ACTA ARITHMETICA
170.1 (2015)

Nearest neighbor spacing distributions for the zeros of the real or imaginary part of the Riemann xi-function on vertical lines

by

Masatoshi Suzuki (Tokyo)

1. Introduction. Let $s = \sigma + it$ ($i = \sqrt{-1}$) be a complex variable, $\zeta(s)$ be the Riemann zeta-function, and

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

be the Riemann xi-function, which is an entire function satisfying the functional equations $\xi(s) = \xi(1-s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$. In this paper, we discuss the distribution of zeros of the entire functions

$$A_\omega(s) := \tfrac{1}{2}(\xi(s+\omega)+\xi(s-\omega)), \qquad B_\omega(s) := \tfrac{1}{2}i(\xi(s+\omega)-\xi(s-\omega)) \tag{1.1}$$

depending on a positive real parameter ω in consideration of the following two relations with the zeros of $\xi(s)$. Firstly, the zeros of $A_\omega(s)$ and $B_\omega(s)$ on the line $\sigma = 1/2$ coincide respectively with the zeros of the real and imaginary parts of $\xi(s)$ on the line $\sigma = 1/2 + \omega$, because

$$\operatorname{Re}\xi(1/2+\omega+it) = A_\omega(1/2+it), \quad \operatorname{Im}\xi(1/2+\omega+it) = -B_\omega(1/2+it) \tag{1.2}$$

by the functional equations of $\xi(s)$. Secondly, for small $\omega > 0$, the zeros of $A_\omega(s)$ and $B_\omega(s)$ (locally) approximate the zeros of $\xi(s)$ and $\xi'(s)$ respectively, because of the asymptotic relations

$$A_\omega(s) = \xi(s) + O(\omega^2), \qquad B_\omega(s) = i\omega\cdot\xi'(s) + O(\omega^3) \qquad (\omega \to 0^+)$$

on compact subsets of $\mathbb{C}$.

The functional equations of $\xi(s)$ imply that $A_\omega(s)$ and $B_\omega(s)$ satisfy

$$A_\omega(s) = A_\omega(1-s), \qquad B_\omega(s) = -B_\omega(1-s)$$

and take real values on the critical line $\sigma = 1/2$. It is known that all zeros of $A_\omega(s)$ and $B_\omega(s)$ are simple zeros lying on the critical line if $\omega \geq 1/2$. This also holds for $0 < \omega < 1/2$ if we assume the Riemann Hypothesis (RH)

2010 *Mathematics Subject Classification*: Primary 11M26; Secondary 11M50, 11N64.
Key words and phrases: Riemann xi-function, zeros, nearest neighbor spacing distributions, logarithmic derivative of the Riemann zeta-function, value distribution.

for $\xi(s)$ [11, Theorem 2.1], or unconditionally, except for a set of zeros up to height T of cardinality $\ll T^{1-a\omega}(\log T)^2$ for any $a < 1$ [12, Theorems 1 and 2]. In this sense, the horizontal distribution of the zeros of $A_\omega(s)$ and $B_\omega(s)$ is well understood. Therefore we focus on their vertical distribution.

Let $X_\omega(s)$ be $A_\omega(s)$ or $B_\omega(s)$. We arrange the zeros $\rho = \beta + i\gamma$ of $X_\omega(s)$ with $\gamma > 0$ into a sequence $\rho_n = \beta_n + i\gamma_n$ so that $\gamma_{n+1} \geq \gamma_n$. Then the distribution of spacings of the normalized imaginary parts

$$\gamma_n^{(1)} := \frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} \tag{1.3}$$

converges to a limiting distribution of equal spacings of length one. This is proved in Lagarias [11, Theorem 4.1] assuming RH if $0 < \omega < 1/2$, and in Li [12, Theorem 1] unconditionally. The above result on the normalized imaginary parts is in contrast to the Montgomery–Odlyzko conjecture and the GUE conjecture which assert that the normalized imaginary parts of the zeros of $\xi(s)$ obey the distribution of eigenvalues of random hermitian matrices from the Gaussian Unitary Ensemble (GUE). Therefore, one might think that the zeros of $A_\omega(s)$ and $B_\omega(s)$ are insignificant objects at least from the viewpoint of their vertical distribution.

However, interestingly enough, it will be proved that the *second normalization* of the imaginary parts defined by

$$\gamma_n^{(2)} := \left(\frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} - n \right) \varrho_\omega^{-1/2} \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e} \tag{1.4}$$

has a remarkable distribution which is related to the Euler product of the Riemann zeta-function, where

$$\varrho_\omega := \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{1+2\omega}},$$

$\Lambda(n)$ is the von Mangoldt function and the series converges absolutely for $\omega > 0$.

In order to state the main theorem, we recall a result on the value distribution of the logarithmic derivative of the Riemann zeta-function on vertical lines. For every $\sigma > 1/2$, there exists a non-negative real valued C^∞-function $M_\sigma(z)$ on $\mathbb{C}$ such that $(2\pi)^{-1} \int_{\mathbb{C}} M_\sigma(z)\, dz = 1$ and

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Phi\left(\frac{\zeta'}{\zeta}(\sigma + it) \right) dt = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(z) \Phi(z)\, dz \tag{1.5}$$

for $\Phi(z)$ being any continuous bounded function on $\mathbb{C}$ or the characteristic function of any compact subset of $\mathbb{C}$ or of the complement of such a subset. Following Ihara [3] we call $M_\sigma(z)$ the *M-function*. Formula (1.5) was obtained by van Kampen–Wintner [8], Kershner–Wintner [10], Guo [2], Ihara [3] and Ihara–Matsumoto [6] (see Appendix for the construction of

$M_\sigma(z)$ and historical details). If $\sigma > 1$, formula (1.5) holds for any continuous function $\Phi(z)$ on $\mathbb{C}$.

Using the M-function, we define the *m-function* by

$$m_\sigma(u) = \int_{-\infty}^{\infty} M_\sigma(u+iv)\,dv \tag{1.6}$$

on the real line. This is well defined because $M_\sigma(z)$ is of rapid decay as $|z| \to \infty$ [3, Theorem 2].

Reflecting the Euler product formula of the Riemann zeta-function, the Fourier transform $\tilde{M}_\sigma(z)$ has an Euler product formula $\tilde{M}_\sigma(z) = \prod_p \tilde{M}_{\sigma,p}(z)$ whose local factors $\tilde{M}_{\sigma,p}(z)$ are some arithmetic Dirichlet series in σ, where p runs over all prime numbers (see Appendix). Therefore, the Fourier transform of the m-function also has an Euler product, since

$$\tilde{m}_\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\sigma(u)e^{ixu}\,du = \frac{1}{2\pi}\int_{\mathbb{C}} M_\sigma(u+iv)e^{ixu}\,du\,dv = \tilde{M}_\sigma(x).$$

Now our main result is stated as follows.

Theorem 1.1. *Let $X_\omega(s)$ be $A_\omega(s)$ or $B_\omega(s)$ for given $\omega > 0$, and let $\gamma_n^{(2)}$ be the secondary normalized imaginary parts of the zeros of $X_\omega(s)$ defined in* (1.4). *Then*

$$\lim_{T\to\infty}\frac{1}{N_\omega(T)}\sum_{0<\gamma_n\le T}\phi(\gamma_{n+1}^{(2)} - \gamma_n^{(2)}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\varrho_\omega^{1/2} m_{1/2+\omega}(\pi\varrho_\omega^{1/2}u)\phi(u)\,du \tag{1.7}$$

for any bounded function $\phi \in C^1(\mathbb{R})$ such that $\phi'(x) \ll 1$ for $|x| \le 1$, $\phi'(x) \ll x^{-2}$ for $|x| \ge 1$ and $u \mapsto \frac{d}{du}\phi\big(\mathrm{Re}\,\frac{\zeta'}{\zeta}(1/2+\omega+iu)\big)$ is bounded on $\mathbb{R}$, where $N_\omega(T)$ is the number of zeros of $X_\omega(s)$ with $0 < t \le T$.

The limit behavior of the integrand of the right-hand side of (1.7) as $\omega \to 0^+$ is obtained as follows by using a result of [4].

Theorem 1.2. *We have*

$$\frac{1}{2\pi}\lim_{\omega\to 0^+}\pi\varrho_\omega^{1/2} m_{1/2+\omega}(\pi\varrho_\omega^{1/2}u) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right).$$

Note that the above two theorems are unconditional.

We now discuss the significance of Theorem 1.1 under RH if $0 < \omega < 1/2$. In this case, all zeros of $X_\omega(s)$ are simple zeros lying on the critical line and

$$N_\omega(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + S_\omega(T) + \frac{7+2\omega}{8} + O\left(\frac{\max\{1, 2\omega-1\}}{T}\right) \tag{1.8}$$

for $T \geq 2$ [11, Theorem 3.1], where the implied constant is independent of ω and

$$S_\omega(t) = \frac{1}{\pi} \arg \zeta(1/2 + \omega + it)$$

is a C^∞-function on the real line obtained by continuous variation along the straight lines joining 2, $2 + it$ and $1/2 + \omega + it$, starting with the value 0. For fixed $\omega > 0$, we have

$$1 = N_\omega(\gamma_{n+1}) - N_\omega(\gamma_n) = \gamma_{n+1}^{(1)} - \gamma_n^{(1)} + S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n) + O(1/\gamma_n),$$

by the simplicity of zeros, (1.3) and (1.8), and thus

$$\gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1 = -(S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)) + O(1/\gamma_n). \tag{1.9}$$

Given this formula, $\gamma_{n+1}^{(1)} - \gamma_n^{(1)} \to 1$ means that the contribution of $S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)$ is smaller than 1 for any fixed $\omega > 0$. In other words, the distribution of spacings of the normalized zeros of $X_\omega(s)$ is dominated by the gamma functor of $\zeta(s)$ only.

On the other hand, it is known that the subtle behavior of the zeros of $\zeta(s)$ as described by the Montgomery–Odlyzko conjecture is caused by the function $S(t)$, which is obtained by $S(t) = \lim_{\omega \to 0^+} S_\omega(t)$ if t is not the ordinate of a zero of $\zeta(s)$, and $S(t) = \frac{1}{2} \lim_{\delta \to 0^+} (S(t + \delta) + S(t - \delta))$ otherwise.

Therefore, from the discussion above, Theorem 1.1 shows that the second normalization (1.4) detects the effect of the arithmetic part $S_\omega(T)$ of the counting function $N_\omega(T)$. The Euler product formula of $\tilde{m}_\sigma(u)$ is a supporting evidence for this observation.

A motivation of this work was L. Weng's question posed to the author. In 2013, Weng–Zagier [18] proved that all high-rank zeta functions for elliptic curves E defined over a finite field satisfy an analogue of the Riemann Hypothesis. Then Weng considered the distribution of the zeros of high-rank zeta functions for E when the rank varies, and observed that the dominant term is very simple but the second dominant term is related to the Sato–Tate measure. He asked the author about an analogue of his observation for the number field case ([16], where he considered another version of (1.4) but it is simplified in [17] and shown to be compatible with (1.4)). For the rational number field $\mathbb{Q}$, high-rank zeta functions $\hat{\zeta}_{\mathbb{Q},n}(s)$ are expressed as linear combinations of products of the Riemann zeta-function and rational functions. The rank one case is $\hat{\zeta}_{\mathbb{Q},1}(s) = \hat{\zeta}(s)$. The rank two case is

$$s(2s-1)(2s-2)\hat{\zeta}_{\mathbb{Q},2}(s) = \xi(2s) - \xi(2s-1) = B_{1/2}(2s - 1/2).$$

Therefore, the second dominant term of the distribution of the zeros is described by $m_1(x)$. The rank three case is

$$3s(3s-1)(3s-2)(3s-3)\hat{\zeta}_{\mathbb{Q},3}(s) = X(s) + X(1-s),$$
$$X(s) = \big(3(2\xi(2)-1)s - 4\xi(2)+3\big)\xi(3s) - \xi(3s-1).$$

This looks similar to $A_1(3s-1) = \xi(3s) + \xi(3s-2)$ in a sense. Therefore, it is expected that the second dominant term of the distribution of the zeros is described by $m_{3/2}(x)$ up to a small correction.

This paper is organized as follows. In Section 2, we prepare some lemmas necessary for the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1 under RH at first, for the simplicity of the argument. Then we prove Theorem 1.1 unconditionally and prove Theorem 1.2. In Section 4, we give several comments and remarks on subjects of the paper. Finally, in the Appendix we provide a review of the construction, basic properties and history of the M-function.

2. Preliminaries. Let $\omega > 0$. We will assume RH if $0 < \omega < 1/2$ throughout this section. Then the imaginary parts of $A_\omega(s)$ and $B_\omega(s)$ are enumerated as

$$\cdots < \gamma_{-1}(B_\omega) < \gamma_{-1}(A_\omega) < \gamma_0(B_\omega) = 0$$
$$< \gamma_1(A_\omega) < \gamma_1(B_\omega) < \gamma_2(A_\omega) < \gamma_2(B_\omega) < \cdots$$

with $\gamma_{-n}(A_\omega) = -\gamma_n(A_\omega)$ and $\gamma_{-n}(B_\omega) = -\gamma_n(B_\omega)$ for $n \geq 1$. Unless stated otherwise, in the rest of this paper we denote by γ_n the nth imaginary part $\gamma_n(A_\omega)$ or $\gamma_n(B_\omega)$ when $n \geq 1$.

LEMMA 2.1. *Let $\Omega > 0$. Then*

$$\gamma_n = \frac{2\pi n}{\log n}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right), \tag{2.1}$$

$$\log\frac{\gamma_n}{2\pi} = \log n\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right), \tag{2.2}$$

where the implied constants are uniform in ω with $0 < \omega \leq \Omega$. These formulas are unconditional.

REMARK. In [11, p. 171] it is claimed that

$$\gamma_n = \frac{2\pi n}{\log n}\left(1 + O\left(\frac{1}{\log n}\right)\right)$$

assuming (1.8) and $S_\omega(t) = O(\log t)$. However, the author does not know how to exclude the factor $\log\log n$ from (2.1).

Proof of Lemma 2.1. Suppose that $\gamma_n = \gamma_n(A_\omega)$. We have $S_\omega(T) = O(\log T)$ unconditionally by [15, Theorem 9.4], where the implied constant does not depend on ω. Therefore,

$$n = N_\omega(\gamma_n) = \frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e}(1+O(1/\gamma_n))$$

by (1.8) and the simplicity of zeros. Substituting this into $2\pi n/\log n$ yields

$$\begin{aligned}\frac{2\pi n}{\log n} &= \gamma_n\left(\frac{1+O(1/\gamma_n)}{1+\frac{\log\log\frac{\gamma_n}{2\pi e}}{\log\frac{\gamma_n}{2\pi}}+O\big(\frac{1}{\gamma_n\log\gamma_n}\big)}+O\bigg(\frac{1}{\log\gamma_n}\bigg)\right)\\ &= \gamma_n\left(1+O\bigg(\frac{\log\log\gamma_n}{\log\gamma_n}\bigg)\right).\end{aligned}$$

This implies $n/\log n \ll \gamma_n$, since $\gamma_n \to \infty$ as $n\to\infty$, and therefore $\log\log\gamma_n/\log\gamma_n \ll \log\log n/\log n$. Hence gives (2.1). Taking log of both sides of (2.1) gives (2.2). The case of $\gamma_n = \gamma_n(B_\omega)$ is proved in a similar way. ■

Lemma 2.2. *The gaps* $\gamma_{n+1}-\gamma_n$ *tend to* 0 *as* $n\to\infty$.

Proof. We will show that $S_\omega(t) = o(\log t)$ for any fixed $\omega > 0$; this implies the conclusion by (1.8). We have

$$\log\zeta(1/2+\omega+it) \ll \begin{cases} 1 & \text{if } \omega > 1/2,\\ \log\log t & \text{if } \omega = 1/2,\\ \log\log\log t & \text{if } \omega = 1/2 \text{ under RH},\\ \dfrac{(\log t)^{1-2\omega}}{\log\log t} & \text{if } 0<\omega<1/2 \text{ under RH},\end{cases}$$

for large $t > 0$, where the first line is a consequence of the absolute convergence of the Dirichlet series of $\log\zeta(s)$, the second line is shown in [13, Theorem 6.7] and the other cases are shown in [15, Theorem 14.5, §14.33]. These estimates imply $S_\omega(t) = o(\log t)$, since $S_\omega(t) \ll |\log\zeta(1/2+\omega+it)|$. ■

Lemma 2.3. *We have*

$$\frac{S_\omega(\gamma_{n+1})-S_\omega(\gamma_n)}{\gamma_{n+1}-\gamma_n} = O(E_{1,\omega}(\gamma_n)) \tag{2.3}$$

with

$$E_{1,\omega}(t) = \begin{cases} 1 & \textit{if } \omega > 1/2,\\ \dfrac{\log t}{\log\log t} & \textit{if } \omega = 1/2,\\ \log\log t & \textit{if } \omega = 1/2 \textit{ under RH},\\ (\log t)^{1-2\omega} & \textit{if } 0<\omega<1/2 \textit{ under RH}.\end{cases} \tag{2.4}$$

Proof. We have $\pi S_\omega'(t) = \mathrm{Re}(\zeta'/\zeta)(1/2+\omega+it)$ by the definition of $S_\omega(t)$, since $\zeta(s)$ has no zeros in $\mathrm{Re}\, s > 1/2$ by RH. Therefore,

$$\pi\left|\frac{S_\omega(\gamma_{n+1})-S_\omega(\gamma_n)}{\gamma_{n+1}-\gamma_n}\right| \le \left|\mathrm{Re}\bigg\{\frac{\zeta'}{\zeta}(1/2+\omega+i\xi)\bigg\}\right| \le \left|\frac{\zeta'}{\zeta}(1/2+\omega+i\xi)\right|$$

for some $\gamma_n < \xi < \gamma_{n+1}$ by Lemma 2.2 and the mean value theorem. On the right-hand side, we have

$$\frac{\zeta'}{\zeta}(1/2+\omega+i\xi) \ll E_{1,\omega}(\xi), \tag{2.5}$$

where the first line of (2.4) is a consequence of the absolute convergence of the Dirichlet series of $(\zeta'/\zeta)(s)$, the second line is shown in [15, (5.14.7)] and the other cases are shown in [15, §14.33]. These estimates imply (2.3), since $\log\xi < \log\gamma_{n+1} = \log\gamma_n + O(\gamma_n^{-1})$ by Lemma 2.2. ∎

Lemma 2.4. *We have*

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e} = 1 + O(E_{2,\omega}(\gamma_n)), \tag{2.6}$$

where $E_{2,\omega}(t) = E_{1,\omega}(t)/\log t$ for the function $E_{1,\omega}(t)$ of (2.4).

Proof. We have

$$N_\omega(t+h) - N_\omega(t) = \frac{h}{2\pi}\log\frac{t}{2\pi} + S_\omega(t+h) - S_\omega(t) + O\left(\frac{1}{t+1}\right)$$

for $0 \le h \le 1$ and $t \ge 2$ by [11, proof of Theorem 4.1], where the implied constant does not depend on h. Applying this to $t = \gamma_n$ and $h = \gamma_{n+1}-\gamma_n$ together with Lemma 2.2, we get

$$1 = N_\omega(\gamma_{n+1}) - N_\omega(\gamma_n) = \frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e} + S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n) + O\left(\frac{1}{\gamma_n}\right)$$

for large n. This implies

$$(\gamma_{n+1}-\gamma_n)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi e}\left(1+O\left(\frac{1}{\log\gamma_n}\left|\frac{S_\omega(\gamma_{n+1})-S_\omega(\gamma_n)}{\gamma_{n+1}-\gamma_n}\right|\right)\right) = 1+O\left(\frac{1}{\gamma_n}\right).$$

Applying (2.3) to the left-hand side, we obtain (2.6). ∎

Lemma 2.5. *Assume that $f \in C^1(\mathbb{R})$ and f' is bounded on $\mathbb{R}$. Then*

$$\frac{1}{\gamma_N}\sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1}-\gamma_n) = \frac{1}{\gamma_N}\int_0^{\gamma_N} f(t)\,dt + O\left(\frac{1}{\log\gamma_N}\right) \tag{2.7}$$

for large $N > 0$.

Proof. We have

$$\frac{1}{\gamma_N}\sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1}-\gamma_n) = \frac{1}{\gamma_N}\int_{\gamma_1}^{\gamma_N} f(t)\,dt + \frac{1}{\gamma_N}\sum_{n=1}^{N-1}\int_{\gamma_n}^{\gamma_{n+1}}(f(\gamma_n)-f(t))\,dt + O(1/\gamma_N).$$

The second sum on the right-hand side is estimated as

$$\Big|\sum_{n=1}^{N-1}\int\limits_{\gamma_n}^{\gamma_{n+1}}(f(\gamma_n)-f(t))\,dt\Big| \le \sum_{n=1}^{N-1}\max_{\gamma_n\le\xi\le\gamma_{n+1}}|f'(\xi)|\int\limits_{\gamma_n}^{\gamma_{n+1}}(t-\gamma_n)\,dt$$
$$\le \frac{1}{2}\max_{\gamma_1\le t<\infty}|f'(t)|\sum_{n=1}^{N-1}(\gamma_{n+1}-\gamma_n)^2.$$

Here the sum on the right-hand side is estimated as

$$\sum_{n=1}^{N-1}(\gamma_{n+1}-\gamma_n)^2 \ll \sum_{n=1}^{N-1}\frac{1}{\log\gamma_n},$$

since $\gamma_{n+1}-\gamma_n \ll (\log\gamma_n)^{-1}$ by (2.6). Using the Stieltjes integral and integration by parts, we get

$$\sum_{n=1}^{N-1}\frac{1}{\log\gamma_n} \ll \int\limits_{\gamma_1}^{\gamma_N}\frac{dN_\omega(t)}{(\log t)^2} \ll \int\limits_{\gamma_1}^{\gamma_N}\frac{dt}{\log t} \ll \frac{\gamma_N}{\log\gamma_N}.$$

Hence we obtain (2.7). ∎

3. Proofs. First, we prove Theorem 1.1 assuming RH if $0<\omega<1/2$ after preparing two propositions based on the results in the previous section.

PROPOSITION 3.1. *Assume that $f\in C^1(\mathbb{R})$ and is bounded on $\mathbb{R}$. Then*

$$\frac{1}{N_\omega(T)}\sum_{0<\gamma_n\le T}f(\gamma_n) = \frac{1}{T}\int\limits_0^T f(t)\,dt + O(E_{2,\omega}(T)) \tag{3.1}$$

for large $T>0$, where $E_{2,\omega}(t)=E_{1,\omega}(t)/\log t$ for the function $E_{1,\omega}(t)$ of (2.4).

Proof. It is sufficient to show that the left-hand side of (2.7) is equal to the left-hand side of (3.1) up to a reasonable error term. We have

$$\frac{1}{\gamma_N}\sum_{n=1}^{N-1}f(\gamma_n)(\gamma_{n+1}-\gamma_n) = \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1}f(\gamma_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_N}{2\pi}$$
$$= \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1}f(\gamma_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}$$
$$+ \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1}f(\gamma_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}\left(\frac{\log\frac{\gamma_N}{2\pi}}{\log\frac{\gamma_n}{2\pi}}-1\right) = S_1+S_2,$$

say. First we consider S_1. We have

$$\left|S_1 - \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1} f(\gamma_n)\right| \ll \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1}\left|\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}-1\right|.$$

For the sum on the right-hand side,

$$\sum_{n=1}^{N-1}\left|\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}-1\right| \ll \sum_{n=1}^{N-1} E_{2,\omega}(\gamma_n) \ll \int_{\gamma_1}^{\gamma_N} E_{2,\omega}(t)\,dN_\omega(t)$$

by using (2.6) and the Stieltjes integral. Here

$$\int_{\gamma_1}^{\gamma_N} E_{2,\omega}(t)\,dN_\omega(t) \ll \int_{\gamma_1}^{\gamma_N} E_{2,\omega}(t)(\log t)\,dt \ll \gamma_N\log\gamma_N E_{2,\omega}(\gamma_N)$$

by integration by parts. Hence

$$\left|S_1 - \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1} f(\gamma_n)\right| \ll E_{2,\omega}(\gamma_N).$$

Next we consider S_2. We have

$$|S_2| \ll \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1}\left(\frac{\log\frac{\gamma_N}{2\pi}}{\log\frac{\gamma_n}{2\pi}}-1\right)$$

by (2.6). Using partial summation for the sum on the right-hand side, we get

$$\begin{aligned}\frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1}\left(\frac{\log\frac{\gamma_N}{2\pi}}{\log\frac{\gamma_n}{2\pi}}-1\right) &= \frac{2\pi}{\gamma_N}\int_{\gamma_1}^{\gamma_N}\Big(\sum_{0<\gamma_n\le x} 1\Big)\frac{1}{x\big(\log\frac{x}{2\pi}\big)^2}\,dx + O(1/\gamma_N)\\ &\ll \frac{1}{\gamma_N}\int_{\gamma_1}^{\gamma_N} x\log x\cdot\frac{1}{x(\log x)^2}\,dx + O(1/\gamma_N)\\ &\ll \frac{1}{\log\gamma_N}.\end{aligned}$$

From the above argument, we obtain

$$\frac{1}{\gamma_N}\sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1}-\gamma_n) = \frac{1}{\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi}}\sum_{n=1}^{N-1} f(\gamma_n) + O(E_{2\omega}(\gamma_N)),$$

since $(\log t)^{-1} \ll E_{2,\omega}(t)$ for every $\omega > 0$. Combining this with (2.7) and

$$\frac{\gamma_N}{2\pi}\log\frac{\gamma_N}{2\pi} = N_\omega(\gamma_N)(1+O(1/\gamma_N)),$$

we obtain (3.1). ∎

Proposition 3.2. *Let $\phi \in C^1(\mathbb{R})$. Assume that $\phi'(x) \ll 1$ for $|x| \le 1$, $\phi'(x) \ll x^{-2}$ for $|x| \ge 1$ and $u \mapsto \frac{d}{du}\phi\big(\mathrm{Re}\,\frac{\zeta'}{\zeta}(1/2+\omega+iu)\big)$ is bounded on $\mathbb{R}$. Define*

$$\ddot{\gamma}_n = \varrho_\omega^{1/2}\gamma_n^{(2)} = \Big(\frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e} - n\Big)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi e}. \tag{3.2}$$

Then

$$\frac{1}{N_\omega(T)}\sum_{0<\gamma_n\le T}\phi(\ddot{\gamma}_{n+1}-\ddot{\gamma}_n) = \frac{1}{N_\omega(T)}\sum_{0<\gamma_n\le T}\phi\Big(-\frac{1}{\pi}\mathrm{Re}\,\frac{\zeta'}{\zeta}(1/2+\omega+i\gamma_n)\Big) + O\Big(\frac{\log\log T}{\log T}\Big) + O(E_{2,\omega}(T)) \tag{3.3}$$

for large $T > 0$.

Proof. On the right-hand side of (1.9), we have

$$S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n) = \frac{1}{\pi}\mathrm{Re}\,\frac{\zeta'}{\zeta}(1/2+\omega+i\xi_n)(\gamma_{n+1}-\gamma_n)$$

for some $\xi_n \in (\gamma_n, \gamma_{n+1})$ by the mean value theorem. Therefore,

$$\begin{aligned}(\gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi e} \\ = -\frac{1}{\pi}\mathrm{Re}\,\frac{\zeta'}{\zeta}(1/2+\omega+i\xi_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e} + O\Big(\frac{\log\gamma_n}{\gamma_n}\Big)\end{aligned} \tag{3.4}$$

by (1.9). On the other hand,

$$\begin{aligned}\ddot{\gamma}_{n+1} - \ddot{\gamma}_n &= (\gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi e} \\ &\quad + (\gamma_{n+1}^{(1)} - (n+1))\frac{1}{2\pi}\Big(\log\frac{\gamma_{n+1}}{2\pi e} - \log\frac{\gamma_n}{2\pi e}\Big)\end{aligned}$$

by the definitions (1.3) and (3.2). The second term of the right-hand side is estimated as

$$\begin{aligned}&(\gamma_{n+1}^{(1)} - (n+1))\frac{1}{2\pi}\Big(\log\frac{\gamma_{n+1}}{2\pi e} - \log\frac{\gamma_n}{2\pi e}\Big) \\ &\quad = (\gamma_{n+1}^{(1)} - (n+1))\frac{1}{2\pi}\log\Big(1 + \frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\Big) \\ &\quad \ll n\frac{\log\log n}{\log n}\cdot\frac{\gamma_{n+1}-\gamma_n}{\gamma_n} \ll \gamma_n\log\log\gamma_n\cdot\frac{1}{\gamma_n\log\gamma_n} = \frac{\log\log\gamma_n}{\log\gamma_n}\end{aligned}$$

by (2.1), (2.2) and (2.6).

By the above argument, we get

$$\ddot{\gamma}_{n+1} - \ddot{\gamma}_n = (\gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1)\frac{1}{2\pi}\log\frac{\gamma_n}{2\pi e} + O\Big(\frac{\log\log\gamma_n}{\log\gamma_n}\Big). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\ddot\gamma_{n+1}-\ddot\gamma_n=-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+\omega+i\xi_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e}+O\biggl(\frac{\log\log\gamma_n}{\log\gamma_n}\biggr) \tag{3.6}$$

for some $\xi_n\in(\gamma_n,\gamma_{n+1})$. Therefore,

$$\begin{aligned}
&\phi(\ddot\gamma_{n+1}-\ddot\gamma_n)\\
&=\phi\biggl(-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+\omega+i\xi_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e}+O\biggl(\frac{\log\log\gamma_n}{\log\gamma_n}\biggr)\biggr)\\
&=\phi\biggl(-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+\omega+i\xi_n)\frac{\gamma_{n+1}-\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi e}\biggr)+O\biggl(\frac{\log\log\gamma_n}{\log\gamma_n}\biggr)\\
&=\phi\biggl(-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+\omega+i\xi_n)(1+E_{2,\omega}(\gamma_n))\biggr)+O\biggl(\frac{\log\log\gamma_n}{\log\gamma_n}\biggr)
\end{aligned}$$

by the mean value theorem and (2.6), since $\phi'(x)$ is bounded.

Now we take $T_0>0$ such that the error term $O(E_{2,\omega}(t))$ of Lemma 2.4 is less than $1/2$ for every $t\ge T_0$. We set $r(t)=-\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+\omega+it)$, $I_1(T)=\{t\in[T_0,T]:|r(t)|\le 2/3\}$ and $I_2(T)=\{t\in[T_0,T]:|r(t)|>2/3\}$ so that $[T_0,T]=I_1(T)\cup I_2(T)$.

If $\gamma_n\ge T_0$ and $\xi_n\in I_1(T)$, we have

$$\begin{aligned}
\phi\bigl(r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))\bigr)-\phi(r(\xi_n))&=\pm\int\limits_{r(\xi_n)}^{r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))}\phi'(u)\,du\\
&\ll|r(\xi_n)|E_{2,\omega}(\gamma_n)\le E_{2,\omega}(\gamma_n),
\end{aligned}$$

since $|r(\xi_n)|\le 1$ and $|r(\xi_n)(1+O(E_{2,\omega}(t)))|\le 1$.

If $\gamma_n\ge T_0$ and $\xi_n\in I_2(T)$, we have

$$\begin{aligned}
\phi\bigl(r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))\bigr)-\phi(r(\xi_n))&=\pm\int\limits_{r(\xi_n)}^{r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))}\phi'(u)\,du\\
&\ll\biggl|\frac{E_{2,\omega}(\gamma_n)}{r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))}\biggr|\ll E_{2,\omega}(\gamma_n),
\end{aligned}$$

since $|r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))|\ge 1/3$. Therefore,

$$\phi\bigl(r(\xi_n)(1+E_{2,\omega}(\gamma_n))\bigr)=\phi(r(\xi_n))+O(E_{2,\omega}(\gamma_n))$$

for every $\gamma_n\ge T_0$ and $\xi_n\in[T_0,T]$. Moreover, we have

$$\phi\bigl(r(\xi_n)(1+E_{2,\omega}(\gamma_n))\bigr)=\phi(r(\gamma_n))+O\biggl(\frac{1}{\log\gamma_n}\biggr)+O(E_{2,\omega}(\gamma_n))$$

by the mean value theorem, since $\frac{d}{du}\phi(r(u))$ is bounded on $\mathbb{R}$, $\gamma_n < \xi_n < \gamma_{n+1}$ and $\gamma_{n+1} - \gamma_n \ll (\log \gamma_n)^{-1}$. Therefore,

$$\frac{1}{N_\omega(T)} \sum_{0<\gamma_n\le T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) = \frac{1}{N_\omega(T)} \sum_{0<\gamma_n\le T} \phi\Big(-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+\omega+i\gamma_n)\Big) + O\Big(\frac{1}{N_\omega(T)} \sum_{0<\gamma_n\le T} \frac{\log\log\gamma_n}{\log\gamma_n}\Big) + O\Big(\frac{1}{N_\omega(T)} \sum_{0<\gamma_n\le T} E_{2,\omega}(\gamma_n)\Big).$$

By the Stieltjes integral and integration by parts, we get

$$\sum_{0<\gamma_n\le T} \frac{\log\log\gamma_n}{\log\gamma_n} = \int_{\gamma_1}^{T} \frac{\log\log t}{\log t}\, dN_\omega(t) \ll \int_{\gamma_1}^{T} \frac{\log\log t}{\log t}(\log t)\, dt \ll T\log\log T$$

and

$$\sum_{0<\gamma_n\le T} E_{2,\omega}(\gamma_n) = \int_{\gamma_1}^{T} E_{2,\omega}(t)\, dN_\omega(t) \ll \int_{\gamma_1}^{T} E_{2,\omega}(t)(\log t)\, dt \ll N_\omega(T)E_{2,\omega}(T).$$

Hence we obtain (3.3). ■

3.1. Proof of Theorem 1.1 under RH. Set $\sigma = 1/2+\omega$. By Propositions 3.1 and 3.2,

$$\frac{1}{N_\omega(T)} \sum_{0<\gamma_n\le T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) = \frac{1}{2T}\int_{-T}^{T} \phi\Big(-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it)\Big)\, dt + O\Big(\frac{\log\log T}{\log T}\Big) + O(E_{2,\omega}(T)) \tag{3.7}$$

for large $T>0$, since $\operatorname{Re}(\zeta'/\zeta)(\sigma+it)$ is an even function of $t\in\mathbb{R}$.

For any continuous and bounded function $\phi(x)$ on $\mathbb{R}$, $\phi(\operatorname{Re} z)$ is a continuous and bounded function on $\mathbb{C}$, because $z \mapsto \frac{1}{2}(z+\bar{z})$ is a continuous function from $\mathbb{C}$ into $\mathbb{R}$. Therefore, by applying formula (1.5) to $\Phi(z) = \phi\big(-\frac{1}{\pi}\operatorname{Re} z\big)$, we have

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \phi\Big(-\frac{1}{\pi}\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it)\Big)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi m_\sigma(\pi u)\phi(u)\, du,$$

since the m-function $m_\sigma(u)$ of (1.6) is even. Hence we obtain

$$\lim_{T\to\infty} \frac{1}{N_\omega(T)} \sum_{0<\gamma_n\le T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi m_\sigma(\pi u)\phi(u)\, du,$$

since $\lim_{T\to\infty} E_{2,\omega}(T) = 0$ for any fixed $\omega>0$. This implies (1.7) by $\gamma_n^{(2)} = \varrho_\omega^{-1/2}\ddot{\gamma}_n$. ■

3.2. Unconditional proof of Theorem 1.1. Let $X_\omega(s)$ be $A_\omega(s)$ or $B_\omega(s)$. We arrange the zeros $\rho = \beta + i\gamma$ of $X_\omega(s)$ with $\gamma > 0$ in a sequence $\rho_n = \beta_n + i\gamma_n$ so that $\gamma_{n+1} \geq \gamma_n$. Firstly, we recall that the number of zeros of $X_\omega(s)$ up to height T and outside the line $\sigma = 1/2$ is bounded by $T^{1-a\omega}(\log T)^2$ for any $a < 1$ [12, Theorem 1]. In addition, for given $0 < \delta < 1$ and $B > 0$, we can take an open subset $E \subset (0, \infty)$ such that

- the measure of $[T, 2T] \cap E$ is bounded by $T/(\log T)^B$ for every $T \geq 2$,
- the number of zeros of $X_\omega(1/2 + it)$ for $t \in [T, 2T] \cap E$ is bounded by $T/(\log T)^B$ for every $T \geq 2$,
- the zeros of $X_\omega(1/2 + it)$ for $t \in [T, 2T] \setminus E$ are simple,
- $[\gamma_n, \gamma_{n+1}] \subset [T, 2T] \setminus E$ if $\gamma_n \in [T, 2T] \setminus E$,
- $\gamma_{n+1} - \gamma_n = O(1/\log T)$ if $\gamma_n \in [T, 2T] \setminus E$,
- $S_\omega(t)$ is of class C^∞ in $(0, \infty) \setminus E$,
- the estimate

$$\frac{\zeta'}{\zeta}(1/2 + \omega + it) \ll (\log T)^{1-\delta} \tag{3.8}$$

  holds for $t \subset [T, 2T] \setminus E$,

by [12, Theorem 1] and the proof of [12, Theorem 2]. Therefore, we have

$$\lim_{T\to\infty} \frac{1}{N_\omega(T)} \sum_{0<\gamma_n\leq T} f(\gamma_n) = \lim_{T\to\infty} \frac{1}{N_\omega(T)} \sum_{\substack{0<\gamma_n\leq T \\ \gamma_n \notin E}} f(\gamma_n).$$

Using (3.8) instead of (2.5) to calculate the right-hand side, we obtain (3.1), (3.3) and (3.7) by replacing $E_{2,\omega}(t)$ by $(\log t)^{-\delta}$ in a way similar to the conditional proof of Theorem 1.1. Hence we obtain Theorem 1.1. ■

3.3. Proof of Theorem 1.2. Let μ_σ be the variance of $M_\sigma(z)$:

$$\mu_\sigma = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(z)|z|^2 \, du \, dv. \tag{3.9}$$

Then

$$\varrho_\omega = \frac{1}{2\pi^2} \, \mu_\sigma$$

for $\sigma = 1/2 + \omega$ by [5, (4.1.8), (4.2.1)] or [4, (1.2.17), (1.2.21)]. Thus, by using the Fourier inversion formula

$$M_\sigma(u + iv) = \frac{1}{2\pi} \int_{\mathbb{C}} \tilde{M}_\sigma(x + iy) e^{-i(xu+yv)} \, dx \, dy,$$

we obtain

$$\lim_{\omega\to 0^+} \pi\varrho_\omega^{1/2} m_{1/2+\omega}(\pi\varrho_\omega^{1/2}u) = \lim_{\omega\to 0^+} \frac{1}{2}\int_{-\infty}^{\infty} \mu_\sigma M_\sigma\left(\mu_\sigma^{1/2}\frac{u+iv}{\sqrt{2}}\right) dv$$
$$= \lim_{\omega\to 0^+} \int_{-\infty}^{\infty} \tilde{M}_\sigma(\sqrt{2}\,\mu_\sigma^{-1/2}x)e^{-ixu}\,dx.$$

The integrand of the right-hand side is estimated as

$$|\tilde{M}_\sigma(\sqrt{2}\,\mu_\sigma^{-1/2}z)| \le \exp(-\sqrt{2}\,|z|/8)$$

by [4, (2.4.2)] if σ is sufficiently close to 1/2. Therefore, by applying Lebesgue's convergence theorem to the right-hand side together with

$$\lim_{\sigma\to 1/2} \tilde{M}_\sigma(\mu_\sigma^{-1/2}z) = \exp(-|z|^2/4),$$

which is a special case of [4, Lemma A], we obtain

$$\lim_{\sigma\to 1/2} \int_{-\infty}^{\infty} \tilde{M}_\sigma(\sqrt{2}\,\mu_\sigma^{-1/2}x)e^{-ixu}\,dx = \int_{-\infty}^{\infty} \exp(-x^2/2)e^{-ixu}\,dx$$
$$= \sqrt{2\pi}\exp(-u^2/2).$$

This implies Theorem 1.2. ■

4. Concluding remarks. Before concluding the main part of the paper, we give several comments and remarks.

4.1. On the range of test functions. In order to extend the range of test functions in formula (1.7), we need to extend the range of test functions in (1.5). An optimistic expectation is that formula (1.5) holds for any continuous function $\Phi(z)$ on $\mathbb{C}$ and the characteristic function of any compact subset of $\mathbb{C}$ or the complement of such a subset if we assume RH. However, the range of test functions in (1.5) could be a much more delicate problem. In fact, if we apply (1.5) formally to the test function $\Phi(w) = |w|^2$ together with (A.3) below, we obtain

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt = \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}} = \mu_\sigma.$$

This agrees with the asymptotic formula

$$\frac{1}{T}\int_{0}^{T} \left|\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt \sim \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}}$$

for $(\sigma - 1/2)\log T \to \infty$, which is a consequence of the estimate $S(T) = O(\log T/\log\log T)$ of Selberg [14, (1.2)], where $f \sim g$ means that the ratio

f/g tends to one. It is easy to see that $\mu_\sigma \sim 1/(2\sigma-1)^2$ as $\sigma \to 1/2$. Thus, we obtain the asymptotic formula

$$\frac{1}{T}\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{a}{\log T}+it\right)\right|^2 dt \sim \frac{1}{4a^2}(\log T)^2$$

as $a \to \infty$ and $T \to \infty$ with $a = o(\log T)$. On the other hand, Goldston–Gonek–Montgomery [1] discovered that, assuming RH,

$$\frac{1}{T}\int_0^T \left|\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\frac{a}{\log T}+it\right)\right|^2 dt \sim \frac{1-e^{-2a}}{4a^2}(\log T)^2$$

as $T \to \infty$ for any fixed $a > 0$ is equivalent to the Montgomery–Odlyzko conjecture. The above facts do not contradict each other, but they suggest a need for a careful consideration of the range of test functions when σ is close to $1/2$.

4.2. On the second normalization. Applying (1.5) formally to the test function $\Phi(w) = |\mathrm{Re}\, w|^2 = w^2 + 2w\bar{w} + \bar{w}^2$ together with (A.3) below, we obtain

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\mathrm{Re}\,\frac{\zeta'}{\zeta}(\sigma+it)\right|^2 dt = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\Lambda(n)^2}{n^{2\sigma}}.$$

Therefore, by (2.1), (2.2) and (3.6), we find that the average size of $\ddot{\gamma}_{n+1}-\ddot{\gamma}_n$ is $\varrho_{\sigma-1/2}$ in spite of (2.5). This is the reason for the normalizing factor $\varrho_\omega^{-1/2}$ of (1.4). The factor $(1/(2\pi))\log(\gamma_n/(2\pi e))$ of (1.4) is a kind of technical adjustment to establish a bridge between the nearest neighbor spacing of normalized zeros and the M-function.

4.3. On a relation to the Montgomery–Odlyzko conjecture. The functions $A_\omega(s)$ and $B_\omega(s)$ are holomorphic in (ω, s) as functions of two complex variables, and all their zeros are simple under RH if ω is a non-zero real number. Hence the sets $\{\gamma_n(\omega) \,|\, \omega > 0\}$ of the imaginary parts of the nth zeros are analytic loci in $(0,\infty)\times(0,\infty)$, and they do not intersect each other. Moreover, assuming the simplicity of zeros of $\xi(s)$, we have $\lim_{\omega\to 0}\gamma_{n+1}(\omega) \neq \lim_{\omega\to 0}\gamma_n(\omega)$ for each $n \geq 1$. Therefore, we expect that the distribution of $\gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega)$ approximates well the distribution of the nearest neighbor spacings $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0)$ if $\omega > 0$ is small enough. In this sense, the distribution of $\ddot{\gamma}_{n+1}(\omega) - \ddot{\gamma}_n(\omega)$ should approximate the distribution of $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0) - 1$ up to a correction factor, since

$$\ddot{\gamma}_{n+1}(\omega) - \ddot{\gamma}_n(\omega) \sim (\gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega) - 1)\frac{1}{2\pi}\log\frac{\gamma_n(\omega)}{2\pi e}$$

for large n by (3.5). Moreover, we have

$$\gamma_{n+1}^{(2)}(\omega) - \gamma_n^{(2)}(\omega) \sim \gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega) - 1$$

when $\sqrt{2}\,\omega \log \gamma_n(\omega) \sim 1$ as $\omega \to 0^+$, since $\rho_\omega \sim 1/(8\pi^2\omega^2)$ as $\omega \to 0^+$. Therefore, for small $\omega > 0$, the distribution of $\gamma_{n+1}^{(2)}(\omega) - \gamma_n^{(2)}(\omega)$ around the height $\exp(1/\omega)$ approximates the -1 shift of the nearest neighbor spacing distribution of $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0) - 1$ in the same range. Conversely, the distribution of $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0) - 1$ around a height $T > 0$ is approximated by the distribution of $\gamma_{n+1}^{(2)}(\omega) - \gamma_n^{(2)}(\omega)$ for $\omega \sim 1/(\sqrt{2}\log T)$.

However, the limit of the density function in Theorem 1.2 is quite different from a shift of the density function $p(u)$ of the nearest neighbor spacing distribution for GUE predicted in the Montgomery–Odlyzko conjecture, where we recall that the density function $p(u)$ is well approximated by the Wigner surmise $p_W(u) = (32u^2/\pi^2)\exp(-4u^2/\pi)$ for GUE [9, Appendix]. In order to fill this gap, we may need a detailed study of the second error term of (3.7), which tends to $O(1)$ as $\omega \to 0^+$, and the effect of the normalizing factor ϱ_ω of (1.4).

4.4. On a possible generalization. Let $L(s, f)$ be a self-dual L-function in the sense of Iwaniec–Kowalski [7, Chap. 5] which includes Dedekind zeta-functions, Dirichlet L-functions associated to real primitive characters, Hecke L-functions associated to self-dual Hecke characters, automorphic L-functions associated to self-dual primitive holomorphic/Maass cusp forms, etc. For such an L-function, the family of functions $A_\omega(s, f)$ and $B_\omega(s, f)$ corresponding to (1.1) is defined as well, and it is established in a way similar to [11] that the distribution of spacings of the normalized imaginary parts of the zeros of $A_\omega(s, f)$ and $B_\omega(s, f)$ converges to a limiting distribution of equal spacings of length one if we assume the Grand Riemann Hypothesis and the Ramanujan–Petersson conjecture for $L(s, f)$. A key ingredient is an analogue of (2.5) and other standard analytic properties of L-functions (see [7, Chap. 5]). Therefore, an analogue of the second normalization (1.4) is defined as well.

However, an analogue of the M-function $M_\sigma(z)$ is not known except for the case of Dedekind zeta functions. It is an interesting problem to find an analogue of $M_\sigma(z)$ for $L(s, f)$; however, it is not obvious what it is, even if it may not be hard to find an analogue of $M_\sigma(z)$ in a way similar to [3] for degree one L-functions like Dirichlet/Hecke L-functions for real/self-dual characters.

Appendix. *M*-function. In this part, we review the construction and basic properties of the M-function $M_\sigma(z)$ in formula (1.5) according to Ihara [3, 4] and Ihara–Matsumoto [5]. See these references for details.

Let $\Lambda : \mathbb{N} \to \mathbb{R}$ be the von Mangoldt function, that is, $\Lambda(n) = \log p$ if $n = p^k$ for some prime number p and integer $k \geq 1$, and $\Lambda(n) = 0$ otherwise. We define arithmetic functions $\Lambda_k : \mathbb{N} \to \mathbb{R}$ by

$$\left(-\frac{\zeta'}{\zeta}(s)\right)^k = \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right)^k = \sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s}$$

for $k \geq 1$, and $\Lambda_0(n) = 1$ if $n = 1$, and $\Lambda_0(n) = 0$ otherwise. For a positive integer n and $z \in \mathbb{C}$, we define

$$\lambda_z(n) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(n)}{k!} z^k.$$

The series converges absolutely and uniformly on every compact subset of $\mathbb{C}$, and it is a polynomial of z by [3, (3.8.5), (3.8.6)]. Moreover, we have

$$\lambda_z(mn) = \lambda_z(m)\lambda_z(n) \quad \text{if } (m, n) = 1 \tag{A.1}$$

(see [3, Prop. 3.8.11(i)]). For a prime number p and complex numbers $s, z \in \mathbb{C}$, we define

$$\tilde{M}_{s,p}(z) = \sum_{j=0}^{\infty} \frac{\lambda_{p^j}(z)\lambda_{p^j}(\bar{z})}{p^{2js}}.$$

The series converges absolutely for all s with $\operatorname{Re} s > 0$ and z in a compact subset of $\mathbb{C}$ by [3, Prop. 3.9.4(i)]. Using $\tilde{M}_{s,p}(z)$, we define $\tilde{M}_s(z)$ by the Euler product

$$\tilde{M}_s(z) = \prod_p \tilde{M}_{s,p}(z), \tag{A.2}$$

where p runs over all prime numbers. The product converges for all s with $\operatorname{Re} s > 1/2$ and z in a compact subset of $\mathbb{C}$ [3, Theorem 5]. We have the Dirichlet series expansion

$$\tilde{M}_s(z) = \sum_{n=1}^{\infty} \frac{\lambda_z(n)\lambda_{\bar{z}}(n)}{n^{2s}}$$

by (A.1), and the series on the right-hand side converges absolutely for all s with $\operatorname{Re} s > 1/2$ and z in a compact subset of $\mathbb{C}$ by [3, Prop. 3.9.4(ii)].

For $\sigma > 1/2$ and $z \in \mathbb{C}$, $\tilde{M}_\sigma(z)$ is a real analytic function of σ and z which does not vanish identically, and satisfies $\tilde{M}_\sigma(z) = \tilde{M}_\sigma(\bar{z}) = \overline{\tilde{M}_\sigma(-\bar{z})}$ and $\tilde{M}_\sigma(z) = O((1+|z|)^{-n})$ for any $n \geq 1$. The M-function in formula (1.5) is defined by the Fourier transform

$$M_\sigma(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)\, dw,$$

where $\psi_z(w) = \exp(i \cdot \operatorname{Re}(\bar{z}w))$. In addition, the M-function is real valued, decays rapidly as $|z| \to \infty$, and the Fourier inversion formula

$$\tilde{M}_\sigma(z) = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)\, dw$$

holds with $\tilde{M}_\sigma(0) = 1$ [3, Theorems 2 and 3, Remark 3.4.6]. In particular, $(2\pi)^{-1}M_\sigma(w)dw$ is a probability measure on $\mathbb{C}$. Corresponding to the Euler product (A.2), the M-function has a convolution Euler product whose p-factor is a certain distribution.

We have

$$\frac{1}{2\pi}\int_{\mathbb{C}} w^a \bar{w}^b M_\sigma(w)\,dw = \sum_{n=1}^{\infty} \frac{\Lambda_a(n)\Lambda_b(n)}{n^{2\sigma}} \tag{A.3}$$

unconditionally together with the absolute convergence of the series if $\sigma > 1$ [3, Theorem 6]. Moreover, we have the limit formula

$$\lim_{\sigma\to 1/2} \mu_\sigma M_\sigma(\mu_\sigma^{1/2} z) = 2e^{-|z|^2},$$

and the convergence is uniform on $|z| \le R$ for any $R > 0$, where μ_σ is the variance in (3.9) [4, Theorem 2]. Theorem 2 is a formal consequence of this formula.

Historically, formula (1.5) was first obtained in 1936 by Kershner–Wintner [10] for $\sigma > 1/2$ in terms of asymptotic distribution functions as an analogue of a 1935 work of Jessen–Wintner for $\log\zeta(s)$. However, they did not explicitly give the density function. The density function $M_\sigma(z)$ was constructed in 1937 by van Kampen–Wintner [8] for $\sigma > 1$ as an infinite convolution Euler product. Then formula (1.5) was rediscovered by Guo [2] in 1993. He constructed $M_\sigma(z)$ for $\sigma > 1/2$ as the Fourier transform of the Euler product $\prod_p \tilde{M}_{\sigma,p}(z)$ but with the test functions in (1.5) restricted to smooth and compactly supported functions. This restriction was relaxed by Ihara–Matsumoto [6] in 2011 which was the culmination of a series of works of Ihara and Matsumoto based on Ihara [3]. In 2008, Ihara [3] studied analytic and arithmetic properties of $M_\sigma(z)$ and $\tilde{M}_\sigma(s)$ systematically and in detail for $\sigma > 1/2$, motivated by a study on Euler–Kronecker constants of global fields. This work was refined in Ihara [4]. The formulation of (1.5) in the introduction depends on [3, Theorem 6] and [6].

Acknowledgements. The author would like to thank Lin Weng and Kohji Matsumoto for their interest and valuable comments on this work. The author also thanks the referee for helpful comments. The author is supported by KAKENHI (Grant-in-Aid for Young Scientists (B)) No. 25800007.

References

[1] D. A. Goldston, S. M. Gonek and H. L. Montgomery, *Mean values of the logarithmic derivative of the Riemann zeta-function with applications to primes in short intervals*, J. Reine Angew. Math. 537 (2001), 105–126.

[2] C. R. Guo, *The distribution of the logarithmic derivative of the Riemann zeta function*, Proc. London Math. Soc. (3) 72 (1996), 1–27.

[3] Y. Ihara, *On "M-functions" closely related to the distribution of L'/L-values*, Publ. RIMS Kyoto Univ. 44 (2008), 893–954.

[4] Y. Ihara, *On certain arithmetic functions $\tilde{M}(s; z_1, z_2)$ associated with global fields: Analytic properties*, Publ. RIMS Kyoto Univ. 47 (2011), 257–305.

[5] Y. Ihara and K. Matsumoto, *On $\log L$ and L'/L for L-functions and the associated "M-functions": Connections in optimal cases*, Moscow Math. J. 11 (2011), 73–111.

[6] Y. Ihara and K. Matsumoto, *On the value-distribution of logarithmic derivatives of Dirichlet L-functions*, in: Analytic Number Theory, Approximation Theory, and Special Functions, in Honor of H. M. Srivastava (G. V. Milovanović and M. Th. Rassias, eds.), Springer, 2014, 79–91.

[7] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. 53, Amer. Math. Soc., Providence, RI, 2004.

[8] E. R. van Kampen and A. Wintner, *Convolutions of distributions on convex curves and the Riemann zeta function*, Amer. J. Math. 59 (1937), 175–204.

[9] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues, and Monodromy*, Amer. Math. Soc. Colloq. Publ. 45, Amer. Math. Soc., Providence, RI, 1999.

[10] R. Kershner and A. Wintner, *On the asymptotic distribution of $\zeta'/\zeta(s)$ in the critical strip*, Amer. J. Math. 59 (1937), 673–678.

[11] J. C. Lagarias, *Zero spacing distributions for differenced L-functions*, Acta Arith. 120 (2005), 159–184.

[12] X. Li, *Variation of the argument of the Riemann ξ function on vertical lines*, Acta Arith. 137 (2009), 277–284.

[13] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Stud. Adv. Math. 97, Cambridge Univ. Press, Cambridge, 2007.

[14] A. Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$*, Avh. Norske Vid. Akad. Oslo. I. 1944, no. 1, 1–27.

[15] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, Oxford Univ. Press, New York, 1986.

[16] L. Weng, *Distributions of zeros of zeta functions*, personal communication, April 2014.

[17] L. Weng, *Distributions of zeros for non-abelian zeta functions*, preprint, http://www2.math.kyushu-u.ac.jp/˜weng/zetazeros.pdf, 2014.

[18] L. Weng and D. Zagier, *Higher rank zeta functions and Riemann hypothesis for elliptic curves*, preprint, http://www2.math.kyushu-u.ac.jp/˜weng/ECRH.pdf, 2013.

Masatoshi Suzuki
Department of Mathematics
Tokyo Institute of Technology
2-12-1 Ookayama, Meguro-ku
Tokyo 152-8551, Japan
E-mail: msuzuki@math.titech.ac.jp

Received on 9.10.2014
and in revised form on 28.1.2015 (7954)