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Equidistribution and the heights of totally real and totally p-adic numbers

by

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1. Introduction and results. Recall that an algebraic number is said to be *totally real* if all of its Galois conjugates lie in the field $\mathbb{R}$ under any choice of embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. For example, if ζ is a root of unity, then $\alpha = \zeta + \zeta^{-1}$ is a totally real number. Schinzel [20] proved the following (1):

Theorem (Schinzel, 1973). *Let $\alpha \in \mathbb{Q}^{\mathrm{tr}}$, $\alpha \neq 0, \pm 1$, be a totally real number. Then*

$$h(\alpha) \geq h\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1}{2}\log\frac{1+\sqrt{5}}{2} = 0.2406059\ldots$$

where h denotes the absolute logarithmic Weil height.

A natural generalization of the concept of a totally real number is that of a *totally p-adic number*, that is, an algebraic number α all of whose Galois conjugates lie in $\mathbb{Q}_p$ for any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, where $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$. Equivalently, we can ask that the minimal polynomial of α split over $\mathbb{Q}_p$. Note that unlike $\mathbb{C}/\mathbb{R}$, the extension $\mathbb{C}_p/\mathbb{Q}_p$ is of infinite degree. Bombieri and Zannier [7] proved the following analogue of the results of Smyth and Flammang:

Theorem (Bombieri and Zannier, 2001). *Let $L/\mathbb{Q}$ be a normal extension (possibly of infinite degree) and S be the set of finite rational primes such that $L_p/\mathbb{Q}_p$ is a finite normal extension. Then*

$$\liminf_{\alpha\in L} h(\alpha) \geq \frac{1}{2}\sum_{p\in S}\frac{\log p}{e_p(p^{f_p}+1)}$$

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(1) In fact Schinzel's result was originally stated only for integers, however, it is easy to see that it generalizes to totally real nonintegers as well; cf. e.g. [15, 16].

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where e_p and f_p denote the ramification and inertial degrees of $L_p/\mathbb{Q}_p$, respectively.

This result was improved and extended by work of the first author and Petsche [14], who not only improved the constants involved but managed to simultaneously handle totally real splitting of infinite primes using potential-theoretic techniques. In the simplified content of the above theorem, under the assumption that our numbers are totally real if $\infty \in S$, they proved that

$$\liminf_{\alpha\in L} h(\alpha) \geq \begin{cases} \dfrac{1}{2}\displaystyle\sum_{p\in S} \dfrac{p^{f_p}\log p}{e_p(p^{2f_p}-1)} & \text{if } \infty \notin S, \\ \dfrac{1}{2}\displaystyle\sum_{\substack{p\in S\\ p\nmid\infty}} \dfrac{p^{f_p}\log p}{e_p(p^{2f_p}-1)} + \dfrac{7\zeta(3)}{4\pi^2} & \text{if } \infty \in S. \end{cases}$$

Bombieri and Zannier, in analogy to the work of Ullmo and Zhang on the Bogomolov conjecture (see [24, 25, 23]), termed this property, that there is a lower bound away from zero for the height of all nonzero numbers which are not roots of unity in the field, the *Bogomolov property*. Let us make the following more precise definitions. Let K be a number field and $\varphi \in K(z)$ a rational map of degree at least 2, and denote by h_φ its associated *canonical height* as defined in [8]. We can then generalize the notion of Bombieri and Zannier and ask which extensions F/K satisfy an analogue of the Bogomolov property with respect to the height h_φ. Specifically:

DEFINITION 1. We say that a field F/K satisfies the *strong Bogomolov property with respect to h_φ* if

$$\liminf_{\alpha\in\mathbb{P}^1(\bar{K})} h_\varphi(\alpha) > 0.$$

We say F/K satisfies the *weak Bogomolov property with respect to h_φ* if

$$\liminf_{\substack{\alpha\in\mathbb{P}^1(\bar{K})\\ h_\varphi(\alpha)>0}} h_\varphi(\alpha) > 0.$$

We note that there are well-known examples of fields which satisfy the weak Bogomolov property, but not the strong; specifically, Amoroso and Dvornicich [1] showed that the maximal abelian extension $\mathbb{Q}^{\mathrm{ab}}$ satisfies the weak Bogomolov property, while it does not satisfy the strong Bogomolov property as it contains all of the roots of unity.

The strong Bogomolov property, unlike the weak, seems to be intimately connected with equidistribution of small points. This is seen as well in the theorem of Zhang on points on an abelian variety defined over the maximal totally real extension [25]: if A is an abelian variety defined over a number field K and $\hat{h}$ is a Néron–Tate height associated to a symmetric ample line bundle on A, then the set of points $A(K\mathbb{Q}^{\mathrm{tr}})$ satisfies the following strong Bogomolov property:

$$\liminf_{x\in A(K\mathbb{Q}^{\mathrm{tr}})} \hat{h}(x) > 0.$$

We will establish an analogous result for the canonical heights associated to the iteration of rational maps on $\mathbb{P}^1$. Since we wish to allow arbitrary number fields as our base fields, we will make the following generalization of the notions of totally real and totally p-adic:

Definition 2. Fix a base number field K, and let S be a set of places of K. For each $v \in S$, we choose a finite normal extension L_v/K_v. We say that α is *totally* L_S/K if, for each $v \in S$, all of the K-Galois conjugates of α lie in L_v for each embedding $\bar{K} \hookrightarrow \mathbb{C}_v$.

Notice that the set of all totally L_S/K numbers forms a field L which is a normal extension of K.

Our main result in this paper is a broad generalization of these results to the dynamical heights arising from the iteration of rational functions. We give a simple measure-theoretic criterion which characterizes the situations above.

Theorem 1. *Let φ be a rational map defined over a number field K, and fix a finite set S of places of K and corresponding finite normal extensions L_v/K_v for each $v \in S$. Let L be the field of numbers all of whose K-Galois conjugates lie in L_v for each $v \in S$, and $\mu_{\varphi,v}$ be the v-adic canonical measure associated to φ. If there exists a place $v \in S$ for which*

$$\mu_{\varphi,v}(L_v) < 1,$$

then L satisfies the strong Bogomolov property with respect to h_φ, that is,

$$\liminf_{\alpha\in L} h_\varphi(\alpha) > 0.$$

Further, if φ is a polynomial, then the converse is also true: if L satisfies the strong Bogomolov property with respect to h_φ, then there exists a place v of L such that $\mu_{\varphi,v}(L_v) < 1$.

In the specific case of the maximal totally real field $\mathbb{Q}^{\mathrm{tr}}$, an independent proof which does not require the assumption that φ is a polynomial in the converse is due to Pottmeyer [17].

Before we prove our theorem, let us state a few interesting corollaries:

Corollary 2. *Let $\varphi \in K(z)$ and L_S/K be as in the theorem, and suppose there exists a place $v \in S$ such that $\mu_{\varphi,v}(L_v) < 1$. Then the set $\mathrm{PrePer}_\varphi(L)$ of preperiodic points of φ which are defined over the field L of all totally L_S/K numbers is finite.*

Corollary 3. *Let $\varphi \in K(z)$ and L_S/K be as in the theorem, and L the field of all totally L_S/K numbers. If there exists a sequence $\{\alpha_n\} \subset L$ of distinct algebraic numbers such that $h_\varphi(\alpha_n) \to 0$, then $\mu_{\varphi,v}(L_v) = 1$ for every place $v \in S$.*

Corollary 4. *Let $L_p/\mathbb{Q}_p$ be a finite normal extension with $L_p = \mathbb{R}$ if $p = \infty$. Then for the usual absolute logarithmic Weil height $h(\alpha)$,*

$$\liminf_{\substack{\alpha\in\mathbb{P}^1(\overline{\mathbb{Q}})\\ \alpha\ is\ totally\ L_p}} h(\alpha) > 0.$$

We note that for all but finitely many primes (namely for those of good reduction), the local dynamics of φ are trivial and resemble those of the standard height, so in particular we get the following:

Corollary 5. *Let φ be a rational map defined over a number field K. Then for almost all finite places v of K, we have*

$$\liminf_{\substack{\alpha\in\mathbb{P}^1(\bar{K})\\ \alpha\ is\ totally\ L_v}} h_\varphi(\alpha) > 0$$

for any finite normal extension L_v/K_v.

2. Proofs

2.1. Background on arithmetic dynamics. Before presenting our proofs we will give a quick review of some background in arithmetic dynamics. Early work on the distribution of numbers by height focused on rational points, with results like the uniform distribution of the Farey fractions, culminating in quantitative generalization of the distribution of rational points at all places by Choi [10, 11]. Shortly afterwards, in a slightly different direction, Bilu [6] formulated an equidistribution theorem for algebraic points of small height at the archimedean place, proving that the Galois conjugates of a sequence of numbers with Weil height tending to zero must equidistribute along the unit circle in $\mathbb{C}$ in the sense of weak convergence of measures. This result was later generalized to abelian varieties by Szpiro, Ullmo, and Zhang [22], and used in the proof of the Bogomolov conjecture by Zhang [25]. Later, these results were vastly generalized to dynamical heights and nonarchimedean places using the techniques of potential theory on Berkovich analytic spaces independently by Baker and Rumely [4], Favre and Rivera-Letelier [12, 13], and Chambert-Loir [9]. The proofs of our main theorems are rooted in this equidistribution result.

We briefly recall now the dynamical equidistribution theorem. Let $\varphi \in K(z)$ be a rational function of degree $d \geq 2$ for K a number field. Then the *dynamical* (or *canonical*) *height associated to* φ, first introduced by Call and Silverman [8], is given by the Tate limit

$$h_\varphi(\alpha) = \lim_{n\to\infty} \frac{1}{d^n} h(\varphi^n(\alpha))$$

where as usual $\varphi^n = \varphi \circ \cdots \circ \varphi$ denotes the n-fold iteration. The dynamical height is characterized by the properties that:

(1) The quantity $|h_\varphi(\alpha) - h(\alpha)|$ is bounded by an absolute constant for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$.
(2) $h_\varphi(\varphi(\alpha)) = dh_\varphi(\alpha)$ for all $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$.

The points $\alpha \in \mathbb{P}^1(\overline{\mathbb{Q}})$ which satisfy $h_\varphi(\alpha) = 0$ are precisely the preperiodic points of φ, that is, those which satisfy $\varphi^n(\alpha) = \varphi^m(\alpha)$ for some $n > m \geq 0$. In the case of $\varphi(z) = z^2$, for which $h_\varphi = h$, the preperiodic points are precisely 0, ∞, and the roots of unity. We refer the reader to [21] for more background on dynamical heights.

Recent work, particularly regarding the equidistribution theorem which we will discuss in more detail below, has made it clear that the proper setting to study nonarchimedean dynamics is the *Berkovich analytic space* $\mathsf{P}^1(\mathbb{C}_v)$, where $\mathbb{C}_v$ is the completion of an algebraic closure of K_v. For background on the Berkovich analytic space, we refer the reader to [5, 3]. While $\mathbb{P}^1(\mathbb{C}_v)$ is totally disconnected, not locally compact, and not spherically complete (there exist nested sequences of closed balls with empty intersection), the Berkovich space $\mathsf{P}^1(\mathbb{C}_v)$ is compact, Hausdorff, and path-connected, and it contains $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subset. As a set, one can construct the affine Berkovich space $\mathsf{A}^1(\mathbb{C}_v)$ as the Gel'fand spectrum consisting of all bounded multiplicative seminorms on the normed ring $\mathbb{C}_v[T]$, so each point in Berkovich space corresponds to such a seminorm. The set $\mathbb{C}_v$ sits inside $\mathsf{A}^1(\mathbb{C}_v)$ as the evaluation seminorms $f \mapsto |f(x)|_v$ for each $x \in \mathbb{C}_v$. When $v \mid \infty$, this is the whole story: all bounded multiplicative seminorms come from evaluation maps at a point and $\mathsf{A}^1(\mathbb{C}) = \mathbb{C}$. For nonarchimedean v, however, there is a wealth of new points, the most important of which in our study is the *Gauss point* $\zeta_{0,1}$, which corresponds to the multiplicative seminorm

$$f \mapsto \sup_{\substack{z\in\mathbb{C}_v \\ |z|_v \leq 1}} |f(z)|_v.$$

One can check by Gauss's lemma that this does in fact define a multiplicative seminorm on $\mathbb{C}_v[T]$. The Berkovich affine line is then endowed with the weakest topology for which the seminorms are continuous.

The equidistribution theorem states that any sequence $\{\alpha_n\}$ of small points with respect to φ, that is, of distinct numbers satisfying $h_\varphi(\alpha_n) \to 0$, is equidistributed with respect to a certain *canonical measure* $\mu_{\varphi,v}$ at each place v. More specifically, we recall the following result (see [4, 12]):

Theorem (Dynamical equidistribution theorem). *Let φ be a rational map of degree at least 2 defined over the number field K. Let $\{\alpha_n\}$ be a sequence of distinct algebraic numbers satisfying $h_\varphi(\alpha_n) \to 0$, and let $G_K = \mathrm{Gal}(\overline{K}/K)$. Let v be a place of K, and for each α_n define the probability measure on $\mathsf{P}^1(\mathbb{C}_v)$ supported equally on each of the G_K-conjugates of α_n:*

$$[\alpha_n] = \frac{1}{\#G_K\alpha_n} \sum_{z\in G_K\alpha_n} \delta_z$$

where δ_z denotes the Dirac measure at z. If $\mu_{\varphi,v}$ denotes the canonical measure on $\mathsf{P}^1(\mathbb{C}_v)$ associated to φ at v, then we have weak convergence of measures:

$$[\alpha_n] \xrightarrow{w} \mu_{\varphi,v} \quad \text{as } n \to \infty,$$

that is, for each continuous real or complex valued function f on $\mathsf{P}^1(\mathbb{C}_v)$ we have

$$\frac{1}{\#G_K\alpha_n} \sum_{z\in G_K\alpha_n} f(z) \to \int_{\mathsf{P}^1(\mathbb{C}_v)} f(z)\, d\mu_{\varphi,v}(z) \quad \text{as } n\to\infty.$$

At a finite place v of good reduction of φ, the measure $\mu_{\varphi,v} = \delta_{\zeta_{0,1}}$ is the Dirac measure supported at the Gauss point $\zeta_{0,1}$. We recall that any given rational φ has good reduction at all but finitely many places of K. Bilu's theorem [6] is the special case of this result for the squaring map $\varphi(z) = z^2$, for which $h_\varphi = h$ and $\mu_{\varphi,\infty}$ is the normalized Haar measure of the unit circle in $\mathbb{C}$. We note that already Bilu's theorem gives a rather immediate proof that totally real numbers cannot have height tending to 0, as they clearly cannot equidistribute around the unit circle in $\mathbb{C}$.

2.2. Proofs. We now proceed to give the proofs of our results.

Proof of Theorem 1. Suppose for the sake of contradiction that there exists a sequence $\{\alpha_n\} \subset \mathbb{P}^1(\overline{K})$ of distinct totally L_S/K algebraic numbers such that $h_\varphi(\alpha_n) \to 0$. Let $\mu_{\varphi,v}$ be the canonical measure and $G_K = \mathrm{Gal}(\overline{K}/K)$. Fix an embedding $\overline{K} \hookrightarrow \mathbb{C}_v \subset \mathsf{P}^1(\mathbb{C}_v)$. Define the usual measures

$$[\alpha_n] = \frac{1}{\#G_K\alpha_n} \sum_{z\in G\alpha_n} \delta_z$$

where, for $z \in \mathbb{C}_v$, δ_z denotes the Dirac measure at z. Then the equidistribution theorem tells us that we have the weak convergence

$$[\alpha_n] \xrightarrow{w} \mu_{\varphi,v}.$$

Recall that we have natural continuous embeddings $\mathbb{P}^1(\mathbb{C}_v) \hookrightarrow \mathsf{P}^1(\mathbb{C}_v)$, and hence we can identify L_v as a subset of $\mathsf{P}^1(\mathbb{C}_v)$. Notice that as $\mathbb{P}^1(L_v) = L_v \cup \{\infty\}$ is compact, it is the closure of L_v in $\mathsf{P}^1(\mathbb{C}_v)$. As $\mu_{\varphi,v}$ is the equilibrium measure of the v-adic Berkovich Julia set, it does not charge any singletons of $\mathbb{P}^1(\mathbb{C}_v)$ (though in the case of potential good reduction, it may charge a nonclassical point of $\mathsf{P}^1(\mathbb{C}_v) \setminus \mathbb{P}^1(\mathbb{C}_v)$), so $\mu_{\varphi,v}(L_v) = \mu_{\varphi,v}(\mathbb{P}^1(L_v))$. By assumption, $\mu_{\varphi,v}(\mathsf{P}^1(\mathbb{C}_v) \setminus \mathbb{P}^1(L_v)) = 1 - \mu_{\varphi,v}(\mathbb{P}^1(L_v)) > 0$. Since $\mu_{\varphi,v}$ is a regular measure, there exists a compact set $E \subset \mathsf{P}^1(\mathbb{C}_v) \setminus \mathbb{P}^1(L_v)$ such that $\mu_{\varphi,v}(E) > 0$ as well. As E and $\mathbb{P}^1(L_v)$ are disjoint closed subsets of $\mathsf{P}^1(\mathbb{C}_v)$, which is a compact Hausdorff space, we know by Urysohn's lemma that there exists a continuous function $f : \mathsf{P}^1(\mathbb{C}_v) \to [0,1]$ which takes the value 1 on E and 0 on $\mathbb{P}^1(L_v)$. But then, as $G_K\alpha_n \subset \mathbb{P}^1(L_v)$,

$$\int_{\mathsf{P}^1(\mathbb{C}_v)} f\, d[\alpha_n] = \frac{1}{\# G_K\alpha_n} \sum_{z \in G_K\alpha_n} f(z) = 0 \quad \text{for all } n,$$

while

$$\int_{\mathsf{P}^1(\mathbb{C}_v)} f\, d\mu_{\varphi,v} \geq \mu_{\varphi,v}(E) > 0.$$

Thus, for this function f,

$$\lim_{n\to\infty} \int_{\mathsf{P}^1(\mathbb{C}_v)} f\, d[\alpha_n] \neq \int_{\mathsf{P}^1(\mathbb{C}_v)} f\, d\mu_{\varphi,v};$$

but this contradicts the weak convergence of measures, and completes the proof of the first part of the theorem.

We will now prove the converse under the assumption that φ is a polynomial. Specifically, we will prove that if at all places $v \in S$, L_v has full $\mu_{\varphi,v}$-measure, then there exists a sequence $\alpha_n \in \mathbb{P}^1(\bar{K})$ such that $h_\varphi(\alpha_n) \to 0$. Since φ is a polynomial, there is a theorem (see for example [2] and [3, Thm. 10.91]) that the v-adic Berkovich Julia sets J_v are compact in $\mathsf{A}^1(\mathbb{C}_v)$, and that the v-adic logarithmic capacity with respect to the point ∞ of each set is given by

$$\gamma_{\infty,v}(J_v) = |a_d|_v^{-1/(d-1)} \tag{1}$$

where $|\cdot|_v$ is normalized in the usual way satisfying the product formula and we have denoted by a_d the leading coefficient of φ, that is, $\varphi(z) = a_d z^d + O(z^{d-1})$. Let M_K denote the places of K, and let

$$\mathbb{J} = \prod_{v \in M_K} J_v$$

be the adelic Berkovich Julia set. Then the (normalized) adelic logarithmic capacity is

$$\gamma_\infty(\mathbb{J}) = \prod_{v \in M_K} \gamma_{\infty,v}(J_v) = 1$$

by (1) and the product formula.

Our basic strategy will be to use the Fekete–Szegő theorem with splitting conditions due to Rumely ([18, Theorem 2.1] and [3, §§6.7, 7.8]; see also [19]) to generate a sequence $\alpha_n \in L$ with $h_\varphi(\alpha_n) \to 0$. For a place $v \in M_K$ we let $h_{\varphi,v}$ denote the usual v-adic local height associated to iteration of φ. Now, $h_{\varphi,v}^{-1}(0) \subset \mathbb{C}_v$ is the filled Julia set, and satisfies

$$J_v \cap \mathbb{C}_v \subset h_{\varphi,v}^{-1}(0).$$

(Note that as J_v denotes the Berkovich Julia set, its intersection with $\mathbb{C}_v$ may be empty, as happens at places of good reduction.)

Let $\mathcal{D}_v(0,1)$ denote the closed v-adic unit disc in $\mathbb{C}_v$. Fix $\epsilon > 0$ and define $\mathbb{E}$ to be the classical adelic set given by $\mathbb{E} = \prod_{v \in M_K} E_v$ where

$$E_v = \begin{cases} \mathcal{D}_v(0,1) & \text{if } v \nmid \infty \text{ and } v \text{ is a place of good reduction of } \varphi, \\ h_{\varphi,v}^{-1}(0) \cap L_v & \text{otherwise,} \end{cases}$$

and we take $L_v = \mathbb{C}_v$ if $v \mid \infty$ and $v \notin S$. Let $\mathbb{U}$ be the classical adelic set given by $\mathbb{U} = \prod_{v \in M_K} U_v$ where

$$U_v = \begin{cases} \mathcal{D}_v(0,1) & \text{if } v \nmid \infty \text{ and } v \text{ is a place of good reduction of } \varphi, \\ h_{\varphi,v}^{-1}([0,\epsilon)) \cap L_v & \text{otherwise.} \end{cases}$$

Since $h_{\varphi,v} : \mathbb{C}_v \to [0,\infty)$ is nonnegative and continuous for each v (see [8, §2] or [3, §10.8]), it follows that $\mathbb{U}$ is an open adelic neighborhood of $\mathbb{E}$.

We claim that for each v, $\gamma_{\infty,v}(E_v) \geq \gamma_{\infty,v}(J_v)$, and thus that

$$\gamma_\infty(\mathbb{E}) = \prod_{v \in M_K} \gamma_{\infty,v}(E_v) \geq \gamma_\infty(\mathbb{J}) \geq 1.$$

First we suppose that $v \in S$ or $v \mid \infty$. We start by noting that $\mathbb{P}^1(L_v)$ is compact and its topology is induced by the topology of the Berkovich space $\mathsf{P}^1(\mathbb{C}_v)$, and thus it is a compact subset of $\mathsf{P}^1(\mathbb{C}_v)$, and so the closure of L_v in $\mathsf{P}^1(\mathbb{C}_v)$ is $\mathbb{P}^1(L_v)$. Now, since $\mu_{\varphi,v}(L_v) = 1$ by assumption, it follows that $\mu_{\varphi,v}$ is fully supported on L_v because the point at infinity cannot have positive measure as $J_v(\varphi)$ is compact in the affine Berkovich line. Since E_v is compact in the affine line, its capacity $\gamma_{\infty,v}(E_v)$ is well defined and achieved by the unique minimal equilibrium measure $\mu_{\varphi,v}$ (see [3, §6.1, Prop. 7.21]), that is, $J_v \subseteq L_v$ and ([2])

$$\log \gamma_{\infty,v}(J_v) = \iint_{J_v \times J_v} \log|x-y|_v \, d\mu_{\varphi,v}(x)\, d\mu_{\varphi,v}(y). \tag{2}$$

Further, as we noted, $J_v \subseteq h_{\varphi,v}^{-1}(0) \cap L_v = E_v$, and thus

$$\log \gamma_{\infty,v}(E_v) \geq \iint_{E_v \times E_v} \log|x-y|_v \, d\mu_{\varphi,v}(x)\, d\mu_{\varphi,v}(y) = \log \gamma_{\infty,v}(J_v)$$

([2]) A similar formula is true for $v \notin S$ if one replaces the kernel on the right hand side with its natural extension to the Berkovich line, which in the terminology of [4, §4] is given by the *Hsia kernel* $-\log \delta(x,y)_\infty$.

by (2). Now, for v a nonarchimedean place of good reduction, we have $E_v = \mathcal{D}_v(0,1)$, and $\gamma_{v,\infty}(\mathcal{D}(0,1)) = 1 = \gamma_{v\infty}(J_v)$, as in this case, $J_v = \{\zeta_{0,1}\}$ is the usual Gauss point, and $\gamma_{\infty,v}(J_v) = 1$ by [3, Thm. 10.91]. The claim that the adelic capacity of $\mathbb{E}$ is at least 1 now follows.

As $\mathbb{U}$ is an open classical adelic neighborhood of $\mathbb{E}$ and $\gamma_\infty(\mathbb{E}) \geq 1$ with compact archimedean components, it now follows from [18, Theorem 2.1] that there are infinitely many totally L_S algebraic numbers all of whose Galois conjugates lie in $\mathbb{U}$. Since any such number has height $h_\varphi(\alpha) \leq |S'|\epsilon$, by taking $\epsilon \to 0$ we can generate an infinite sequence of mutually distinct totally L_v algebraic numbers for all $v \in S$ with $h_\varphi(\alpha) \to 0$, which is the desired conclusion. ∎

Proof of Corollary 2. It is clear that the set L of all totally L_S/K numbers is a field which is normal over K. To see that the set of preperiodic points is finite, merely note that if an infinite number of such points existed, the limit infimum of the φ-canonical height would necessarily be 0, contradicting Theorem 1. ∎

Proof of Corollary 3. This is immediate as it is the contrapositive of the first part of the theorem. ∎

Proof of Corollary 4. Let $\varphi(z) = z^2$ and $L_\infty = \mathbb{R}$. Then the equilibrium measure of the (Berkovich) complex Julia set is $\mu_{\varphi,\infty} = \lambda$, the normalized Haar measure of the unit circle S^1 in $\mathbb{C}^\times \subset \mathsf{P}^1(\mathbb{C}) = \mathbb{P}^1(\mathbb{C})$. Clearly $\lambda(\mathbb{R}) = \lambda(\mathbb{R} \cap S^1) = \lambda(\{\pm 1\}) = 0$. Thus the theorem applies, and we see that all totally $\mathbb{R}$ numbers α are in $\overline{\mathbb{Q}}$, that is, all totally real numbers which are not zero or roots of unity (the preperiodic points of φ) must have Weil height bounded away from zero by an absolute constant.

Now consider the case where $\varphi(z) = z^2$ and $L_p/\mathbb{Q}_p$ is a finite normal extension for a finite prime p. The equilibrium measure $\mu_{\varphi,p} = \delta_{\zeta_{0,1}}$ is the Dirac measure supported on the Gauss point $\zeta_{0,1}$. Since $\zeta_{0,1} \notin L_p$, we have $\mu_{\varphi,p}(L_v) = 0 < 1$, and the theorem again applies. ∎

Proof of Corollary 5. We merely note that for almost all places of K, φ has good reduction, which means that the Berkovich v-adic Julia set is the Gauss point. But again as in the proof of the above corollary, $\mu_{\varphi,v}(L_p) = 0$, and so the theorem applies. ∎

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