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Note on the paper "An extension of a theorem of Euler" by Hirata-Kohno et al.

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by

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1. Introduction. Let $n, d, k > 2$ and y be positive integers such that $\gcd(n, d) = 1$. For an integer $\nu > 1$, we denote by $P(\nu)$ the greatest prime factor of ν and we put $P(1) = 1$. Let b be a squarefree positive integer such that $P(b) \leq k$. We consider the equation

$$n(n+d)\cdots(n+(k-1)d) = by^2 \tag{1}$$

in n, d, k and y.

A celebrated theorem of Erdős and Selfridge [7] states that the product of consecutive positive integers is never a perfect power. An old, difficult conjecture states that even a product of consecutive terms of an arithmetic progression of length $k > 3$ and difference $d \geq 1$ is never a perfect power. Euler proved (see [6, pp. 440 and 635]) that a product of four terms in arithmetic progression is never a square solving equation (1) with $b = 1$ and $k = 4$. Obláth [10] obtained a similar statement for $b = 1$, $k = 5$. Bennett, Bruin, Győry and Hajdu [1] solved (1) with $b = 1$ and $6 \leq k \leq 11$. For more results on this topic see [1], [8] and the references given there.

We write

$$n + id = a_i x_i^2 \quad \text{for } 0 \leq i < k \tag{2}$$

where a_i are squarefree integers such that $P(a_i) \leq \max(P(b), k-1)$ and x_i are positive integers. Every solution to (1) yields a k-tuple $(a_0, a_1, \ldots, a_{k-1})$. Recently Hirata-Kohno, Laishram, Shorey and Tijdeman [8] proved the following theorem.

THEOREM A (Hirata-Kohno, Laishram, Shorey, Tijdeman). *Equation* (1) *with* $d > 1, P(b) = k$ *and* $7 \leq k \leq 100$ *implies that* $(a_0, a_1, \ldots, a_{k-1})$

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is among the following tuples or their mirror images:

$$\begin{aligned} k=7: &\quad (2,3,1,5,6,7,2),(3,1,5,6,7,2,1),(1,5,6,7,2,1,10),\\ k=13: &\quad (3,1,5,6,7,2,1,10,11,3,13,14,15),\\ &\quad (1,5,6,7,2,1,10,11,3,13,14,15,1),\\ k=19: &\quad (1,5,6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22),\\ k=23: &\quad (5,6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22,23,6,1,26,3),\\ &\quad (6,7,2,1,10,11,3,13,14,15,1,17,2,19,5,21,22,23,6,1,26,3,7). \end{aligned}$$

For $k=5$ Mukhopadhyay and Shorey [9] proved the following result.

THEOREM B (Mukhopadhyay, Shorey). *If n and d are coprime nonzero integers, then the Diophantine equation*

$$n(n+d)(n+2d)(n+3d)(n+4d)=by^2$$

has no solutions in nonzero integers b, y and $P(b)\le 3$.

In this article we solve (1) with $k=5$ and $P(b)=5$, and we handle the eight special cases mentioned in Theorem A. We prove the following theorems.

THEOREM 1. *Equation* (1) *with $d>1$, $P(b)=k$ and $7\le k\le 100$ has no solutions.*

THEOREM 2. *Equation* (1) *with $d>1$, $k=5$ and $P(b)=5$ implies that* $(n,d)\in\{(-12,7),(-4,3)\}$.

2. Preliminary lemmas. In the proofs of Theorems 2 and 1 we need several results using the elliptic Chabauty method (see [4], [5]). Bruin's routines related to the elliptic Chabauty method are contained in Magma [2]. Here we only indicate the main steps without explaining the background theory. To see how the method works in practice, in particular with the help of Magma, [3] is an excellent source. For the method to work, the rank of the elliptic curve (defined over the number field K) should be strictly less than the degree of K. In the present cases it turns out that the ranks of the elliptic curves are either 0 or 1, so the elliptic Chabauty method is applicable. Further, the procedure `PseudoMordellWeilGroup` of Magma is able to find a subgroup of the Mordell–Weil group of finite odd index. We also need to check that the index is not divisible by some prime numbers provided by the procedure `Chabauty`. This last step can be done by the inbuilt function `IsPSaturated`.

LEMMA 1. *Equation* (1) *with $k=7$ and $(a_0,a_1,\ldots,a_6)=(1,5,6,7,2,1,10)$ implies that $n=2$, $d=1$.*

Proof. Using the fact that $n = x_0^2$ and $d = (x_5^2 - x_0^2)/5$ we obtain the following system of equations:

$$x_5^2 + 4x_0^2 = 25x_1^2,$$
$$4x_5^2 + x_0^2 = 10x_4^2,$$
$$6x_5^2 - x_0^2 = 50x_6^2.$$

The second equation implies that x_0 is even, say $x_0 = 2z$ with $z \in \mathbb{Z}$. By standard factorization argument in the Gaussian integers we get

$$(x_5 + 4iz)(x_5 + iz) = \delta\square,$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$. Thus putting $X = x_5/z$ it is sufficient to find all points (X, Y) on the curves

$$C_\delta : \quad \delta(X + i)(X + 4i)(3X^2 - 2) = Y^2, \tag{3}$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$, for which $X \in \mathbb{Q}$ and $Y \in \mathbb{Q}(i)$. Note that if $(X, Y) \in C_\delta$ then $(X, iY) \in C_{-\delta}$. We will use this isomorphism later on to reduce the number of curves to be examined. Hence we need to consider the curve C_δ for $\delta \in \{1 - 3i, 1 + 3i, 3 - i, 3 + i\}$.

I. $\delta = 1 - 3i$. In this case C_{1-3i} is isomorphic to the elliptic curve

$$E_{1-3i} : \quad y^2 = x^3 + ix^2 + (-17i - 23)x + (2291i + 1597).$$

Using Magma we find that the rank of E_{1-3i} is 0 and there is no point on C_{1-3i} for which $X \in \mathbb{Q}$.

II. $\delta = 1+3i$. Here $E_{1+3i} : y^2 = x^3 - ix^2 + (17i - 23)x + (-2291i + 1597)$. The rank of this curve is 0 and there is no point on C_{1+3i} for which $X \in \mathbb{Q}$.

III. $\delta = 3 - i$. Then $E_{3-i} : y^2 = x^3 + x^2 + (-17i + 23)x + (-1597i - 2291)$. We have $E_{3-i}(\mathbb{Q}(i)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$ as an Abelian group. Applying elliptic Chabauty with $p = 13$, we get $x_5/z = -3$. Thus $n = 2$ and $d = 1$.

IV. $\delta = 3 + i$. Then $E_{3+i} : y^2 = x^3 + x^2 + (17i + 23)x + (1597i - 2291)$. The rank of this curve is 1 and applying elliptic Chabauty again with $p = 13$ we obtain $x_5/z = 3$. This implies that $n = 2$ and $d = 1$. ∎

LEMMA 2. *Equation* (1) *with* $k = 7$ *and* $(a_0, a_1, \ldots, a_6) = (2, 3, 1, 5, 6, 7, 2)$ *implies that* $n = 2$, $d = 1$.

Proof. In this case we have the following system of equations:

$$x_4^2 + x_0^2 = 2x_1^2,$$
$$9x_4^2 + x_0^2 = 10x_3^2,$$
$$9x_4^2 - x_0^2 = 2x_6^2.$$

The same argument as in the proof of Theorem 1 shows that it is sufficient to find all points (X, Y) on the curves

$$C_\delta : \quad 2\delta(X + i)(3X + i)(9X^2 - 1) = Y^2, \tag{4}$$

where $\delta \in \{-4 \pm 2i, -2 \pm 4i, 2 \pm 4i, 4 \pm 2i\}$, for which $X \in \mathbb{Q}$ and $Y \in \mathbb{Q}(i)$. We summarize the results obtained by elliptic Chabauty in the following table. In each case we used $p = 29$.

δ	Curve	x_4/x_0
$2-4i$	$y^2 = x^3 + (-12i - 9)x + (-572i - 104)$	$\{-1, \pm 1/3\}$
$2+4i$	$y^2 = x^3 + (12i - 9)x + (-572i + 104)$	$\{1, \pm 1/3\}$
$4-2i$	$y^2 = x^3 + (-12i + 9)x + (-104i - 572)$	$\{\pm 1/3\}$
$4+2i$	$y^2 = x^3 + (12i + 9)x + (-104i + 572)$	$\{\pm 1/3\}$

Thus $x_4/x_0 \in \{\pm 1, \pm 1/3\}$. From $x_4/x_0 = \pm 1$ it follows that $n = 2$, $d = 1$, while $x_4/x_0 = \pm 1/3$ does not yield any solutions. ∎

LEMMA 3. *Equation* (1) *with* $k = 7$ *and* $(a_0, a_1, \ldots, a_6) = (3, 1, 5, 6, 7, 2, 1)$ *implies that* $n = 3$, $d = 1$.

Proof. Here we get the following system of equations:

$$\begin{aligned} 2x_3^2 + 2x_0^2 &= x_1^2, \\ 4x_3^2 + x_0^2 &= 5x_2^2, \\ 12x_3^2 - 3x_0^2 &= x_6^2. \end{aligned}$$

Again it is sufficient to find all points (X, Y) on the curves

$$C_\delta: \quad \delta(X + i)(2X + i)(12X^2 - 3) = Y^2, \tag{5}$$

where $\delta \in \{-3 \pm i, -1 \pm 3i, 1 \pm 3i, 3 \pm i\}$, for which $X \in \mathbb{Q}$ and $Y \in \mathbb{Q}(i)$. We summarize the results obtained by elliptic Chabauty in the following table. In each case we used $p = 13$.

δ	Curve	x_3/x_0
$1-3i$	$y^2 = x^3 + (27i + 36)x + (243i - 351)$	$\{-1, \pm 1/2\}$
$1+3i$	$y^2 = x^3 + (-27i + 36)x + (243i + 351)$	$\{1, \pm 1/2\}$
$3-i$	$y^2 = x^3 + (27i - 36)x + (-351i + 243)$	$\{\pm 1/2\}$
$3+i$	$y^2 = x^3 + (-27i - 36)x + (-351i - 243)$	$\{\pm 1/2\}$

Thus $x_3/x_0 \in \{\pm 1, \pm 1/2\}$. From $x_4/x_0 = \pm 1$ it follows that $n = 3, d = 1$, while $x_3/x_0 = \pm 1/2$ does not yield any solutions. ∎

LEMMA 4. *Equation* (1) *with* $(a_0, a_1, \ldots, a_4) = (-3, -5, 2, 1, 1)$ *and* $k = 5$, $d > 1$ *implies that* $n = -12, d = 7$.

Proof. From (2) we have

$$\begin{aligned} \tfrac{1}{4}x_4^2 - \tfrac{9}{4}x_0^2 &= -5x_1^2, \\ \tfrac{1}{2}x_4^2 - \tfrac{3}{2}x_0^2 &= 2x_2^2, \\ \tfrac{3}{4}x_4^2 - \tfrac{3}{4}x_0^2 &= x_3^2. \end{aligned}$$

Clearly, $\gcd(x_4, x_0) = 1$ or 2. In both cases we get the system

$$\begin{aligned} X_4^2 - 9X_0^2 &= -5\square, \\ X_4^2 - 3X_0^2 &= \square, \\ X_4^2 - X_0^2 &= 3\square, \end{aligned}$$

where $X_4 = x_4/\gcd(x_4, x_0)$ and $X_0 = x_0/\gcd(x_4, x_0)$. The curve in this case is

$$C_\delta: \quad \delta(X + \sqrt{3})(X + 3)(X^2 - 1) = Y^2,$$

where δ is from a finite set. The elliptic Chabauty method applied with $p = 11, 37$ and 59 provides all points for which the first coordinate is rational. These coordinates are $\{-3, -2, -1, 1, 2\}$. We obtain the arithmetic progression with $(n, d) = (-12, 7)$. ∎

Lemma 5. *Equation* (1) *with* $(a_0, a_1, \ldots, a_4) = (2, 5, 2, -1, -1)$ *and* $k = 5$, $d > 1$ *implies that* $n = -4$, $d = 3$.

Proof. We use x_3 and x_2 to get a system of equations as in the previous lemmas. The elliptic Chabauty method applied with $p = 13$ yields $x_3/x_2 = \pm 1$, hence $(n, d) = (-4, 3)$. ∎

Lemma 6. *Equation* (1) *with* $(a_0, a_1, \ldots, a_4) = (6, 5, 1, 3, 2)$ *and* $k = 5$, $d > 1$ *has no solutions.*

Proof. In this case we have

$$\delta(x_3 + \sqrt{-1}\,x_0)(x_3 + 2\sqrt{-1}\,x_0)(2x_3^2 - x_0^2) = \square,$$

where $\delta \in \{1 \pm 3\sqrt{-1}, 3 \pm \sqrt{-1}\}$. Chabauty's argument gives $x_3/x_0 = \pm 1$, which corresponds to arithmetic progressions with $d = \pm 1$. ∎

3. Remaining cases of Theorem A. In this section we prove Theorem 1. First note that Lemmas 1, 2 and 3 imply the statement of the theorem for $k = 7, 13$ and 19. The remaining two possibilities can be eliminated in a similar way; we present the argument for the tuple

$$(5, 6, 7, 2, 1, 10, 11, 3, 13, 14, 15, 1, 17, 2, 19, 5, 21, 22, 23, 6, 1, 26, 3).$$

We have the system of equations

$$\begin{aligned} n + d &= 6x_1^2, \\ n + 3d &= 2x_3^2, \\ n + 5d &= 10x_5^2, \\ n + 7d &= 3x_7^2, \\ n + 9d &= 14x_9^2, \\ n + 11d &= x_{11}^2, \\ n + 13d &= 2x_{13}^2. \end{aligned}$$

We find that x_7, x_{11} and $n+d$ are even integers. Dividing all equations by 2 we obtain an arithmetic progression of length 7 and $(a_0, a_1, \dots, a_6) = (3,1,5,6,7,2,1)$. This is not possible by Lemma 3 and Theorem 1 is proved.

4. The case $k=5$. In this section we prove Theorem 2. As 5 divides one of the terms, by symmetry we may assume that $5 \mid n+d$ or $5 \mid n+2d$. First we compute the set of possible tuples $(a_0, a_1, a_2, a_3, a_4)$ for which appropriate congruence conditions hold ($\gcd(a_i, a_j) \in \{1, P(j-i)\}$ for $0 \le i < j \le 4$) and the number of sign changes is at most 1 and the product $a_0a_1a_2a_3a_4$ is positive. Then we eliminate tuples by using elliptic curves of rank 0. We consider elliptic curves

$$(n+\alpha_1 d)(n+\alpha_2 d)(n+\alpha_3 d)(n+\alpha_4 d) = \prod_i a_{\alpha_i} \square,$$

where α_i, $i \in \{1,2,3,4\}$, are distinct integers in $\{0,1,2,3,4\}$. If the rank is 0, then we obtain all possible values of n/d. Since $\gcd(n,d)=1$ we get all possible values of n and d. It turns out that it remains to deal with the following tuples:

$$\begin{aligned}
&(-3,-5,2,1,1),\\
&(-2,-5,3,1,1),\\
&(-1,-15,-1,-2,3),\\
&(2,5,2,-1,-1),\\
&(6,5,1,3,2).
\end{aligned}$$

In the case of $(-3,-5,2,1,1)$ Lemma 4 implies that $(n,d) = (-12,7)$.

If $(a_0, a_1, \dots, a_4) = (-2,-5,3,1,1)$, then $\gcd(n,d)=1$ implies that $\gcd(n,3)=1$. Since $n = -2x_0^2$ we obtain $n \equiv 1 \pmod 3$. From the equation $n+2d = 3x_2^2$ we get $d \equiv 1 \pmod 3$. Finally, the equation $n+4d = x_4^2$ leads to a contradiction.

If $(a_0, a_1, \dots, a_4) = (-1,-15,-1,-2,3)$, then we obtain $\gcd(n,3)=1$. From the equations $n = -x_0^2$ and $n+d = -15x_1^2$ we get $n \equiv 2 \pmod 3$ and $d \equiv 1 \pmod 3$. Now the contradiction follows from the equation $n+2d = -x_2^2$.

In the case of $(2,5,2,-1,-1)$, Lemma 5 implies that $(n,d) = (-4,3)$. The last tuple is eliminated by Lemma 6.

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