is it true that

\[
S(x, y) \geq (1 + o(1))S(0, y) \quad \text{as } y \to \infty?
\]

Namely, is it true that the average value of \( f(n) \) in the interval \((0, y]\) is an asymptotic lower bound for the average value of \( f(n) \) in any interval of length \( y \) as \( y \to \infty \)?

Concerning Question 2 above, observe that our proof indicates that this is indeed the case except when \( x \in [yz, y^2] \), where \( z = (\log y)^{1/2}(\log \log y)^{-2} \).
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