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Explicit evaluations of quadruple Euler sums

by

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1. Introduction. For a pair of positive integers p, q with $q \geq 2$, the classical Euler sum is defined as

$$S_{p,q} := \sum_{k=1}^{\infty} \frac{1}{k^q} \sum_{j=1}^{k} \frac{1}{j^p}. \tag{1.1}$$

The number $w = p+q$ is the weight of $S_{p,q}$. The evaluation of $S_{p,q}$ in terms of values of the Riemann zeta function at positive integers is known when $p = 1$ or $(p,q) = (2,4)$ or $(p,q) = (4,2)$ or $p = q$ or $p+q$ is odd [1, 2, 3, 5]. Further generalizations of extensions of Euler sums have been developed [6, 8–10, 12, 13, 20, 21]. Here we state results which are related to the evaluation of $S_{p,q}$.

Theorem A ([1, 2, 6]). *For each positive integer n with $n \geq 2$, we have*

$$S_{1,n} = \frac{n+2}{2}\zeta(n+1) - \frac{1}{2}\sum_{\ell=2}^{n-1} \zeta(\ell)\zeta(n+1-\ell). \tag{1.2}$$

Theorem B ([2, 6]). *For an odd weight $w = p+q$ with $p, q \geq 2$, we have*

$$\begin{aligned} S_{p,q} = {} & \frac{1}{2}\zeta(p+q) + \frac{1+(-1)^{p+1}}{2}\zeta(p)\zeta(q) \\ & + (-1)^p \sum_{\ell=0}^{[p/2]} \binom{w-2\ell-1}{q-1} \zeta(2\ell)\zeta(w-2\ell) \\ & + (-1)^p \sum_{\ell=0}^{[q/2]} \binom{w-2\ell-1}{p-1} \zeta(2\ell)\zeta(w-2\ell). \end{aligned} \tag{1.3}$$

Here $\zeta(s)$ is the well-known Riemann zeta function and $\zeta(0) = -1/2$.

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The multiple zeta function or r-fold Euler sum [3, 4, 15] for positive integers $\alpha_1, \ldots, \alpha_r$ with $\alpha_r \geq 2$, defined by

$$\zeta(\alpha_1, \ldots, \alpha_r) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} \frac{1}{n_1^{\alpha_1}(n_1+n_2)^{\alpha_2} \cdots (n_1+\cdots+n_r)^{\alpha_r}} \tag{1.4}$$

can be rewritten as a nested r-fold sum

$$\sum_{n_r=1}^{\infty} \frac{1}{n_r^{\alpha_r}} \sum_{n_{r-1}=1}^{n_r-1} \frac{1}{n_{r-1}^{\alpha_{r-1}}} \cdots \sum_{n_2=1}^{n_3-1} \frac{1}{n_2^{\alpha_2}} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^{\alpha_1}}. \tag{1.5}$$

Its weight is the positive integer $|\alpha| = \alpha_1 + \cdots + \alpha_r$. So these multiple zeta functions are just multiple version of the classical Euler sums. In particular, when $r = 2$, we have

$$\zeta(p, q) = S_{p,q} - \zeta(p+q). \tag{1.6}$$

Around 1994, D. Borwein [2] and C. Markett [18] proved for $n \geq 3$ that

$$\begin{aligned} \zeta(1,1,n) = {} & \frac{n(n+1)}{6}\zeta(n+2) - \frac{n-1}{2}\zeta(2)\zeta(n) \\ & - \frac{n}{4}\sum_{k=0}^{n-4} \zeta(n-k-1)\zeta(k+3) \\ & + \frac{1}{6}\sum_{k=0}^{n-1} \zeta(n-k-2)\sum_{j=0}^{k} \zeta(k-j+2)\zeta(j+2). \end{aligned} \tag{1.7}$$

As pointed out in [3], such a formula can also be obtained from the power series expansion of the function $1 - \Gamma(1-x)\Gamma(1-y)/\Gamma(1-x-y)$. Let $\{1\}^m$ be the m repetitions of 1, for example,

$$\zeta(\{1\}^3, n) = \zeta(1,1,1,n) \quad \text{and} \quad \zeta(\{1\}^4, n) = \zeta(1,1,1,1,n).$$

Then as given in [3], the generating function for multiple zeta values of the form $\zeta(\{1\}^m, n+2)$ is given by

$$\begin{aligned} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^{n+1}y^{m+1}\zeta(\{1\}^m, n+2) &= 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \\ &= 1 - \exp\left\{\sum_{k=2}^{\infty}(x^k + y^k - (x+y)^k)\frac{\zeta(k)}{k}\right\}. \end{aligned} \tag{1.8}$$

Here we give the value of $\zeta(\{1\}^3, n)$ in a different way.

THEOREM 1. *For positive integers n, we have*

$$(1.9)\quad 2\zeta(1,1,1,2n) = \zeta(2n+3) + \zeta(1,2n+2) + \zeta(2,2n+1) + \zeta(3,2n) + \sum_{\ell=2}^{2n-1} (-1)^{\ell+1}\zeta(2n+1-\ell)\zeta(1,1,\ell)$$

and

$$(1.10)\quad \zeta(1,1,1,2n+1) = -\zeta(2n+4) + \sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)\zeta(1,1,\ell) - \frac{1}{2}\sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(1,2n+2-\ell)\zeta(1,\ell).$$

The proof of Theorem 1 is given in Section 2.

To give evaluations of general Euler sums, we have to introduce two kinds of multiple zeta values with variables:

$$(1.11)\quad H_{\alpha}(\mathbf{x}) = \sum_{n_r=0}^{\infty} \frac{1}{(n_r+x_r)^{\alpha_r}} \cdots \sum_{n_2=0}^{\widetilde{n}_3} \frac{1}{(n_2+x_2)^{\alpha_2}} \sum_{n_1=0}^{\widetilde{n}_2} \frac{1}{(n_1+x_1)^{\alpha_1}},$$

$$(1.12)\quad G_{\alpha}(\mathbf{x}) = \sum_{n_r=0}^{\infty} \frac{1}{(n_r+x_r)^{\alpha_r}} \cdots \sum_{n_2=0}^{\widetilde{n}_3} \frac{1}{(n_2+x_2)^{\alpha_2}} \sum_{n_1=0}^{\widetilde{n}_r} \frac{1}{(n_1+x_1)^{\alpha_1}},$$

where $x_1, \ldots, x_r$ are positive numbers no greater than 1 and

$$\widetilde{n}_j = \begin{cases} n_j & \text{if } x_{j-1} \le x_j, \text{ or } x_1 \le x_r \text{ when } j=r, \\ n_j - 1 & \text{if } x_{j-1} > x_j, \text{ or } x_1 > x_r \text{ when } j=r. \end{cases}$$

THEOREM 2. *For each positive integer n and positive real numbers x, y, z with $0 < x < y < z \le 1$, we have*

$$(1.13)\quad \begin{aligned} &G_{1,1,1,2n}(z,x,y,z) - H_{1,1,1,2n}(z,z-x,z-y,1) \\ &\quad = \sum_{\ell=2}^{2n-1} (-1)^{\ell+1}\zeta(2n+1-\ell)H_{1,1,\ell}(x,y,z) \\ &\qquad - H_{1,1,2n}(x,y,z)\{\gamma+\psi(z)\} \\ &\qquad + H_{1,1,2n}(z-x,z-y,1)\{-\psi(x)+\psi(z)\} \\ &\qquad + H_{1,2n}(z-y,1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{z-y}{(j+x)(k+j+y)(k+j+z)} \end{aligned}$$

and

$$\begin{aligned}(1.14)\quad & G_{1,1,1,2n+1}(z,x,y,z)+H_{1,1,1,2n+1}(z,z-x,z-y,1)\\ &=\sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)H_{1,1,\ell}(x,y,z)\\ &\quad-H_{1,1,2n+1}(x,y,z)\{\gamma+\psi(z)\}\\ &\quad-H_{1,1,2n+1}(z-x,z-y,1)\{-\psi(x)+\psi(z)\}\\ &\quad-H_{1,2n+1}(z-y,1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{z-y}{(j+x)(k+j+y)(k+j+z)}.\end{aligned}$$

Here $\psi(x)$ is the digamma function defined by $\Gamma'(x)/\Gamma(x)$ and γ is the Euler constant.

THEOREM 3. *For each positive integer n and positive real numbers x, y, z with $0 < x, y < z \le 1$, we have*

$$\begin{aligned}(1.15)\quad & G_{1,1,1,2n}(x,z,z-y,z)-G_{1,1,1,2n}(y,z,z-x,1)\\ &=\sum_{\ell=2}^{2n-1}(-1)^{\ell+1}H_{1,2n+1-\ell}(y,1)H_{1,\ell}(x,z)\\ &\quad+G_{1,1,2n}(z-x,y,1)\{-\psi(x)+\psi(z)\}\\ &\quad-G_{1,1,2n}(z-y,x,z)\{-\psi(y)+\psi(z)\}\\ &\quad-H_{1,2n}(x,z)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{z-1}{(k+y)(k+j+z)(k+j+1)}\end{aligned}$$

and

$$\begin{aligned}(1.16)\quad & G_{1,1,1,2n+1}(x,z,z-y,z)+G_{1,1,1,2n+1}(y,z,z-x,1)\\ &=\sum_{\ell=2}^{2n}(-1)^{\ell}H_{1,2n+2-\ell}(y,1)H_{1,\ell}(x,z)\\ &\quad-G_{1,1,2n+1}(z-x,y,1)\{-\psi(x)+\psi(z)\}\\ &\quad-G_{1,1,2n+1}(z-y,x,z)\{-\psi(y)+\psi(z)\}\\ &\quad-H_{1,2n+1}(x,z)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{z-1}{(k+y)(k+j+z)(k+j+1)}.\end{aligned}$$

As both sides of the identities in Theorems 2 and 3 are continuous functions of x, y and z, by a limit process if necessary, we let $x = y = z = 1$ and get the evaluations of $\zeta(1,1,1,2n)$ and $\zeta(1,1,1,2n+1)$. If we perform partial differentiations with respect to x, y, z and then let x, y, z approach to 1, we obtain linear equations among multiple zeta values of the same weight. By solving the system of linear equations we get evaluations of quadruple Euler sums of odd weight in terms of triple, double, or single Euler sums.

2. The evaluation of $\zeta(1,1,1,n)$. The evaluation of $\zeta(1,1,1,n)$ is a direct consequence of the following propositions.

PROPOSITION 1. *For each positive integer n, we have*

$$2\zeta(1,1,1,2n) = \zeta(2n+3) + \zeta(1,2n+2) + \zeta(2,2n+1) + \zeta(3,2n) + \sum_{\ell=2}^{2n-1} (-1)^{\ell+1} \zeta(2n+1-\ell)\zeta(1,1,\ell)$$

and

$$\begin{aligned} 4\zeta(1,1,1,2n{+}1) = & -\zeta(2n{+}4) - \{\zeta(1,2n{+}3)+\zeta(2,2n{+}2)+\zeta(3,2n{+}1)\} \\ & -2\{\zeta(1,1,2n{+}2)+\zeta(1,2,2n{+}1)+\zeta(2,1,2n{+}1)\} \\ & +\sum_{\ell=2}^{2n} (-1)^{\ell}\zeta(2n{+}2-\ell)\zeta(1,1,\ell). \end{aligned}$$

Proof. For each positive integer $n \geq 2$, we consider the particular zeta value

$$\zeta(1,1,2n-1,2) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\sum_{n_4=1}^{\infty} \frac{1}{n_1(n_1+n_2)(n_1+n_2+n_3)^{2n-1}(n_1+n_2+n_3+n_4)^2}. \tag{2.1}$$

In the partial fraction decomposition [15, 16, 17]

$$\frac{1}{X^{2n-1}(X+T)^2} = \sum_{\ell=2}^{2n-1} (-1)^{\ell+1}(2n-\ell)\frac{1}{T^{2n+1-\ell}X^{\ell}} - \frac{1}{T^{2n-1}(X+T)^2} + (2n-1)\frac{1}{T^{2n}}\left\{\frac{1}{X} - \frac{1}{X+T}\right\} \tag{2.2}$$

we set $X = n_1+n_2+n_3$, $T = n_4$, then multiply both sides of the resulting identity by $\frac{1}{n_1(n_1+n_2)}$ and sum over all positive integers n_1, n_2, n_3, n_4, to get the following expression for $\zeta(1,1,2n-1,2)$:

$$\begin{aligned} \zeta(1,1,2n{-}1,2) = & \sum_{\ell=2}^{2n-1} (-1)^{\ell+1}(2n-\ell)\zeta(2n{+}1-\ell)\zeta(1,1,\ell) - \eta(1,1,2n{-}1,2) \\ & +(2n-1)\sum_{n_4=1}^{\infty}\frac{1}{n_4^{2n}}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{1}{n_1(n_1+n_2)} \\ & \times\sum_{n_3=1}^{\infty}\left\{\frac{1}{n_1+n_2+n_3} - \frac{1}{n_1+n_2+n_3+n_4}\right\}, \end{aligned} \tag{2.3}$$

where

$$\eta(1,1,2n-1,2) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\sum_{n_4=1}^{\infty}\frac{1}{n_1(n_1+n_2)n_4^{2n-1}(n_1+n_2+n_3+n_4)^2}. \tag{2.4}$$

Let $A(n)$ be the final term of $\zeta(1,1,2n-1,2)$ in (2.3). Then $A(n)$ is just an ordinary quadruple sum in disguise. By repeatedly using the identity

$$\sum_{n=1}^{\infty}\left\{\frac{1}{n+x}-\frac{1}{n+k+x}\right\} = \sum_{n=1}^{k}\frac{1}{n+x}, \quad x>0, \tag{2.5}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=1}^{m}\frac{1}{n(n+k)} = \sum_{k=1}^{m}\frac{1}{k}\sum_{n=1}^{\infty}\left\{\frac{1}{n}-\frac{1}{n+k}\right\} = \sum_{k=1}^{m}\frac{1}{k}\sum_{n=1}^{k}\frac{1}{n}, \tag{2.6}$$

we get

$$A(n) = \sum_{n_4=1}^{\infty}\frac{1}{n_4^{2n}}\sum_{n_3=1}^{n_4}\frac{1}{n_3}\sum_{n_2=1}^{n_3}\frac{1}{n_2}\sum_{n_1=1}^{n_2}\frac{1}{n_1}. \tag{2.7}$$

On the other hand, if we begin with $\eta(1,1,2n-1,2)$ and employ the same partial fraction decomposition (2.2) with $X=n_4$, $T=n_1+n_2+n_3$, we get

$$\begin{aligned}\eta(1,1,2n-1,2) &= \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}(2n-\ell)\zeta(\ell)\zeta(1,1,2n+1-\ell)-\zeta(1,1,2n-1,2)\\ &\quad+(2n-1)\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=1}^{\infty}\frac{1}{n_1(n_1+n_2)(n_1+n_2+n_3)^{2n}}\sum_{n_4=1}^{n_1+n_2+n_3}\frac{1}{n_4}.\end{aligned} \tag{2.8}$$

Let $B(n)$ be the quadruple sum above. Then $B(n)$ can be rewritten as

$$B(n) = \sum_{n_3=1}^{\infty}\frac{1}{n_3^{2n}}\left\{\sum_{n_2=1}^{n_3-1}\frac{1}{n_2}\sum_{n_1=1}^{n_2-1}\frac{1}{n_1}\right\}\sum_{n_4=1}^{n_3}\frac{1}{n_4} \tag{2.9}$$

with a change of variables. The difference of (2.3) and (2.8) gives

$$B(n)-A(n) = \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}\zeta(2n+1-\ell)\zeta(1,1,\ell). \tag{2.10}$$

The sum $B(n)$ in (2.9) is divided into a sum of triple sums and quadruple sums according to the range of n_4:

- For $n_4=n_3$, the corresponding subseries is $\zeta(1,1,2n+1)$.
- For $n_2<n_4<n_3$, the corresponding subseries is $\zeta(1,1,1,2n)$.

- For $n_4 = n_2$, the corresponding subseries is $\zeta(1,2,2n)$.
- For $n_4 < n_2$, the corresponding subseries is $2\zeta(1,1,1,2n)+\zeta(2,1,2n)$.

Consequently, we have

$$B(n) = 3\zeta(1,1,1,2n)+\{\zeta(1,1,2n+1)+\zeta(1,2,2n)+\zeta(2,1,2n)\}. \tag{2.11}$$

With a similar consideration, we get

$$\begin{aligned} A(n) = \zeta(1,1,1,2n)&+\{\zeta(1,1,2n+1)+\zeta(1,2,2n)+\zeta(2,1,2n)\} \\ &+\{\zeta(1,2n+2)+\zeta(2,2n+1)+\zeta(3,2n)\}+\zeta(2n+3). \end{aligned} \tag{2.12}$$

By (2.10)–(2.12), we obtain

$$\begin{aligned} 2\zeta(1,1,1,2n) = \zeta(2n+3)&+\{\zeta(1,2n+2)+\zeta(2,2n+1)+\zeta(3,2n)\} \\ &+\sum_{\ell=2}^{2n-1}(-1)^{\ell+1}\zeta(2n+1-\ell)\zeta(1,1,\ell). \end{aligned} \tag{2.13}$$

If we begin with the quadruple zeta function $\zeta(1,1,2n,2)$ and employ the partial fraction decomposition [18]

$$\begin{aligned} \frac{1}{X^{2n}(X+T)^2} = &\sum_{\ell=2}^{2n}(-1)^{\ell}(2n+1-\ell)\frac{1}{T^{2n+2-\ell}X^{\ell}}+\frac{1}{T^{2n}(X+T)^2} \\ &-2n\,\frac{1}{T^{2n+1}}\left\{\frac{1}{X}-\frac{1}{X+T}\right\}, \end{aligned} \tag{2.14}$$

we get

$$A(n+1/2)+B(n+1/2) = \sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)\zeta(1,1,\ell). \tag{2.15}$$

The evaluation of $\zeta(1,1,1,2n+1)$ then follows from the facts that

$$\begin{aligned} A(n+1/2) = \zeta(1,1,1,2n+1)& \\ &+\{\zeta(1,1,2n+2)+\zeta(1,2,2n+1)+\zeta(2,1,2n+1)\} \\ &+\{\zeta(1,2n+3)+\zeta(2,2n+2)+\zeta(3,2n+1)\}+\zeta(2n+4) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} B(n+1/2) = 3\zeta(1,1,1,2n+1)& \\ &+\{\zeta(1,1,2n+2)+\zeta(1,2,2n+1)+\zeta(2,1,2n+1)\}. \quad \blacksquare \end{aligned}$$

Proposition 2. *For each positive integer n, we have*

$$\begin{aligned} 3\zeta(1,1,1,2n+1) = &-2\{\zeta(1,1,2n+2)+\zeta(1,2,2n+1)+\zeta(2,1,2n+1)\} \\ &-\{\zeta(1,2n+3)+\zeta(2,2n+2)+\zeta(3,2n+1)\} \\ &+\frac{1}{2}\sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(1,2n+2-\ell)\zeta(1,\ell). \end{aligned}$$

Proof. We consider the multiple series

$$\xi(n)=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\sum_{k_4=1}^{\infty}\frac{1}{k_1(k_1+k_2)^{2n}k_3(k_1+k_2+k_3+k_4)^2}.$$

Applying the partial fraction decomposition

$$\begin{aligned}\frac{1}{X^{2n}(X+T)^2}&=\sum_{\ell=2}^{2n}(-1)^{\ell}(2n+1-\ell)\frac{1}{T^{2n+2-\ell}X^{\ell}}+\frac{1}{T^{2n}(X+T)^2}\\&\quad-2n\,\frac{1}{T^{2n+1}}\left\{\frac{1}{X}-\frac{1}{X+T}\right\}\end{aligned}$$

with $X=k_1+k_2$ and $T=k_3+k_4$, we get

$$\begin{aligned}\xi(n)&=\sum_{\ell=2}^{2n}(-1)^{\ell}(2n+1-\ell)\zeta(1,2n+2-\ell)\zeta(1,\ell)+\xi(n)\\&\quad-2n\sum_{k_3=1}^{\infty}\sum_{k_4=1}^{\infty}\frac{1}{(k_3+k_4)^{2n+1}k_3}\sum_{k_1=1}^{\infty}\frac{1}{k_1}\sum_{k_2=1}^{\infty}\left(\frac{1}{k_1+k_2}-\frac{1}{k_1+k_2+k_3+k_4}\right).\end{aligned}$$

By the same argument as in Proposition 1, the above series is equal to

$$\sum_{k_1=1}^{\infty}\frac{1}{k_1^{2n+1}}\sum_{k_2=1}^{k_1}\frac{1}{k_2}\sum_{k_3=1}^{k_2}\frac{1}{k_3}\sum_{k_4=1}^{k_1-1}\frac{1}{k_4},$$

which can be decomposed as

$$\begin{aligned}&3\zeta(1,1,1,2n+1)+\{\zeta(1,2n+3)+\zeta(2,2n+2)+\zeta(3,2n+1)\}\\&\qquad+2\{\zeta(1,1,2n+2)+\zeta(1,2,2n+1)+\zeta(2,1,2n+1)\}.\end{aligned}$$

Consequently, our assertion follows by elementary considerations. ∎

Remark. The evaluation of $\zeta(\{1\}^m,n)$ for $m\ge 4$ needs more relations similar to those given in Propositions 1 and 2. However, it can be done by considering multiple series of the form

$$\sum_{\mathbf{k}\in\mathbb{N}^{p+1}}\sum_{\mathbf{j}\in\mathbb{N}^{q}}\frac{1}{s_1\cdots s_p s_{p+1}^{\alpha}\sigma_1\cdots\sigma_{q-1}(s_{p+1}+\sigma_q)^2},\qquad \alpha=2n+1 \text{ or } 2n,\ p+q=m,$$

where $s_m=k_1+\cdots+k_m$ and $\sigma_\ell=j_1+\cdots+j_\ell$. The identities obtained are

$$G_{2n}(p,q)-G_{2n}(q-1,p+1)=\sum_{\alpha=2}^{2n-1}(-1)^{\alpha-1}\zeta(\{1\}^p,\alpha)\zeta(\{1\}^{q-1},2n+1-\alpha)$$

and

$$G_{2n+1}(p,q)+G_{2n+1}(q-1,p+1)=\sum_{\alpha=2}^{2n}(-1)^{\alpha}\zeta(\{1\}^p,\alpha)\zeta(\{1\}^{q-1},2n+2-\alpha).$$

Finally, the decompositions of $G_n(p,q)$ then lead to explicit evaluations of $\zeta(\{1\}^m, n)$ in terms of weighted alternating sum of the products

$$\zeta(\{1\}^p, n_1)\zeta(\{1\}^{m-p-1}, n_2)$$

with $0 \le p \le m-1$ and $n_1+n_2 = n+1$. See [7] for the details.

3. The proofs of Theorems 2 and 3. The well-known digamma function $\psi(x)$ is defined as the logarithmic derivative of the gamma function $\Gamma(x)$, i.e.

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad x > 0.$$

We need the following properties of $\psi(x)$.

(1) For positive numbers x and y, we have

$$-\psi(x)+\psi(y) = \sum_{j=0}^{\infty}\left(\frac{1}{j+x} - \frac{1}{j+y}\right).$$

In particular, $\psi(1) = -\gamma$ with

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right),$$

the Euler constant.

(2) A functional equation of $\psi(x)$ is given by

$$\psi(x+1) = \frac{1}{x}+\psi(x) \qquad \text{for } x > 0.$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. We begin with the multiple zeta value with variables

$$H_{1,1,2n-1,2}(x,y,z,z) - H_{1,1,2n+1}(x,y,z) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=1}^{\infty}\frac{1}{(s_1+x)(s_2+y)(s_3+z)^{2n-1}(s_4+z)^2}$$

with $0 < x < y < z \le 1$ and

$$T_{2n-1}(x,y,z) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=1}^{\infty}\frac{1}{(s_1+x)(s_2+z)^{2n-1}(k_3+y)(s_4+z)^2}$$

with $0 < x, y < z \leq 1$, where $s_j = k_1+\cdots+k_j$. Employing the partial fraction decomposition (2.2) with $X = s_3+z$, $T = k_4$, we get the relation

$$\begin{aligned}
&H_{1,1,2n-1,2}(x,y,z,z) - H_{1,1,2n+1}(x,y,z) \\
&\quad = \sum_{\ell=2}^{2n-1} (-1)^{\ell+1}(2n-\ell) H_{1,1,\ell}(x,y,z)\zeta(2n+1-\ell) - T_{2n-1}(x,y,z) \\
&\qquad + (2n-1) \sum_{k_4=1}^{\infty} \frac{1}{k_4^{2n+1}} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{1}{(s_1+x)(s_2+y)} \sum_{k_3=0}^{\infty} \left(\frac{1}{s_3+z} - \frac{1}{s_4+z} \right).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{k_3=0}^{\infty} \left(\frac{1}{s_3+z} - \frac{1}{s_4+z} \right) &= \psi(k_1+k_2+z) - \psi(k_1+k_2+k_4+z) \\
&= \sum_{k_3=0}^{k_4-1} \frac{1}{k_1+k_2+k_3+z}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k_2=0}^{\infty} \frac{1}{s_2+y} \sum_{k_3=0}^{k_4-1} \frac{1}{k_1+k_2+k_3+z} \\
&\qquad = \sum_{k_3=0}^{k_4-1} \frac{1}{k_3+z-y} \sum_{k_2=0}^{\infty} \left(\frac{1}{k_1+k_2+y} - \frac{1}{k_1+k_2+k_3+z} \right) \\
&\qquad = \sum_{k_3=0}^{k_4-1} \frac{1}{k_3+z-y} \sum_{k_2=0}^{\infty} \left(\frac{1}{k_1+k_2+y} - \frac{1}{k_1+k_2+z} \right) \\
&\qquad\quad + \sum_{k_3=1}^{k_4-1} \frac{1}{k_3+z-y} \sum_{k_2=0}^{k_3-1} \frac{1}{k_1+k_2+z}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\sum_{k_1=0}^{\infty} \frac{1}{k_1+x} \sum_{k_2=0}^{k_3-1} \frac{1}{k_1+k_2+z} \\
&\qquad = \sum_{k_2=0}^{k_3-1} \frac{1}{k_2+z-x} \sum_{k_1=0}^{\infty} \left(\frac{1}{k_1+x} - \frac{1}{k_1+k_2+z} \right) \\
&\qquad = \sum_{k_2=0}^{k_3-1} \frac{1}{k_2+z-x} (-\psi(x)+\psi(z)) + \sum_{k_2=1}^{k_3-1} \frac{1}{k_2+z-x} \sum_{k_1=0}^{k_2-1} \frac{1}{k_1+z}.
\end{aligned}$$

Consequently, the final series in $H_{1,1,2n-1,2}(x,y,z,z)-H_{1,1,2n-1}(x,y,z)$ can be rewritten as

$$H_{1,2n+1}(z-y,1)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{z-y}{(k_1+x)(k_1+k_2+y)(k_1+k_2+z)}$$
$$+H_{1,1,2n+1}(z-x,z-y,1)\{-\psi(x)+\psi(y)\}$$
$$+H_{1,1,1,2n+1}(z,z-x,z-y,1).$$

On the other hand, if we employ the same partial fraction decomposition with $X=k_4$ and $T=k_1+k_2+k_3+k_4$, we get

$$\begin{aligned}T_{2n-1}(x,y,z) = &\sum_{\ell=2}^{2n-1}(-1)^{\ell+1}(2n-\ell)H_{1,1,2n+1-\ell}(x,y,z)\zeta(\ell)\\ &-\{H_{1,1,2n-1,2}(x,y,z,z)-H_{1,1,2n+1}(x,y,z)\}\\ &+(2n-1)\{G_{1,1,1,2n+1}(z,x,y,z)+H_{1,1,2n+1}(x,y,z)[\psi(z)-\gamma]\}.\end{aligned}$$

The difference of the two expressions for

$$H_{1,1,2n-1,2}(x,y,z,z)-H_{1,1,2n+1}(x,y,z)+T_{2n-1}(x,y,z)$$

then leads to the first part of Theorem 2.

To obtain the second part, we begin with

$$H_{1,1,2n,2}(x,y,z,z)-H_{1,1,2n+2}(x,y,z)\quad\text{and}\quad T_{2n}(x,y,z,z)$$

and employ the partial fraction decomposition (2.14). Again we get two different expressions for the same multiple series

$$H_{1,1,2n,2}(x,y,z,z)-H_{1,1,2n+2}(x,y,z)-T_{2n}(x,y,z),$$

so the difference gives our assertion.

Theorem 3 can be obtained in the same way if we consider the multiple series

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\frac{1}{(s_1+x)(s_2+z)^{\alpha}(k_3+y)(s_4+z)^2}$$

with $\alpha=2n-1$ and $2n$ and $0<x,y<z\le 1$ ∎

4. Immediate consequences. Applying the partial differential operators

$$\left(\frac{\partial}{\partial x}\right)^{p-1}\left(\frac{\partial}{\partial y}\right)^{q-1}\qquad (p,q\ge 1)$$

to (1.13) and (1.15) when $p+q$ is even and to (1.14) and (1.16) when $p+q$ is odd, we get the following theorems.

Theorem 4. *For positive integers p, q, n, we have*

$$\begin{aligned}(4.1)\quad &\zeta(p,q,1,2n)+\zeta(p,1,q,2n)\\ &= \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}\zeta(2n+1-\ell)H_{p,q,\ell}(1,1,1)\\ &\quad+\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(\ell+1)\zeta(p-\ell,q,2n)+\sum_{\ell=1}^{q-1}(-1)^{p+\ell+1}\zeta(q-\ell,2n)S_{p,\ell+1}\\ &\quad+(-1)^{p+1}\zeta(p+1)\zeta(q,2n)+\zeta(2n)S_{p,q+1}-C_{p,q}(n)\end{aligned}$$

when $p+q$ is even and

$$\begin{aligned}(4.2)\quad &\zeta(p,q,1,2n+1)+\zeta(p,1,q,2n+1)\\ &= \sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)H_{p,q,\ell}(1,1,1)\\ &\quad+\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(\ell+1)\zeta(p-\ell,q,2n+1)\\ &\quad+\sum_{\ell=1}^{q-1}(-1)^{p+\ell+1}\zeta(q-\ell,2n+1)S_{p,\ell+1}\\ &\quad+(-1)^{p+1}\zeta(p+1)\zeta(q,2n+1)-\zeta(2n+1)S_{p,q+1}-C_{p,q}(n+1/2),\end{aligned}$$

when $p+q$ is odd, where

$$\begin{aligned}C_{p,q}(n) = {}&\zeta(1,p+q,2n)+\zeta(1,p,2n+q)+\zeta(p,q+1,2n)\\ &+\zeta(p,1,2n+q)+\zeta(p,q,2n+1)+\zeta(p+q,1,2n)\\ &+\zeta(p+q+1,2n)+\zeta(1,2n+p+q)+\zeta(p+1,2n+q)\\ &+\zeta(p+q,2n+1)+\zeta(p,2n+q+1)+\zeta(2n+p+q+1).\end{aligned}$$

In particular, for $q = 1$ we have

$$\begin{aligned}(4.3)\quad &2\zeta(2m+1,1,1,2n)\\ &= \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}\zeta(2n+1-\ell)H_{2m+1,1,\ell}(1,1,1)\\ &\quad+\sum_{\ell=1}^{2m}(-1)^{\ell}\zeta(\ell+1)\zeta(2m+1-\ell,1,2n)\\ &\quad+\zeta(2n)S_{2m+1,2}+\zeta(2m+2)\zeta(1,2n)-C_{2m+1,1}(n)\end{aligned}$$

and

$$(4.4)\qquad 2\zeta(2m,1,1,2n+1)=\sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)H_{2m,1,\ell}(1,1,1)$$
$$+\sum_{\ell=1}^{2m-1}(-1)^{\ell}\zeta(\ell+1)\zeta(2m-\ell,1,2n+1)$$
$$-\zeta(2n+1)S_{2m,2}-\zeta(2m+1)\zeta(1,2n+1)-C_{2m,1}(n+1/2)$$

for every positive integer m.

THEOREM 5. *For positive integers* p,q,n, *we have*

$$(4.5)\qquad \zeta(p,1,q,2n)-\zeta(q,1,p,2n)=\sum_{\ell=2}^{2n-1}(-1)^{p+\ell}S_{q,2n+1-\ell}S_{p,\ell}$$
$$+\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(\ell+1)\{G_{p-\alpha,q,2n}(1,1,1)-S_{q,p+2n-\alpha}\}$$
$$-\sum_{\ell=1}^{q-1}(-1)^{\ell}\zeta(\ell+1)\{G_{q-\alpha,p,2n}(1,1,1)-S_{p,q+2n-\alpha}\}$$
$$+(-1)^{p-1}\zeta(p+1)S_{q,2n}-(-1)^{p-1}\zeta(q+1)S_{p,2n}+\widetilde{C}_{p,q}(n)$$

when $p+q$ *is even, and*

$$(4.6)\qquad \zeta(p,1,q,2n+1)-\zeta(q,1,p,2n+1)=\sum_{\ell=2}^{2n}(-1)^{p+\ell}S_{q,2n+2-\ell}S_{p,\ell}$$
$$+\sum_{\ell=1}^{p-1}(-1)^{\ell}\zeta(\ell+1)\{G_{p-\alpha,q,2n+1}(1,1,1)-S_{q,p+2n+1-\alpha}\}$$
$$-\sum_{\ell=1}^{q-1}(-1)^{\ell}\zeta(\ell+1)\{G_{q-\alpha,p,2n+1}(1,1,1)-S_{p,q+2n+1-\alpha}\}$$
$$+(-1)^{p-1}\zeta(p+1)S_{q,2n+1}+(-1)^{p-1}\zeta(q+1)S_{p,2n+1}+\widetilde{C}_{p,q}(n+1/2)$$

when $p+q$ *is odd, where*

$$\widetilde{C}_{p,q}(n)=\zeta(1,p,q+2n)+\zeta(q,p+1,2n)+\zeta(p+1,q+2n)$$
$$-\zeta(1,q,p+2n)-\zeta(p,q+1,2n)-\zeta(q+1,p+2n).$$

The above theorems are enough to obtain the values of $\zeta(1,1,2m+1,2n)$, $\zeta(1,1,2m,2n+1)$, $\zeta(1,2m+1,1,2n)$ and $\zeta(1,2m,1,2n+1)$ by using the following relations:

$$(4.7)\quad \zeta(1,1,2m+1,2n)-\zeta(2m+1,1,1,2n) = \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}S_{2m+1,2n+1-\ell}S_{1,\ell} + \sum_{\ell=1}^{2m}(-1)^{\ell+1}\zeta(\ell+1)\{G_{2m+1-\alpha,1,2n}(1,1,1)-S_{1,2m+2n+1-\alpha}\} + \zeta(2)S_{2m+1,2n}-\zeta(2m+2)S_{1,2n}+\widetilde{C}_{1,2m+1}(n),$$

$$(4.8)\quad \zeta(1,1,2m,2n+1)-\zeta(2m,1,1,2n+1) = \sum_{\ell=2}^{2n}(-1)^{\ell+1}S_{2m,2n+2-\ell}S_{1,\ell} + \sum_{\ell=1}^{2m-1}(-1)^{\ell+1}\zeta(\ell+1)\{G_{2m-\alpha,1,2n+1}(1,1,1)-S_{1,2m+2n+1-\alpha}\} + \zeta(2)S_{2m,2n+1}+\zeta(2m+1)S_{1,2n+1}+\widetilde{C}_{1,2m}(n+1/2),$$

$$(4.9)\quad \zeta(1,2m+1,1,2n)+\zeta(1,1,2m+1,2n) = \sum_{\ell=2}^{2n-1}(-1)^{\ell+1}\zeta(2n+1-\ell)H_{1,2m+1,\ell}(1,1,1) + \sum_{\ell=1}^{2m}(-1)^{\ell}\zeta(2m+1-\ell,2n)S_{1,\ell+1} + \zeta(2)\zeta(2m+1,2n)+\zeta(2n)S_{1,2m+2}-C_{1,2m+1}(n)$$

and

$$(4.10)\quad \zeta(1,2m,1,2n+1)+\zeta(1,1,2m,2n+1) = \sum_{\ell=2}^{2n}(-1)^{\ell}\zeta(2n+2-\ell)H_{1,2m,\ell}(1,1,1) + \sum_{\ell=1}^{2m-1}(-1)^{\ell}\zeta(2m-\ell,2n+1)S_{1,\ell+1} + \zeta(2)\zeta(2m,2n+1)-\zeta(2n+1)S_{1,2m+1}-C_{1,2m}(n+1/2).$$

Furthermore, for every positive integer m, applying the partial differential operators $\partial^{2m}/\partial z\partial x^{2m-1}$ and $\partial^{2m}/\partial z\partial x^{2m}$ to (1.13) and (1.14) respectively, $\partial^{2m}/\partial z\partial x^{2m-1}$ or $\partial^{2m}/\partial z\partial y^{2m-1}$ and $\partial^{2m+1}/\partial z\partial x^{2m}$ or $\partial^{2m+1}/\partial z\partial y^{2m}$ to (1.15) and (1.16) respectively, we obtain the values of $\zeta(1,2,p,q)$, $\zeta(1,p,2,q)$,

$\zeta(2,1,p,q), \zeta(2,p,1,q), \zeta(p,1,2,q)$ and $\zeta(p,2,1,q)$ by solving the linear system

$$\begin{bmatrix} 2 & 1 & 2 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \zeta(1,2,p,q) \\ \zeta(1,p,2,q) \\ \zeta(2,1,p,q) \\ \zeta(2,p,1,q) \\ \zeta(p,1,2,q) \\ \zeta(p,2,1,q) \end{bmatrix} = \begin{bmatrix} \Delta_1(p,q) \\ \Delta_2(p,q) \\ \Delta_3(p,q) \\ \Delta_4(p,q) \\ \Delta_5(p,q) \\ \Delta_6(p,q) \end{bmatrix},$$

where $(p,q) = (2m+1, 2n+1)$ or $(2m, 2n)$ and each $\Delta_i(p,q)$ is a linear combination of products of single, double and triple Euler sums with sum of depths less than or equal to 4.

5. A concluding remark. Below we list quadruple Euler sums in terms of single zeta values when the weight is less than or equal to 7. This is based on the application of our theorems and the restricted sum formula [11], [14].

$$\begin{aligned}
\zeta(1,1,1,2) &= \zeta(5),\\
\zeta(1,1,1,3) &= \tfrac{3}{4}\zeta(6) - \tfrac{1}{2}\zeta^2(3),\\
\zeta(1,1,2,2) &= -\tfrac{4}{3}\zeta(6) + \zeta^2(3),\\
\zeta(1,2,1,2) &= -\tfrac{1}{2}\zeta(6) + \tfrac{1}{2}\zeta^2(3),\\
\zeta(2,1,1,2) &= \tfrac{25}{12}\zeta(6) - \zeta^2(3),\\
\zeta(1,1,1,4) &= 5\zeta(7) - 2\zeta(2)\zeta(5) - \tfrac{5}{4}\zeta(3)\zeta(4),\\
\zeta(1,1,2,3) &= -\tfrac{221}{16}\zeta(7) + \tfrac{11}{2}\zeta(2)\zeta(5) + \tfrac{7}{2}\zeta(3)\zeta(4),\\
\zeta(1,2,1,3) &= \tfrac{61}{8}\zeta(7) - \tfrac{9}{2}\zeta(2)\zeta(5),\\
\zeta(2,1,1,3) &= -\tfrac{109}{16}\zeta(7) + 5\zeta(2)\zeta(5) - \tfrac{5}{4}\zeta(3)\zeta(4),\\
\zeta(1,1,3,2) &= \tfrac{5}{8}\zeta(7) + \tfrac{5}{2}\zeta(2)\zeta(5) - \tfrac{15}{4}\zeta(3)\zeta(4),\\
\zeta(1,3,1,2) &= -\tfrac{1}{4}\zeta(7) + \tfrac{1}{4}\zeta(3)\zeta(4),\\
\zeta(3,1,1,2) &= \tfrac{61}{8}\zeta(7) - \tfrac{11}{2}\zeta(2)\zeta(5) + \tfrac{7}{4}\zeta(3)\zeta(4),\\
\zeta(1,2,2,2) &= \tfrac{157}{16}\zeta(7) - \tfrac{15}{2}\zeta(2)\zeta(5) + \tfrac{9}{4}\zeta(3)\zeta(4),\\
\zeta(2,1,2,2) &= \tfrac{75}{8}\zeta(7) - \tfrac{11}{2}\zeta(2)\zeta(5),\\
\zeta(2,2,1,2) &= -\tfrac{291}{16}\zeta(7) + 12\zeta(2)\zeta(5) - \tfrac{3}{2}\zeta(3)\zeta(4).
\end{aligned}$$

By the duality theorem [19], quadruple Euler sums can be reduced to triple Euler sums at weight 7. For example, $\zeta(1,3,1,2) = \zeta(3,1,3)$. Clearly, the above results are precisely those mentioned by Markett and Broadhurst [4, 18].

We also recall our theorem used above:

Theorem 6 (The restricted sum formula [11]). *For positive integers m, q with $m \geq q$ and every nonnegative integer p, we have*

$$\sum_{\substack{\alpha_i \geq 1, \forall i \\ |\boldsymbol{\alpha}|=m}} \zeta(\{1\}^p, \alpha_1, \ldots, \alpha_q+1) = \sum_{\substack{c_i \geq 1, \forall i \\ |\boldsymbol{c}|=p+q}} \zeta(c_1, \ldots, c_{p+1}+(m-q)+1).$$

Remark. For

$$\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \{1\}^{a_2-1}, b_2+1, \ldots, \{1\}^{a_r-1}, b_r+1)$$

with positive integers $a_1, b_1, \ldots, a_r, b_r$, the dual of $\mathbf{k}$ is given by

$$\mathbf{k}' = (\{1\}^{b_r-1}, a_r+1, \ldots, \{1\}^{b_2-1}, a_2+1, \{1\}^{b_1-1}, a_1+1).$$

Then Ohno's generalization of the duality theorem and sum formula [19] asserts that for any nonnegative integers ℓ,

$$\sum_{\substack{\alpha_i \geq 0, \forall i \\ |\boldsymbol{\alpha}|=\ell}} \zeta(\mathbf{k}+\boldsymbol{\alpha}) = \sum_{\substack{\beta_i \geq 0, \forall i \\ |\boldsymbol{\beta}|=\ell}} \zeta(\mathbf{k}'+\boldsymbol{\beta}).$$

For $\ell = 0$, this is just the Drinfeld duality theorem $\zeta(\mathbf{k}) = \zeta(\mathbf{k}')$.

Finally, although we are unable to prove that our method leads to evaluations of all multiple zeta values when the sum of weight and depth is odd, we are able to extend our formulas to evaluate multiple zeta values of arbitrary depth. For example, for real numbers $\{x_i\}_{i=1}^p$ and z with $0 < x_1 < \cdots < x_p < z \leq 1$, we always have the first kind of identities as shown in Theorem 2 with left hand sides being

$$G_{\{1\}^{p+1},2n}(z, x_1, \ldots, x_p, z) - H_{\{1\}^{p+1},2n}(z, z-x_1, \ldots, z-x_p, 1)$$

and

$$G_{\{1\}^{p+1},2n+1}(z, x_1, \ldots, x_p, z) + H_{\{1\}^{p+1},2n+1}(z, z-x_1, \ldots, z-x_p, 1).$$

Next, by applying the partial differential operators

$$\frac{1}{(2m)!}\left(\frac{\partial}{\partial x_1}\right)^{2m} \quad \text{and} \quad \frac{-1}{(2m-1)!}\left(\frac{\partial}{\partial x_1}\right)^{2m-1}$$

to the above respectively and then letting $x_1, \ldots, x_p$ and z approach 1, we derive the values of

$$\zeta(2m+1, \{1\}^p, 2n) \quad \text{and} \quad \zeta(2m, \{1\}^p, 2n+1).$$

It is worth noting that as in the case of depth 4, we also obtain two kinds of identities like Theorems 2 and 3 for depth 5. In general, we have k kinds of identities for depth $2k$ and $2k+1$. We believe that these identities are enough to reduce multiple zeta values to lower depth provided the sum of weight and depth is odd.

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