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Relative Bogomolov extensions

by

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1. Introduction. We work within a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ throughout this paper. We write h for the usual absolute logarithmic height on algebraic numbers. If K is a subfield of $\overline{\mathbb{Q}}$, then K satisfies the *Bogomolov property*, (B), if there exists some $\varepsilon > 0$ such that there is no element $\alpha \in K^\times$ such that $0 < h(\alpha) < \varepsilon$. This definition was first stated in [9]. Recall that $h(\alpha) = 0$ if and only if α is a root of unity [8, Theorem 1.5.9]. We introduce the following generalization of (B) to relative extensions.

DEFINITION 1.1. Let $\mathbb{Q} \subseteq K \subseteq L \subseteq \overline{\mathbb{Q}}$ be fields. We say that L/K is *Bogomolov*, or that L/K satisfies the *relative Bogomolov property*, (RB), if there exists $\varepsilon > 0$ such that

$$\{\alpha \in L^\times \mid 0 < h(\alpha) < \varepsilon\} \subseteq K.$$

In other words, L/K satisfies (RB) if and only if there is no sequence $\{\alpha_n\} \subseteq L^\times \setminus K^\times$ with $0 < h(\alpha_n) \to 0$ as $n \to \infty$. The following facts are immediate from the definition.

PROPOSITION 1.2. *Suppose $K \subseteq L \subseteq M$ are subfields of $\overline{\mathbb{Q}}$.*

(a) *If K satisfies* (B) (*in particular if $K/\mathbb{Q}$ is finite*), *then L/K is Bogomolov if and only if L satisfies* (B).
(b) *M/K is Bogomolov if and only if M/L and L/K are both Bogomolov.*
(c) *If $L \setminus K$ contains a root of unity and L/K is Bogomolov, then K satisfies* (B).

Part (c) follows because multiplying an algebraic number by a root of unity does not affect the height. Therefore, if $K^\times$ contains a sequence with positive height tending to zero, then so does $L^\times \setminus K^\times$.

It has already been shown that finite extensions may not satisfy (RB). For instance, it is demonstrated in [2, Example 5.3] that $\mathbb{Q}^{\mathrm{tr}}(i)/\mathbb{Q}^{\mathrm{tr}}$ is not

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Bogomolov. Here $\mathbb{Q}^{\mathrm{tr}}$ denotes the maximal totally real extension of $\mathbb{Q}$, which satisfies (B) (see [18]). Interestingly, Pottmeyer [17] has recently stated a bound that implies that every finite extension of $\mathbb{Q}^{\mathrm{tr}}(i)$ (the so-called "maximal CM field") satisfies (RB), using an archimedean estimate of Garza [12].

One of the main aims of the present paper is simply to construct examples of extensions L/K which satisfy (RB) even though K does not satisfy (B). Example 4.2 is one where L/K is infinite—this construction uses our results from Section 3. Example 4.1 shows a finite extension L/K which does not satisfy (RB). This example is quite elementary and does not rely on other results in this paper.

It is natural to ask what conditions can be placed on a field K of algebraic numbers to ensure that there exists at least one relative Bogomolov extension L/K. In this respect we prove the following, our main result.

THEOREM 1.3. *Let $K/\mathbb{Q}$ be an algebraic extension. Assume there exists a (finite) rational prime ℓ and a number field $F \subseteq K$ such that no prime of $\mathcal{O}_F$ lying over ℓ is ramified in K/F (in particular this holds if $K/\mathbb{Q}$ is Galois and some prime ℓ has finite ramification index in K). Then there exist relative Bogomolov extensions L/K. These extensions can be constructed explicitly of the form $K(\sqrt[\ell]{\alpha})$ for appropriately chosen elements $\alpha \in K$.*

This theorem should be compared with [9, Theorem 2], which states that a Galois extension of the rationals with a bounded *local degree* (ramification index times inertial degree) has the Bogomolov property.

We briefly describe what is known on fields with the Bogomolov property in order to put our results in context. Schinzel [18] showed in 1973 that there is a positive lower bound on the height of totally real numbers outside of $\{\pm 1\}$, establishing (B) for the maximal totally real field $\mathbb{Q}^{\mathrm{tr}}$. This can be described as an "archimedean" height estimate, and was generalized by Garza to a lower bound on the height of algebraic numbers with at least one real conjugate [12]. Another common approach that has been used (for example for the archimedean part of the argument in [13]) for archimedean estimates is equidistribution, starting with Bilu's theorem [7], but these techniques will not be used in the current paper in favor of the Schinzel–Garza inequality.

One non-archimedean strategy originates in Amoroso and Dvornicich's paper [3], where it is shown that (B) is enjoyed by the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ of $\mathbb{Q}$, which was generalized to relative extensions and strengthened considerably in [4] and [5]. Their strategy involves estimating how close a certain automorphism in a Galois group is to the action of raising an element to a power, with respect to a place lying over some auxilliary prime. This strategy is quite powerful and is also used in [13], the elliptic curve analogue of [3] ([1]), and in [2], where it is summarized nicely by their Lemma 2.2. The main theorem (Theorem 1.2) of the latter paper generalizes both the results on abelian extensions and [9, Theorem 2], which states that (B) is satisfied by a field having bounded local degrees above some rational prime.

In our present efforts to prove that a relative extension L/K is Bogomolov when K may not satisfy (B), it is not clear that the Amoroso–Dvornicich technique can be used to produce any new results. Instead we appeal to more classical bounds in terms of ramification. Our main tool is the lower bound [19, Theorem 2], due to Silverman. This bound is written in notation more similar to ours in [20, Section 3], where the author uses it effectively to give examples of fields satisfying the closely related *Northcott property*, (N). This stronger property, first defined along with (B) in [9], is satisfied by a field K if for any T at most finitely many points in K have height at most T. Silverman's inequality generalizes to the relative case a type of bound going back to Theorem 1 of Mahler [15], which is exactly the lower bound used in [9, Theorem 2], where as mentioned before the authors exploit the existence of a bound on local degrees above some finite rational prime. Our Theorem 1.3 has the related hypothesis of finite ramification above a prime—for this theorem we also require an archimedean estimate coming from the above-stated theorem of Garza.

The rest of this paper is organized as follows. In Section 2 we introduce notation and prove a criterion, Theorem 2.4, for when we can use ramification information to conclude that a finite relative extension L/K is Bogomolov. In Section 3 we describe how to apply these techniques to bound below the heights of elements properly contained in an extension of the form $K(\sqrt[\ell]{\alpha})$, using Hecke's classical theory of ramification in Kummer extensions. We combine this with the archimedean Schinzel–Garza inequality to prove Theorem 1.3. Finally, in Section 4 we construct the aforementioned explicit examples.

We conclude the introduction by mentioning a few questions for further investigation. As mentioned above, if $L \setminus K$ contains a root of unity ζ and if K does not satisfy (B), then $L \setminus K$ contains elements of arbitrarly small positive height of the form $\zeta\alpha$, with $\alpha \in K$. If one could construct such an extension where the *only* elements of small height in $L \setminus K$ were obtained by multiplying elements of K by roots of unity, this would suggest a weaker version of (RB) that could be explored. Pottmeyer has shown that all finite extensions of the maximal CM field are Bogomolov. In this same spirit,

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([1]) The theorem from [3] is a result about heights on $\mathbb{G}_m(\mathbb{Q}^{ab}) = \mathbb{G}_m(\mathbb{Q}(\mathbb{G}_{m,tors}))$. Theorem 1 of [13] replaces the *inner* $\mathbb{G}_m$ with an elliptic curve. Another well-known analogue of [3] is the main result of [6], which is the analogous result for $A(\mathbb{Q}^{ab})$, where A is an abelian variety.

it would be interesting to exhibit fields $K \subsetneq \overline{\mathbb{Q}}$ admitting *no* Bogomolov extensions. One easy example of this can be found if K is the subfield of $\overline{\mathbb{Q}}$ fixed by complex conjugation, i.e. $\overline{\mathbb{Q}} \cap \mathbb{R}$ (if we first embed $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$), but one might expect this to happen for other fields K that are sufficiently "big", for example pseudo-algebraically closed (a "PAC field", a field K such that every geometrically irreducible variety over K has a K-rational point—see [11, Chapter 11] for more; see [2, Section 6] for speculations on PAC fields and property (B)). If this occurs for a field K satisfying (B), this field would be maximal with respect to the Bogomolov property.

Note added in revision. Considering the question in the previous paragraph of whether there exists an extension L/K where $L \setminus K$ contains elements of arbitrarily small height, but only elements which are elements of K multiplied by roots of unity: during revision the author was made aware of the following example of an extension with this property, which comes from a recent article of Amoroso. Let $K = \mathbb{Q}(2^{1/3}, 2^{1/9}, 2^{1/27}, \dots)$, choosing the real roots, say, and let $L = K(\zeta_3, \zeta_9, \zeta_{27}, \dots)$, where ζ_n denotes a primitive nth root of unity. According to [1, Theorem 1.3], the only elements α of L with height less than $\log(3/2)/18$ are those where α^n is a power of 2 for some integer n, and all such elements are roots of unity times elements of K.

2. Lower bounds and a ramification criterion for (RB). First we establish some notation. For a finite extension of number fields M/F, we write $D_{M/F}$ for the relative discriminant, $\mathcal{D}_{M/F}$ for the relative different, and $N_{M/F}$ for the relative ideal norm. Recall that $D_{M/F}$ is an ideal in the ring of integers $\mathcal{O}_F$, while $\mathcal{D}_{M/F}$ is an ideal of $\mathcal{O}_M$, and we have

$$D_{M/F} = N_{M/F}(\mathcal{D}_{M/F}).$$

(This is often taken as the definition of the discriminant.) It is an elementary result [16, Theorem 4.16] that $\mathcal{D}_{M/F}$ is generated by the set of differents $\delta_{M/F}(\beta)$ of integral generators β of the extension M/F. Here if $\beta \in \mathcal{O}_M$ and $M = F(\beta)$, the different $\delta_{M/F}(\beta)$ is defined to be $f'(\beta)$, where f is the minimal polynomial for β over F. For a tower of number fields $M'/M/F$ we will make use of the well-known identity

$$D_{M'/F} = D_{M/F}^{[M':M]} N_{M/F}(D_{M'/M}) \quad \text{[16, Proposition 4.15].}$$

A prime $\mathfrak{p}$ of F will mean a prime ideal in the ring of integers $\mathcal{O}_F$, with corresponding non-archimedean valuation $v_{\mathfrak{p}}$. If π is a uniformizing parameter for the associated place v, and if $\mathfrak{p}$ divides the rational prime ℓ, we normalize the absolute value $|\cdot|_v$ so that $|\pi|_v^{[F:\mathbb{Q}]} = \ell^f$, where f is the associated residue class degree.

The *absolute logarithmic height* of an algebraic number α is given by

$$h(\alpha) = \sum_v \log^+ |\alpha|_v,$$

the sum being taken over the places of any number field containing α. We define the *multiplicative height* $H(\alpha) = \exp h(\alpha)$. We will often use basic facts about the height such as [8, Lemma 1.5.18 and Proposition 1.5.15] without specific reference.

Let F be a number field of degree d over $\mathbb{Q}$, and let K/F be an algebraic extension. We define

$$\rho(K/F) = \limsup\{\delta(M)/[M : F] \mid F \subseteq M \subseteq K,\ [M : F] < \infty\},$$

where $\delta(M)$ denotes the number of archimedean places of M. In this context the limit superior is taken over the directed set of finite subextensions of K/F. In other words, $\rho(K/F)$ is the least real number ρ such that for any finite extension M/F contained in K, there is a finite extension M'/M with $M' \subseteq K$ such that $\delta(M')/[M' : F] \leq \rho$. Note that $d/2 \leq \rho(K/F) \leq d$, and that $\rho(L/F) \leq \rho(K/F)$ for any tower $L/K/F$. Of course if K/F is finite, then $\rho(K/F) = \delta(K)/[K : F]$.

We will apply the following inequality of Silverman [19, Theorem 2] (cf. [20, Section 3]) to produce a ramification criterion for (RB).

THEOREM 2.1 (Silverman). *If γ generates a relative extension of number fields B/M, where $[B : M] = s$ and $[M : \mathbb{Q}] = m$, then*

$$H(\gamma) \geq s^{-\frac{\delta(M)}{2m(s-1)}} N_{M/\mathbb{Q}}(D_{B/M})^{\frac{1}{2ms(s-1)}}.$$

This is a relative field discriminant version of a bound of Mahler [15, Theorem 10]. Widmer exploited the dependence only on relative ramification in this bound to produce a ramification criterion for the Northcott property [20]. The following proposition illustrates our use of Silverman's inequality.

PROPOSITION 2.2. *Let $M/F/\mathbb{Q}$ be a tower of finite extensions, and let $d = [F : \mathbb{Q}]$ and $e = [M : F]$. Assume α generates an extension F'/F and that F' and M are linearly disjoint over F. Let $M' = M(\alpha)$. Suppose $\gamma \in M'^{\times} \setminus M^{\times}$. Let $B = M(\gamma)$, $C = B \cap F'$, and $s = [B : M] = [C : F]$. We have*

$$H(\gamma) \geq s^{-\frac{\rho(M/F)}{2d(s-1)}} N_{F/\mathbb{Q}}\left(\frac{D^e_{C/F}}{\gcd(D^e_{C/F}, D^s_{M/F})}\right)^{\frac{1}{2des(s-1)}}. \tag{2.1}$$

In particular, if no prime ramifying in F'/F is ramified in M/F, then

$$H(\gamma) \geq s^{-\frac{\rho(M/F)}{2d(s-1)}} N_{F/\mathbb{Q}}(D_{C/F})^{\frac{1}{2ds(s-1)}}. \tag{2.2}$$

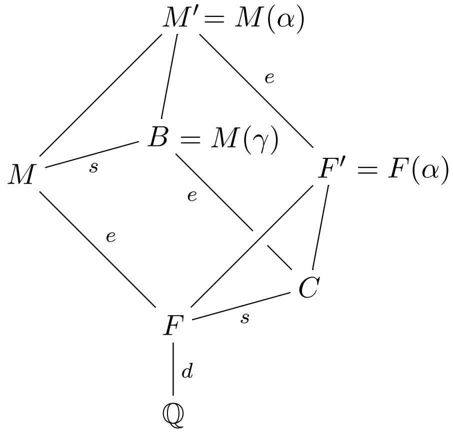

Fig. 2.3. Diagram of the fields described in Proposition 2.2

Proof. We apply Silverman's inequality to the extension B/M. Since M/F is finite, we have $\delta(M)/e = \rho(M/F)$, and thus we obtain

$$H(\gamma) \geq s^{-\frac{\rho(M/F)}{2d(s-1)}} N_{M/\mathbb{Q}}\left(D_{B/M}\right)^{\frac{1}{2des(s-1)}}. \tag{2.3}$$

Using basic properties of relative norms and discriminants, we get

$$N_{M/\mathbb{Q}}(D_{B/M}) = N_{F/\mathbb{Q}}(N_{M/F}(D_{B/M})) = N_{F/\mathbb{Q}}\left(\frac{D_{B/F}}{D_{M/F}^s}\right).$$

Since $D_{B/F}$ is divisible by both $D_{C/F}^e$ and $D_{M/F}^s$, we now have

$$\begin{aligned} N_{M/\mathbb{Q}}(D_{B/M}) &\geq N_{F/\mathbb{Q}}\left(\frac{\operatorname{lcm}(D_{C/F}^e, D_{M/F}^s)}{D_{M/F}^s}\right) \\ &= N_{F/\mathbb{Q}}\left(\frac{D_{C/F}^e}{\gcd(D_{C/F}^e, D_{M/F}^s)}\right). \end{aligned}$$

Combining this inequality with (2.3) completes the proof of (2.1). Inequality (2.2) follows immediately. ∎

Now we move from the case of a finite extension M/F to that of an infinite extension K/F, which leads to a criterion for a finite relative extension to satisfy (RB).

THEOREM 2.4. *Let $K/\mathbb{Q}$ be an infinite algebraic extension, and let $L = K(\alpha)$ be a finite extension of K. Let $f(x)$ denote the minimal polynomial for α over K. Let F be a number field such that $F \subseteq K$ and $[F(\alpha) : F] = [L : K]$ (2). Let $d = [F : \mathbb{Q}]$, $\rho = \rho(K/F)$, and $F' = F(\alpha)$. Assume that F' and K are linearly disjoint over F, and that no prime ramifying in F'/F is*

(2) This is satisfied, for example, if F contains the coefficients of $f(x)$.

ramified in K/F. If $\gamma \in L^\times \setminus K^\times$, then

$$H(\gamma) \geq \min\{(N_{F/\mathbb{Q}}(D_{C/F})s^{-\rho s})^{\frac{1}{2ds(s-1)}} \mid F \subsetneq C \subset F',\ s = s(C) = [C : F]\}. \tag{2.4}$$

In particular, if for each field C with $F \subsetneq C \subseteq F'$ we have

$$N_{F/\mathbb{Q}}\left(D_{C/F}\right) > s^{\rho s}, \tag{2.5}$$

where $s = [C : F]$ and $\rho = \rho(K/F)$, then L/K is Bogomolov.

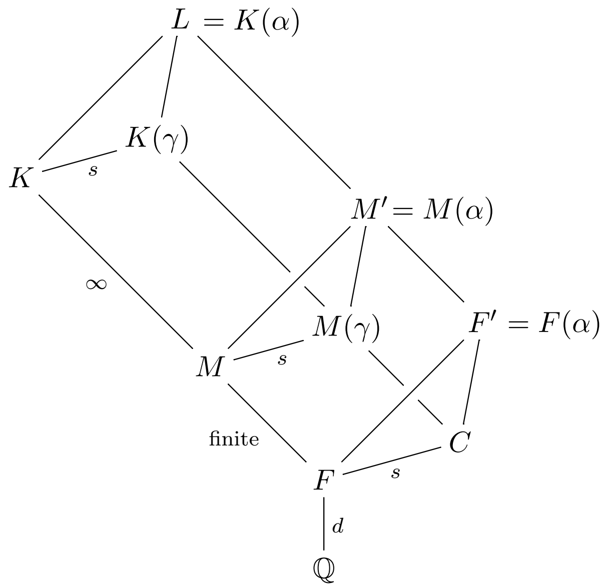

Fig. 2.5. Diagram of the fields involved in the proof of Theorem 2.4

Proof. Let M/F be a finite extension with $M \subseteq K$ and $[M(\gamma) : M] = [K(\gamma) : K]$; for example, we may create M by adjoining to F the coefficients of the minimal polynomial for γ over K. Notice that, since no prime ramifying in F'/F is ramified in K/F, none of these primes is ramified in M/F either. Let $C = M(\gamma) \cap F'$. Then Proposition 2.2 gives

$$H(\gamma) \geq s^{-\frac{\rho(M/F)}{2d(s-1)}} N_{F/\mathbb{Q}}(D_{C/F})^{\frac{1}{2ds(s-1)}},$$

and since $\rho(M/F) \leq \rho$, inequality (2.4) follows. Moreover, if inequality (2.5) is satisfied for all fields C with $F \subsetneq C \subseteq F'$, then the lower bound in (2.4) is greater than 1 and depends only on K and L, and therefore L/K is Bogomolov ∎

Remark 2.6. Notice that the lower bound in Theorem 2.4 depends on the choice of a primitive element α. In fact α could be replaced by any collection of elements which generate the finite extension L/K.

3. Adjoining ℓth roots and the proof of Theorem 1.3. Extensions formed by adjoining an ℓth root of an element, where ℓ is a prime, are an easy source of examples in which we can successfully apply the bounds of the previous section. An extension of prime degree has no intermediate extensions, so application of Theorem 2.4 becomes much cleaner. Furthermore, the discriminants of such extensions when the base field contains a primitive ℓth root of unity (Kummer extensions) are completely understood thanks to classical work of Hecke (see [14, §39]; cf. [10, Section 10.2.3]). We now illustrate how we can exploit this theory. We begin with the following very general elementary lemma, which we will need to apply Hecke's theory to extensions where the base field may not contain an ℓth root of unity. For understanding the statement and proof of this lemma we refer the reader to the beginning of the previous section for relevant notation and definitions.

LEMMA 3.1. *Let F_1 and F_2 be finite extensions of a number field F, and assume that $m = [F_1 : F]$ and $n = [F_2 : F]$ are relatively prime. Let F_3 denote the compositum F_1F_2. If $\mathfrak{a}$ is an ideal of $\mathcal{O}_F$ such that*

$$\mathfrak{a}\mathcal{O}_{F_2} \mid D_{F_3/F_2}, \tag{3.1}$$

then $\mathfrak{a} \mid D_{F_1/F}$.

Proof. Since m and n are relatively prime, we know that $F_1 \cap F_2 = F$. Any element $\beta \in \mathcal{O}_{F_1}$ which generates the extension F_1/F, i.e. such that $F_1 = F(\beta)$, is also a primitive generator of the extension F_3/F_2, and has the same minimal polynomial over F_2 as over F. Since $\mathcal{D}_{F_1/F}$ is generated as an ideal of $\mathcal{O}_{F_1}$ by the differents of all such integral generators, and $\mathcal{D}_{F_3/F_2}$ is likewise generated by the differents of the integral elements which generate the field extension F_3/F_2, we immediately conclude that $\mathcal{D}_{F_1/F}\mathcal{O}_{F_3} \subseteq \mathcal{D}_{F_3/F_2}$, or equivalently

$$\mathcal{D}_{F_3/F_2} \mid \mathcal{D}_{F_1/F}\mathcal{O}_{F_3}. \tag{3.2}$$

Notice that

$$N_{F_3/F}(\mathcal{D}_{F_3/F_2}) = N_{F_2/F}(N_{F_3/F_2}(\mathcal{D}_{F_3/F_2})) = N_{F_2/F}(D_{F_3/F_2}),$$

while

$$\begin{aligned} N_{F_3/F}(\mathcal{D}_{F_1/F}\mathcal{O}_{F_3}) &= N_{F_1/F}(N_{F_3/F_1}(\mathcal{D}_{F_1/F}\mathcal{O}_{F_3})) \\ &= N_{F_1/F}(\mathcal{D}_{F_1/F}^{[F_3:F_1]}) = D_{F_1/F}^n. \end{aligned}$$

Therefore taking the norm $N_{F_3/F}$ of each side of (3.2) preserves this divisibility condition and yields

$$N_{F_2/F}(D_{F_3/F_2}) \mid D_{F_1/F}^n. \tag{3.3}$$

Now we take the norm $N_{F_2/F}$ of both sides of (3.1) and apply (3.3) to achieve

$$\mathfrak{a}^n \mid N_{F_2/F}(D_{F_3/F_2}) \mid D^n_{F_1/F};$$

the lemma follows. ■

Our next lemma illustrates how we can apply Hecke's theory and will be used in our proof of Theorem 1.3.

LEMMA 3.2. *Let F be a number field, ℓ a rational prime, and let π be an element of $\mathcal{O}_F$ such that $v_{\mathfrak{p}}(\pi) = 1$ for all primes $\mathfrak{p}$ of F lying over ℓ. Let $\pi^{1/\ell}$ denote a root of the polynomial $f(x) = x^\ell - \pi$, and set $F' = F(\pi^{1/\ell})$. Then*

$$\ell^\ell \mathcal{O}_F \mid D_{F'/F}.$$

Proof. The polynomial $f(x) \in \mathcal{O}[x]$ is irreducible, since by construction it is Eisenstein with respect to any prime $\mathfrak{p}$ lying over ℓ. Let $F_2 = F(\zeta_\ell)$, where ζ_ℓ is a primitive ℓth root of unity, and let $F_3 = F_2F'$. Let $\mathfrak{p}$ be any prime of F lying over ℓ, and note that $\mathfrak{p}$ is totally ramified in F_2/F. We let $\mathfrak{P}$ denote the prime of F_2 lying over $\mathfrak{p}$, so we have $\mathfrak{p}\mathcal{O}_{F_2} = \mathfrak{P}^n$, where $n = [F_2 : F]$. Note that $n \mid (\ell - 1)$, so n is relatively prime to $\ell = [F' : F]$.

We wish to apply Hecke's theorem, as stated in [10, Theorem 10.2.9], to the extension $F_3 = F_2(\pi^{1/\ell})$ and the prime $\mathfrak{P}$. Let $e(\mathfrak{p}|\ell)$ and $e(\mathfrak{P}|\ell)$ denote respectively the absolute ramification indices of $\mathfrak{p}$ and $\mathfrak{P}$, so that $e(\mathfrak{P}|\ell) = ne(\mathfrak{p}|\ell)$. Since $v_{\mathfrak{P}}(\pi) = n$ is not divisible by ℓ, we are in case (1) of the mentioned theorem, and we have

$$v_{\mathfrak{P}}(D_{F_3/F_2}) = \ell - 1 + \ell e(\mathfrak{P}|\ell) = \ell - 1 + \ell n e(\mathfrak{p}|\ell) \geq \ell n e(\mathfrak{p}|\ell).$$

This means that $\mathfrak{P}^{\ell n e(\mathfrak{p}|\ell)} = \mathfrak{p}^{\ell e(\mathfrak{p}|\ell)}\mathcal{O}_{F_2}$ divides D_{F_3/F_2}. By Lemma 3.1 (with $F_1 = F'$) we then have $\mathfrak{p}^{\ell e(\mathfrak{p}|\ell)} \mid D_{F'/F}$. Combining this for each prime $\mathfrak{p}$ lying over ℓ, we get

$$D_{F'/F} \subseteq \prod_{\mathfrak{p}|\ell} \mathfrak{p}^{\ell e(\mathfrak{p}|\ell)} = \ell^\ell \mathcal{O}_F. \ ■$$

Proof of Theorem 1.3. Let $K/\mathbb{Q}$ be an algebraic extension, and let ℓ be a rational prime. Let F be a number field such that $F \subseteq K$ and no primes of F lying over ℓ are ramified in K/F, and set $d = [F : \mathbb{Q}]$. Let π be an element of $\mathcal{O}_F$ such that $v_{\mathfrak{p}}(\pi) = 1$ for each prime $\mathfrak{p}$ of F lying over ℓ. Such an element exists by the Chinese Remainder Theorem (see for example [16, Corollary 2 of Proposition 1.14]). We let $F' = F(\pi^{1/\ell})$ for some choice of the root. We note as in the proof of Lemma 3.2 that $[F' : F] = \ell$, since the polynomial $x^\ell - \pi$ is irreducible over F. Lemma 3.2 tells us that

$$\ell^\ell \mathcal{O}_F \mid D_{F'/F},$$

and therefore

$$N_{F/\mathbb{Q}}(D_{F'/F}) \geq \ell^{\ell d}.$$

Let $L = K(\pi^{1/\ell})$. We want to show that L/K is Bogomolov. If $\mathfrak{p}$ is a prime of F lying over ℓ, we know $\mathfrak{p}$ is unramified in K/F and totally ramified in F'/F, so K and F' are linearly disjoint over F.

First suppose that $\rho(K/\mathbb{Q}) < 1$, so that $\rho := \rho(K/F) < d$. Our construction now gives

$$N_{F/\mathbb{Q}}(D_{F'/F}) \geq \ell^{d\ell} > \ell^{\rho\ell}, \tag{3.4}$$

and therefore L/K is Bogomolov by Theorem 2.4. (Note that there are no intermediate fields between F and F', since $[F' : F] = \ell$.) More specifically, let $\gamma \in L \setminus K$, so that $\ell = [K(\gamma) : K]$. Then, combining (2.4) and (3.4) we obtain

$$\begin{aligned} H(\gamma) &\geq \{N_{F/\mathbb{Q}}(D_{F'/F})\ell^{-\rho\cdot\ell}\}^{\frac{1}{2d\ell(\ell-1)}} \\ &\geq \{\ell^{d\ell}\ell^{-\rho\cdot\ell}\}^{\frac{1}{2d\ell(\ell-1)}} \geq \ell^{\frac{d-\rho}{2d(\ell-1)}} > 1, \end{aligned} \tag{3.5}$$

and in this case we are done using only our ramification criterion.

If $\rho = d$, we will have to use the following archimedean estimate of Garza.

Theorem 3.3 (Garza [12]). *Let K be a number field of degree d over $\mathbb{Q}$ with r real places and r' complex places. If $K = \mathbb{Q}(\gamma)$, then*

$$H(\gamma) \geq \left(2^{-d/r} + \sqrt{1 + 4^{-d/r}}\right)^{\frac{r}{2d}}. \tag{3.6}$$

Now we fix an arbitrary real number $\theta \in \left(\frac{2\ell-1}{2\ell}, 1\right)$. If $\rho(M/\mathbb{Q}) \leq \theta$, then as in (3.5) we have

$$H(\gamma) > \ell^{\frac{1-\theta}{2(\ell-1)}} > 1. \tag{3.7}$$

On the other hand, if $\rho(M/\mathbb{Q}) > \theta$, let r and s denote the number of real and complex archimedean places of M, respectively. Notice that $M(\gamma) = M(\sqrt[\ell]{\alpha})$ has r real places and $r(\ell - 1)/2 + s\ell$ complex places. This means that

$$\begin{aligned} \rho(\mathbb{Q}(\gamma)/\mathbb{Q}) \geq \rho(M(\gamma)/\mathbb{Q}) &= \frac{r + r(\ell-1)/2 + s\ell}{\ell[M : \mathbb{Q}]} \\ &= \rho(M/\mathbb{Q}) - \frac{r_{\mathrm{m}}}{[M : \mathbb{Q}]}\left(\frac{\ell-1}{2\ell}\right) > \theta - \frac{\ell-1}{2\ell}. \end{aligned}$$

If $\mathbb{Q}(\gamma)$ has r' real places and s' complex places, then

$$\frac{r'}{d'} = 2\rho(\mathbb{Q}(\gamma)/\mathbb{Q}) - 1 > 2\theta - 1 - \frac{\ell-1}{\ell} > 0$$

by our choice of $\theta > \frac{2\ell-1}{2\ell}$.

Now we may bound below the height of γ by an absolute constant using (3.6). Explicitly, writing $\phi = 2\theta - 1 - (\ell - 1)/\ell$, we have

$$H(\gamma) \geq \big(2^{-1/\phi} + \sqrt{1 + 4^{-1/\phi}}\big)^{\phi/2} > 1. \tag{3.8}$$

Either (3.7) or (3.8) must hold. Taking the minimum of these bounds, we see that $H(\gamma)$ is bounded below by a constant greater than 1 which depends only on ℓ, and our proof of Theorem 1.3 is complete. ∎

4. Examples. After establishing the following two examples, it is clear that, even if K does not satisfy (B), an extension L/K may or may not satisfy (RB), independent of whether L/K is finite or infinite.

Example 4.1 (L/K not Bogomolov). Let b be a non-square rational number and let $K = \mathbb{Q}(b^{1/2}, b^{1/4}, b^{1/8}, \dots)$, for any choices of the roots. Notice that $b^{1/3} \notin K$, and let $L = K(b^{1/3})$. The extension L/K is not Bogomolov. To see this, consider the elements $b^{1/3}b^x \in L \setminus K$, where x is a rational number close to $-1/3$ with denominator a power of 2. Notice that $h(b^{1/3}b^x) = h(b^{x+1/3}) = (x + 1/3)h(b) \to 0$ as $x \to -1/3$. Many similar examples can be constructed easily, including of course infintie relative extensions.

Example 4.2 (L/K Bogomolov). Let $K = \mathbb{Q}(3^{1/3}, 3^{1/9}, 3^{1/27}, \dots)$, and note that 3 is the only rational prime that ramifies in K. Let $3 < p_1 < p_2 < \cdots$ be an infinite sequence of primes congruent to 3 (mod 4). Set $K_0 = K$, and for each $n \geq 1$ set $K_n = K_{n-1}(\sqrt{p_n})$. For a given $n \geq 1$, we wish to apply Proposition 2.2 to estimate the height of an element $\gamma \in K_n^\times \setminus K_{n-1}$. To match the notation of Proposition 2.2 we set $F = \mathbb{Q}$ and choose $M \subseteq K_{n-1}$ to be a number field containing $\sqrt{p_1}, \dots, \sqrt{p_{n-1}}$ and the coefficients for the minimal polynomial of γ over K_n. We use $C = F' = \mathbb{Q}(\sqrt{p_n})$. Note that in this case $N_{F/\mathbb{Q}}(D_{C/F})$ is simply the discriminant of the quadratic field, which in this case is p_n. We have the trivial estimate $\rho(M/F) \leq d$, so applying (2.2), we get

$$H(\gamma) \geq 2^{-1/2} p_n^{1/4} = \left(\frac{p_n}{4}\right)^{1/4} \geq \left(\frac{p_1}{4}\right)^{1/4}.$$

Letting $L = \bigcup_n K_n$, we now see that L/K is an infinite Bogomolov extension, and in fact L can be constructed so that the lower bound on the height of an element of $L^\times \setminus K^\times$ is arbitrarily large.

If the roots $3^{1/3^i}$ are chosen in a compatible way (e.g. if we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and impose that the roots are all real), then K has the property that all of its proper subfields are finite extensions of $\mathbb{Q}$. (The interested reader will verify that the only subfields of $\mathbb{Q}(3^{1/3^n})$ are $\mathbb{Q}(3^{1/3^i})$,

$0 \leq i \leq n$.) Therefore, not only does K fail to satisfy (B), but it is not a Bogomolov extension of any subfield.

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