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Spacing of zeros of Hecke L-functions and the class number problem

by

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1. A bit of history and results. The group of ideal classes $\mathcal{C}\ell(K)$ of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-q})$ is the most fascinating finite group in arithmetic. Here we are faced with one of the most challenging problems in analytic number theory, that is, to estimate the order of the group $h = |\mathcal{C}\ell(K)|$. C. F. Gauss conjectured (in a parallel setting of binary quadratic forms) that the class number $h = h(-q)$ tends to infinity as $-q$ runs over the negative discriminants. Hence there are only a finite number of imaginary quadratic fields with a given class number. But how many of these fields are there exactly for $h = 1$, or $h = 2$, etc.? To answer this question one needs an effective lower bound for h in terms of q (a fast computer could be helpful as well).

The problem was linked early on to the L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \tag{1.1}$$

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for the real character χ of conductor q (the Kronecker symbol)

$$\chi(n) = \left(\frac{-q}{n}\right). \tag{1.2}$$

In this connection P. G. L. Dirichlet established the formula

$$h = \pi^{-1}\sqrt{q}\, L(1,\chi) \tag{1.3}$$

(we assume that $-q$ is a fundamental discriminant, $q > 4$, so there are two units ± 1 in the ring of integers $\mathcal{O}_K \subset K$). Rather than estimating the class number, Dirichlet inferred from (1.3) that $L(1,\chi)$ does not vanish, which property he needed to establish the equidistribution of primes in arithmetic progressions. Truly the lower bound

$$L(1,\chi) \geq \pi/\sqrt{q} \tag{1.4}$$

follows from (1.3), because $h \geq 1$.

The Grand Riemann Hypothesis for $L(s,\chi)$ implies

$$(\log\log q)^{-1} \ll L(1,\chi) \ll \log\log q, \tag{1.5}$$

whence the class number varies only slightly about $\sqrt{q}$:

$$\sqrt{q}/\log\log q \ll h \ll \sqrt{q}\log\log q. \tag{1.6}$$

But sadly enough we may not see proofs of such estimates (which are best possible in order of magnitude) in the near future.

At present we know that $L(s,\chi) \neq 0$ for $s = \sigma + it$ in the region

$$\sigma > 1 - c/\log q(|t|+1) \tag{1.7}$$

where c is a positive constant, for any character $\chi \pmod q$ with at most one exception. The exceptional character $\chi \pmod q$ is real and the exceptional zero of $L(s,\chi)$ in the region (1.7) is real and simple, say β if it exists, with

$$\beta > 1 - c/\log q. \tag{1.8}$$

Using complex function theory one can translate various zero-free regions of $L(s,\chi)$ which are stretched along the line $\operatorname{Re} s = 1$ to lower bounds for $|L(s,\chi)|$ on the line $\operatorname{Re} s = 1$. In the case of a real character (1.2), H. Hecke (see [L1]) showed that if $L(s,\chi)$ has no exceptional zero, then $L(1,\chi) \gg (\log q)^{-1}$, whence

$$h \gg \sqrt{q}\,(\log q)^{-1} \tag{1.9}$$

by the Dirichlet formula (1.3). Moreover if $L(s,\chi)$ does have an exceptional zero $s = \beta$ satisfying (1.8), then we have quite precise relations between β and h (see [GSc], [G1], [GS]). In particular one can derive from the Dirichlet estimate (1.4) that

$$\beta \leq 1 - c/\sqrt{q}. \tag{1.10}$$

Back to the history we should point out that E. Landau [L1] first came up with ideas which pushed the exceptional zero further to the left of (1.10). Generalizing slightly in this context we owe to Landau the product (a quadratic lift L-function)

$$\mathrm{Lan}(s,f)=L(s,f)L(s,f\otimes\chi)=\sum_n a_f(n)n^{-s} \tag{1.11}$$

where

$$L(s,f)=\sum_n \lambda_f(n)n^{-s}$$

can be any decent L-function and $L(s,f\otimes\chi)$ is derived from $L(s,f)$ by twisting its coefficients $\lambda_f(n)$ with $\chi(n)$. If $L(s,f)$ does have an Euler product so do $L(s,f\otimes\chi)$ and $\mathrm{Lan}(s,f)$. The key point is that the prime coefficients of $\mathrm{Lan}(s,f)$ are

$$a_f(p)=\lambda_f(p)(1+\chi(p)).$$

Assuming the class number h is small (or equivalently that $L(s,\chi)$ has an exceptional zero) we find that $\chi(p)=-1$ and $a_f(p)=0$ quite often if $p\ll\sqrt{q}$. In other words $\chi(m)$ pretends to be the Möbius function $\mu(m)$ on squarefree numbers. Therefore, under this ficticious assumption, $L(s,f\otimes\chi)$ approximates $L(s,f)^{-1}$, and $\mathrm{Lan}(s,f)$ behaves like a constant (no matter what s and f are!).

Landau worked with $\mathrm{Lan}(s,\chi')=L(s,\chi')L(s,\chi\chi')$ where $\chi'\ (\mathrm{mod}\,q')$ is any real primitive character other than $\chi\ (\mathrm{mod}\,q)$. He [L1] proved that for any real zeros β,β' of $L(s,\chi),L(s,\chi')$ respectively,

$$\min(\beta,\beta')\le 1-c/\log qq'. \tag{1.12}$$

This shows that the exceptional zero occurs very rarely (if at all?).

Next a repulsion property of the exceptional zero was discovered, notably in the works by M. Deuring [D] and H. Heilbronn [H]. This says: the closer β is to the point $s=1$ the further away from $s=1$ are the other zeros, not only of $L(s,\chi)$, but of any L-function for a character of comparable conductor. The power of repulsion is masterly exploited in the celebrated work of Yu. V. Linnik [L] on the least prime in an arithmetic progression.

A cute logical play with repulsion led E. Landau [L2] to the lower bound

$$h\gg q^{1/8-\varepsilon} \tag{1.13}$$

for any $\varepsilon>0$, the implied constant depending on ε. Slightly later by the same logic, but with more precise estimates for relevant series, C. L. Siegel [S] proved that

$$h\gg q^{1/2-\varepsilon}. \tag{1.14}$$

Both estimates suffer from the serious defect of having the implied constant not computable. For that reason the Landau–Siegel estimates do not help to determine all quadratic imaginary fields with a fixed class number. The case $h = 1$ was eventually solved by arithmetical means (complex multiplication) by K. Heegner [He] and H. M. Stark [S1] and by transcendental means (linear forms in logarithms) by A. Baker [B] (see also the notes [S2] about earlier attempts by A. O. Gelfond and Yu. V. Linnik [GL]).

By way of the repulsion one may still hope to produce effective results provided an "exceptional" zero is given numerically. But, believing in GRH one cannot expect to find a real zero of any decent L-function other than at the central point $s = 1/2$. Hence the question: Does the central zero have an effect on the class number? Yes it does, and the impact depends on the order of the zero. This effect was first revealed in conversations by J. Friedlander in the early 70's. Soon after J. V. Armitage gave an example of the zeta function of a number field which vanishes at the central point, Friedlander [F] succeeded in estimating effectively the class number of relative quadratic extensions. Then D. Goldfeld [G2] went quite further by employing L-functions of elliptic curves. These L-functions are suspected to have central zero of order equal to the rank of the group of rational points on the curve (the Birch and Swinnerton-Dyer conjecture). Subsequently B. Gross and D. Zagier [GZ] provided an elliptic curve of analytic rank three which completed Goldfeld's work with the estimate

$$h \gg \prod_{p|q} (1 - 1/\sqrt{p})^2 \log q. \tag{1.15}$$

This is the first and so far the only unconditional estimate (apart from the implied constant, see [O]) which shows that $h \to \infty$ effectively. Recently P. Sarnak and A. Zaharescu [SZ] used the same elliptic curve to show that $h \gg q^{1/10}$ with an effective constant. However their result is conditional; they assume (among a few minor restrictions on q) that $\mathrm{Lan}_E(s) = L_E(s) L_E(s, \chi)$ has no complex zeros off the critical line, whereas the real zeros can be anywhere.

After having exploited the power of the central zero it seems promising to focus on the critical line and ask if some clustering of zeros has any effect on the class number. In fact this possibility was contemplated in the literature independently of the central zero effects. In this paper we derive quite strong and effective lower bounds for h, though conditionally subject to the existence of many small (subnormal) gaps between zeros of the L-function associated with a class group character. Let

$$L(s, \psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})(N\mathfrak{a})^{-s} = \sum_n \lambda(n) n^{-s} \tag{1.16}$$

for $\psi \in \widehat{\mathcal{C}\ell}(K)$, where $\mathfrak{a}$ runs over the non-zero integral ideals. This Hecke L-function does not factor as the Landau product (1.11) (unless ψ is a genus character), yet the crucial feature—the lacunarity of the coefficients

$$\lambda(n) = \sum_{N\mathfrak{a}=n} \psi(\mathfrak{a}) \tag{1.17}$$

—appears if the class number is assumed to be relatively small. One can show that the number of zeros of $L(s,\psi)$ in the rectangle $s=\sigma+it$ with $0\le\sigma\le 1,\ 0<t\le T$ satisfies

$$N(T,\psi) = \frac{T}{\pi}\log\frac{T\sqrt{q}}{2\pi} - \frac{T}{\pi} + O(\log qT). \tag{1.18}$$

This indicates (assuming GRH for $L(s,\psi)$) that the average gap between consecutive zeros $\varrho = 1/2+i\gamma$ and $\varrho' = 1/2+i\gamma'$ is about $\pi/\log\gamma$.

We prove that if the gap is somewhat smaller than the average for sufficiently many pairs of zeros on the critical line (no Riemann hypothesis is required) then $h \gg \sqrt{q}\,(\log q)^{-A}$ for some constant $A>0$. Actually we establish various more general results among which are the following two theorems. Let $\varrho = 1/2+i\gamma$ denote the zeros of $L(s,\psi)$ on the critical line and $\varrho' = 1/2+i\gamma'$ denote the nearest zero to ϱ on the critical line (we assume that $\varrho'\ne\varrho$ except when ϱ is a multiple zero, in which case $\varrho'=\varrho$). Note that we do not count zeros off the critical line, but we allow them to exist. For $0<\alpha\le 1$ and $T\ge 2$ we put

$$D(\alpha,T) = \#\Big\{\varrho;\ 2\le\gamma\le T,\ |\gamma-\gamma'|\le\frac{\pi(1-\alpha)}{\log\gamma}\Big\}. \tag{1.19}$$

Theorem 1.1. *Let $A\ge 0$ and $\log T\ge(\log q)^{A+6}$. Suppose*

$$D(\alpha,T) \ge \frac{cT\log T}{\alpha(\log q)^A} \tag{1.20}$$

for some $0<\alpha\le 1$, where c is a large absolute constant. Then

$$L(1,\chi) \ge (\log T)^{-2}(\log q)^{-2A-6}. \tag{1.21}$$

This result is a special case of Proposition 10.1. Taking $A=12$ and $\log T = (\log q)^{18}$ we get $L(1,\chi)\ge(\log q)^{-66}$, provided $D(\alpha,T)\ge\alpha^{-1}cT(\log T)^{1/3}$.

An interesting case is $\zeta_K(s)=\zeta(s)L(s,\chi)$ (where $\zeta(s)$ is the Riemann zeta function), that is, the case of the trivial class group character. Since we do not need all the zeros we choose only those of $\zeta(s)$ and state the conditions in absolute terms (without mentioning the conductor q, see Corollary 10.2).

Theorem 1.2. *Let $\varrho=1/2+i\gamma$ be the zeros of $\zeta(s)$ on the critical line and $\varrho'=1/2+i\gamma'$ be the nearest zero to ϱ on the critical line ($\varrho'=\varrho$ if ϱ is multiple). Suppose*

$$\#\Big\{\varrho;\ 0<\gamma\le T,\ |\gamma-\gamma'|\le\frac{\pi}{\log\gamma}\Big(1-\frac{1}{\sqrt{\log\gamma}}\Big)\Big\} \gg T(\log T)^{4/5} \tag{1.22}$$

for any $T \geq 2001$. *Then*

$$L(1,\chi) \gg (\log q)^{-90} \tag{1.23}$$

where the implied constant is effectively computable.

Many other results can be inferred from Proposition 10.1. We selected our points ϱ, ϱ' from zeros on the critical line. However it is not hard to include other zeros in the critical strip, or even points where $L(s,\psi)$ or even $L'(s,\psi)$ is small. As an illustration, the following assertion follows immediately from Proposition 10.1.

Corollary 1.3. *Suppose there are points* $2 \leq t_1 < \ldots < t_R \leq T$ *with* $t_{r+1} - t_r \geq 1$ *such that*

$$|L'(1/2 + it_r, \psi)| \leq (\log q)^{7/2}$$

for $r = 1, \ldots, R$, *where* $R \gg T = \exp(\log q)^6$. *Then*

$$L(1,\chi) \gg (\log q)^{-18}.$$

Considerations of Random Matrix Theory (see [Hu]) suggest that the hypothesis above is likely to be achieved.

Many sections of this paper are valid for arbitrary points in the strip (not necessarily zeros of $L(s,\psi)$); it is only in the last four sections that we select the points on the line $\operatorname{Re} s = 1/2$ to simplify the arguments. Thus, if (1.22) were established unconditionally for pairs of zeros $\varrho = \beta + i\gamma$, $\varrho' = \beta' + i\gamma'$, which may or may not be on the critical line, then (1.23) would hold. On the other hand one should be careful of charging the Riemann hypothesis. Although Theorem 1.2 does not require the Riemann hypothesis, we can imagine that someone shows the condition (1.22) using the Riemann hypothesis. In this scenario one still cannot conclude an unconditional, effective bound (1.23).

Note that the average gap between consecutive zeros of $\zeta(s)$ is $2\pi/\log\gamma$, so we count in (1.22) the gaps which are slightly smaller than half the average. In view of the implications for the class number, one has a good reason to search for small gaps between zeros of $\zeta(s)$. This task was undertaken long ago. Let us assume the Riemann hypothesis for $\zeta(s)$. First H. L. Montgomery [M] showed that

$$|\gamma - \gamma'| < \frac{2\pi\theta}{\log\gamma} \tag{1.24}$$

infinitely often with $\theta = 0.68$. This was subsequently lowered to $\theta = 0.5179$ by Montgomery and Odlyzko [MO], to $\theta = 0.5171$ by Conrey, Ghosh and Gonek [CGG], and to $\theta = 0.5169$ by Conrey and Iwaniec (work in progress). We doubt that the current technology is capable to reduce (1.24) down to $\theta = 1/2$. Nevertheless it is an attractive and realistic proposition to get (1.24) with any $\theta > 1/2$.

The well justified Pair Correlation Conjecture (PCC) of H. L. Montgomery [M] does imply (1.24) with any $\theta > 0$ for a positive density of zeros. Precisely one expects that

$$\#\left\{m \neq n;\ 0 < \gamma_m, \gamma_n \leq T,\ \frac{2\pi\alpha}{\log T} < \gamma_m - \gamma_n < \frac{2\pi\beta}{\log T}\right\} \sim \frac{T}{2\pi}(\log T)\int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du$$

as $T \to \infty$, for any fixed $\beta > \alpha$.

Remarks. Montgomery says he was led to formulate the PCC when looking for small gaps between zeros of $\zeta(s)$ in connection to the class number problem. We cannot guess how precise the connection he established at that time. However, Montgomery did publish a joint paper with P. J. Weinberger [MW] in which they used zeros of fixed real L-functions close to the central point $s = 1/2$ to derive explicit estimates and to perform extensive numerical computations for the imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-q})$ with the class number $h = 1, 2$.

In a similar fashion M. Jutila [J] considered a large family of Dirichlet L-functions $L(s, \chi)$ for all $\chi \pmod k$ with $k \leq X$, and he showed that their zeros near the central point tend to form an arithmetic progression if the class number of $K = \mathbb{Q}(\sqrt{-q})$ is relatively small. Our principal idea in this paper is reminiscent of that of Jutila. However, as we are dealing with a single L-function (no averaging over characters) our arguments are quite intricate, especially when we have to deal with the off-diagonal terms in the mean value of $|L(1/2 + it, \psi)|^2$ (see Theorem 6.1). Jutila's arguments do not go that far.

Our general result in Proposition 10.1 would also imply approximate periodicity in the distribution of most of the zeros of $\zeta(s)$ (which is inherited from oscillation of the root number (7.26)), if we have assumed that the class number was small. This clearly violates the distribution law of zeros according to the PCC.

At the meeting in Seattle of August 1996 R. Heath-Brown gave a lecture "Small Class Number and the Pair Correlation of Zeros" in which he communicated results (still unpublished) some of which are similar to ours, yet they are more restrictive. Heath-Brown requires $L(1, \chi) \ll q^{-\delta}$ for some constant $\delta > 1/4$, which condition contradicts the Siegel bound $L(1, \chi) \gg q^{-\varepsilon}$, but his arguments are effective so the results remain valid.

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Note added in May 2001. We found our results in Section 3 and Section 4 to be similar to these in Appendix A and Appendix B of the paper "Rankin–Selberg L-functions in the level aspect" by E. Kowalski, P. Michel and J. Vanderkam (to appear). Had we known their results earlier we would gladly incorporate them to reduce our arguments. However, we decided not to modify our original parts to preserve the self-contained presentation.

2. Basic automorphic forms. We are mainly interested in L-functions for characters on ideals in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-q})$. Every such L-function is associated with a holomorphic automorphic form of level q and the real primitive character $\chi \pmod q$ (the Nebentypus). However, to get better perspective, we begin by reviewing the whole spectrum of real-analytic forms. In particular we focus on the Eisenstein series, because they are most important automorphic forms for our applications to Dirichlet L-functions (they correspond to genus characters of the class group of K). Some more details and proofs can be found in [I] and [DFI2].

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper-half plane $\mathbb{H}$ by the linear fractional transformations $\gamma z = (az+b)/(cz+d)$ if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. We put

$$j_\gamma(z) = \frac{cz+d}{|cz+d|}.$$

Note that $j_{\beta\gamma}(z) = j_\beta(\gamma z) j_\gamma(z)$. Next we fix a positive integer k and put

$$J_\gamma(z,s) = j_\gamma^{-k}(z)(\mathrm{Im}\,\gamma z)^s = \left(\frac{cz+d}{|cz+d|}\right)^{-k}\left(\frac{y}{|cz+d|^2}\right)^s$$

for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, $z \in \mathbb{H}$ and $s \in \mathbb{C}$. Note that $J_{\beta\gamma}(z,s) = j_\gamma^{-k}(z) J_\beta(z,s)$. Since $J_\gamma(z,s)$ depends only on the lower row (c,d) of γ we shall write $J_{(c,d)}(z,s)$ in place of $J_\gamma(z,s)$. Actually $J_{(c,d)}(z,s)$ is defined by the last expression for any pair of real numbers c,d, not both zero. Note that for $u>0$ we have $J_{(uc,ud)}(z,s) = u^{-2s} J_{(c,d)}(z,s)$.

Throughout $\Gamma = \Gamma_0(q)$ denotes the Hecke congruence group of level q; its index in the modular group is

$$\nu(q) = [\Gamma_0(1) : \Gamma_0(q)] = q\prod_{p\mid q}(1+1/p). \tag{2.1}$$

To simplify the presentation (without compromising our applications) we restrict q to odd, squarefree numbers. Let $\chi = \chi_q$ be the real primitive character of conductor q, i.e. $\chi_q(n) = \left(\frac{n}{q}\right)$ is the Jacobi–Legendre symbol.

This induces a character on Γ by

$$\chi(\gamma)=\chi(d)\qquad \text{if } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma. \tag{2.2}$$

We are interested in the space $\mathcal{A}_k(\Gamma,\chi)$ of automorphic functions of weight $k\geq 1$ for the group Γ and character χ, i.e. the functions $f:\mathbb{H}\to\mathbb{C}$ satisfying

$$f(\gamma z)=\chi(\gamma)j_\gamma^k(z)f(z)\qquad \text{if } \gamma\in\Gamma. \tag{2.3}$$

We assume $\chi(-1)=(-1)^k$, as otherwise $\mathcal{A}_k(\Gamma,\chi)$ consists only of the zero function. The Laplace operator

$$\Delta_k=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-iky\frac{\partial}{\partial x}$$

acts on $\mathcal{A}_k^\infty(\Gamma,\chi)$, the subspace of smooth automorphic functions. Any $f\in\mathcal{A}_k^\infty(\Gamma,\chi)$ which is eigenfunction of Δ_k, say $(\Delta_k+\lambda)f=0$, is called a *Maass form* of eigenvalue λ.

Our primary examples of Maass forms are the Eisenstein series associated with cusps of Γ. Let $\Gamma_\mathfrak{a}=\{\gamma\in\Gamma;\ \gamma\mathfrak{a}=\mathfrak{a}\}$ be the stability group of the cusp $\mathfrak{a}$. There exists $\sigma_\mathfrak{a}\in\mathrm{SL}_2(\mathbb{R})$ such that $\sigma_\mathfrak{a}\infty=\mathfrak{a}$ and $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a}=\Gamma_\infty$, the group of translations by integers. We call $\sigma_\mathfrak{a}$ a *scaling matrix* of $\mathfrak{a}$. The *Eisenstein series* associated with $\mathfrak{a}$ is defined by

$$E_\mathfrak{a}(z,s)=\sum_{\gamma\in\Gamma_\mathfrak{a}/\Gamma}\chi(\gamma)J_{\sigma_\mathfrak{a}^{-1}\gamma}(z,s). \tag{2.4}$$

This series converges absolutely for $\mathrm{Re}\, s>1$, it does not depend on the choice of $\sigma_\mathfrak{a}$, nor on the choice of $\mathfrak{a}$ in its equivalence class. The Eisenstein series $E_\mathfrak{a}(z,s)$ is a Maass form of eigenvalue $\lambda=s(1-s)$.

Any cusp $\mathfrak{a}$ of $\Gamma=\Gamma_0(q)$ is equivalent to a rational point $1/v$, where v is a divisor of q (recall that q is squarefree). Put

$$q=uw \tag{2.5}$$

so w is the width of the cusp $\mathfrak{a}\sim 1/v$. As a scaling matrix of $\mathfrak{a}\sim 1/v$ we can choose

$$\sigma_\mathfrak{a}=\begin{pmatrix}\sqrt{w} & 0\\ v\sqrt{w} & 1/\sqrt{w}\end{pmatrix}.$$

Next, according to (2.5), we factor the character $\chi_q=\chi_v\chi_w$. Then the Eisenstein series (2.4) can be written explicitly as follows:

$$E_\mathfrak{a}(z,s)=\frac{1}{2w^s}\sum\sum_{(c,d)=1}\chi_v(d)\chi_w(-c)J_{(cv,d)}(z,s) \tag{2.6}$$

where c,d run over co-prime integers. Hence applying Poisson's summation one can derive a Fourier expansion of $E_\mathfrak{a}(z,s)$ (in terms of the Whittaker function) from which one can see (among other things) that $E_{\mathfrak{a}}(z,s)$ is meromorphic in the whole complex s-plane without poles in $\operatorname{Re} s \geq 1/2$ (see (7.12) and (7.13) of [DFI2]).

The Eisenstein series $E_{\mathfrak{a}}(z,s)$ on the line $\operatorname{Re} s = 1/2$ yield an eigenpacket of the continuous spectrum of Δ_k in the subspace $\mathcal{L}_k(\Gamma,\chi)$ of square-integrable functions $f(z) \in \mathcal{A}_k(\Gamma,\chi)$ with respect to the invariant measure $y^{-2}dxdy$. The continuous spectrum covers the segment $[1/4,\infty)$ with multiplicity $\tau(q)$ (the number of inequivalent cusps equals the number of divisors of q). Let $\mathcal{E}_k(\Gamma,\chi) \subset \mathcal{L}_k(\Gamma,\chi)$ be the subspace of the continuous spectrum (it is a linear space spanned by a kind of incomplete Eisenstein series). Let $\mathcal{C}_k(\Gamma,\chi)$ be the orthogonal complement of $\mathcal{E}_k(\Gamma,\chi)$ in $\mathcal{L}_k(\Gamma,\chi)$, so $\mathcal{L}_k(\Gamma,\chi) = \mathcal{E}_k(\Gamma,\chi) \oplus \mathcal{C}_k(\Gamma,\chi)$. The Laplace operator Δ_k acts on $\mathcal{C}_k(\Gamma,\chi)$, and it has an infinite, purely discrete spectrum in the segment $\left[\frac{k}{2}\left(1-\frac{k}{2}\right),\infty\right)$. In other words $\mathcal{C}_k(\Gamma,\chi)$ is spanned by square-integrable automorphic forms. These are characterized by vanishing at every cusp (because they are orthogonal to every incomplete Eisenstein series), and are called *Maass cusp forms*.

From now on we take only the Maass cusp forms $f(z)$ of the Laplace eigenvalue $\lambda = \frac{k}{2}\left(1-\frac{k}{2}\right)$, and if $k=1$ we also take the Eisenstein series $E_{\mathfrak{a}}(z,s)$ at $s=1/2$. All these forms come from the classical holomorphic forms of weight k; precisely we have

$$F(z) = y^{-k/2} f(z) \in S_k(\Gamma,\chi), \qquad E_{\mathfrak{a}}(z) = y^{-1/2} E_{\mathfrak{a}}(z,1/2) \in M_1(\Gamma,\chi).$$

For any $n \geq 1$ the *Hecke operator* T_n is defined on $M_k(\Gamma,\chi)$ by

$$(T_n F)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \chi(a) \left(\frac{a}{d}\right)^{k/2} \sum_{b \,(\mathrm{mod}\, d)} F\left(\frac{az+b}{d}\right).$$

There is a basis of $S_k(\Gamma,\chi)$ which consists of eigenforms of the Hecke operators T_n with $(n,q)=1$. Moreover, by the multiplicity-one property (which holds in $S_k(\Gamma,\chi)$ because χ is primitive of conductor equal to the level) these forms are automatically eigenfunctions of all T_n. Consequently, we may assume that

$$T_n F = \lambda_F(n) F \quad \text{for all } n \geq 1. \tag{2.7}$$

After a normalization of F by a suitable scalar the Hecke eigenvalues $\lambda_F(n)$ agree with the coefficients in the Fourier series

$$F(z) = \sum_{n=1}^{\infty} \lambda_F(n) n^{(k-1)/2} e(nz). \tag{2.8}$$

Such an F is called a *primitive cusp form* of *weight* k, *level* q and *character* χ.

One can show that the modified Eisenstein series $y^{-k/2}E_{\mathfrak{a}}(z,s)$ are also eigenfunctions of all the Hecke operators T_n (see Section 6 of [DFI2]), but we are only interested in $E_{\mathfrak{a}}(z) = y^{-1/2}E_{\mathfrak{a}}(z, 1/2)$. In this case $(k = 1,\ s = 1/2)$ we have

$$T_n E_{\mathfrak{a}} = \lambda_{\mathfrak{a}}(n) E_{\mathfrak{a}} \quad \text{for all } n \geq 1 \tag{2.9}$$

with

$$\lambda_{\mathfrak{a}}(n) = \sum_{n_1 n_2 = n} \chi_v(n_1)\chi_w(n_2). \tag{2.10}$$

Moreover the Hecke eigenvalues $\lambda_{\mathfrak{a}}(n)$ are proportional to the Fourier coefficients of $E_{\mathfrak{a}}(z)$; specifically we have (see [I] and [DFI2])

$$E_{\mathfrak{a}}(z) = \overline{\varepsilon}_v \frac{2i}{h} \sum_{n=0}^{\infty} \lambda_{\mathfrak{a}}(n) e(nz) \tag{2.11}$$

where $\varepsilon_v = \tau(\chi_v)/\sqrt{v}$, so $\varepsilon_v = 1$ or i according as $v \equiv 1$ or $3 \pmod 4$ and h is the class number of $K = \mathbb{Q}(\sqrt{-q})$,

$$h = \pi^{-1}\sqrt{q}\, L(1, \chi). \tag{2.12}$$

The zero coefficient is given by

$$\lambda_{\mathfrak{a}}(0) = \begin{cases} h/2 & \text{if } \mathfrak{a} \sim \infty, 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Our particular Eisenstein series $E_{\mathfrak{a}}(z)$ (recall that in this case we have $k = 1$ and $\chi_q(-1) = -1$ so $q \equiv 3 \pmod 4$) can be expressed by theta functions for ideal classes of $K = \mathbb{Q}(\sqrt{-q})$. For every class $\mathcal{A} \in \mathcal{C}\ell(K)$ we put

$$\theta_{\mathcal{A}}(z) = \frac{1}{2} + \sum_{\mathfrak{a} \in \mathcal{A}} e(zN\mathfrak{a}) \tag{2.14}$$

where $\mathfrak{a}$ runs over integral ideals in $\mathcal{A}$ and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$ (the number 2 stands for the number of units, we assume $q \neq 3$). This theta function is also given by

$$\theta_{\mathcal{A}}(z) = \frac{1}{2} \sum_m \sum_n e(z\varphi_{\mathcal{A}}(m, n)) \tag{2.15}$$

where $\varphi_{\mathcal{A}}(x, y) = ax^2 + bxy + cy^2$ is the corresponding quadratic form. Specifically we have $a > 0$, $(a, b, c) = 1, b^2 - 4ac = -q$ and

$$\mathfrak{a} = a\mathbb{Z} + \frac{b + i\sqrt{q}}{2}\,\mathbb{Z}$$

is an integral primitive ideal representing the class $\mathcal{A}$. One shows that the theta function $\theta_{\mathcal{A}}(z)$ for any class $\mathcal{A}$ belongs to $M_1(\Gamma, \chi)$. Hence for any character $\psi \in \widehat{\mathcal{C}\ell}(K)$,

$$\theta(z; \psi) = \sum_{\mathcal{A} \in \mathcal{C}\ell(K)} \psi(\mathcal{A})\theta_{\mathcal{A}}(z) \tag{2.16}$$

is an automorphic form of weight one, level q and character $\chi = \chi_q$. Note that $\theta(z;\overline{\psi}) = \theta(z;\psi)$. This has the Fourier expansion

$$\theta(z;\psi) = \sum_{n=0}^{\infty} \lambda_\psi(n) e(nz) \tag{2.17}$$

with $\lambda_\psi(0) = \delta_\psi h/2$, and for $n \geq 1$,

$$\lambda_\psi(n) = \sum_{N\mathfrak{a}=n} \psi(\mathfrak{a}). \tag{2.18}$$

In particular the Eisenstein series $E_\mathfrak{a}(z)$ are obtained from theta functions for real class group characters. Any real character $\psi \in \widehat{\mathcal{C}\ell}(K)$ is given uniquely by

$$\psi(\mathfrak{p}) = \begin{cases} \chi_v(N\mathfrak{p}) & \text{if } p \nmid v, \\ \chi_w(N\mathfrak{p}) & \text{if } p \nmid w, \end{cases} \tag{2.19}$$

where $\chi_v \chi_w = \chi_q$ (note that $\psi(\mathfrak{a})$ is well defined by (2.19) because $\chi_q(N\mathfrak{a}) = 1$ if $(\mathfrak{a}, q) = 1$). Interchanging v and w we obtain the same ψ. However different factorizations $vw = q$ up to the order yield distinct real class group characters. Therefore we have exactly $\tau(q)/2$ such characters; they are called the *genus characters*. If $\mathfrak{a} \sim 1/v$ then

$$E_\mathfrak{a}(z) = \overline{\varepsilon}_v \frac{2i}{h} \theta(z;\psi) \tag{2.20}$$

where $\psi \in \widehat{\mathcal{C}\ell}(K)$ is the genus character given by (2.19) and $\lambda_\mathfrak{a}(n) = \lambda_\psi(n)$ for all $n \geq 0$ (see (2.11)). Note that the Eisenstein series $E_\mathfrak{a}(z)$ and $E_{\mathfrak{a}'}(z)$ for the "transposed" cusps $\mathfrak{a} \sim 1/v$ and $\mathfrak{a}' \sim 1/w$ are linearly dependent, in fact $\varepsilon_v E_\mathfrak{a}(z) = \varepsilon_w E_{\mathfrak{a}'}(z)$ (this is true only for the Eisenstein series at the central point $s = 1/2\,!$).

If $\psi \in \widehat{\mathcal{C}\ell}(K)$ is not real, then the theta function $\theta(z;\psi)$ is a primitive cusp form of weight one with Hecke eigenvalues $\lambda_\psi(n)$ given by (2.18).

Cusp forms of any odd weight can be constructed from the class group characters as follows. Let k be odd, $k > 1$ and $q \equiv 3 \pmod 4$, $q > 3$. Let ψ be a character on ideals in $K = \mathbb{Q}(\sqrt{-q})$ such that

$$\psi((\alpha)) = \left(\frac{\alpha}{|\alpha|}\right)^{k-1} \tag{2.21}$$

for any $\alpha \in K^*$. All such characters are obtained by multiplying a fixed character with the class group characters so we have exactly $h = h(-q)$ characters of type (2.21) (we say of *frequency* $k-1$). With every ψ of frequency $k-1$ we associate the function

$$\theta(z;\psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) (N\mathfrak{a})^{(k-1)/2} e(zN\mathfrak{a}) \tag{2.22}$$

where $\mathfrak{a}$ runs over the non-zero integral ideals. One shows that $\theta(z;\psi) \in S_k(\Gamma,\chi)$ and that $\theta(z;\psi)$ is a primitive cusp form with Hecke eigenvalues $\lambda_\psi(n)$ given by (2.18) (see Section 12.3 of [I]).

Besides (2.3) and (2.7) the primitive forms satisfy some bilateral modular equations which are obtained by certain transformations $\omega \in \mathrm{SL}_2(\mathbb{R})$ not in the group $\Gamma_0(q)$. For any $\omega = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ the *ω-stroke operator* is defined on functions $F : \mathbb{H} \to \mathbb{C}$ by

$$F_{|\omega}(z) = (cz+d)^{-k}F(z). \tag{2.23}$$

Note that $(F_{|\tau})_{|\sigma} = F_{|\tau\sigma}$ for any $\tau,\sigma \in \mathrm{SL}_2(\mathbb{R})$.

Let $q = rs$ (recall that q is squarefree so $(r,s) = 1$). We are interested in the ω-stroke operator for

$$\omega = \begin{pmatrix} \alpha\sqrt{r} & \beta/\sqrt{r} \\ \gamma s\sqrt{r} & \delta\sqrt{r} \end{pmatrix} \tag{2.24}$$

with $\alpha,\beta,\gamma,\delta$ integers such that $\det\omega = \alpha\delta r - \beta\gamma s = 1$.

First for the ω given by (2.24) one checks that the ω-stroke maps $S_k(\Gamma,\chi)$ to itself. Next note that the ω-stroke on $S_k(\Gamma,\chi)$ is a pseudo-involution, precisely

$$F_{|\omega^2} = \chi_r(-1)\chi_s(r)F \tag{2.25}$$

where $\chi_r\chi_s = \chi_q$. Moreover the ω-stroke on $S_k(\Gamma,\chi)$ almost commutes with the Hecke operators T_n for $(n,q) = 1$, precisely

$$T_n(F_{|\omega}) = \chi_r(n)(T_nF)_{|\omega} \quad \text{if } (n,q) = 1. \tag{2.26}$$

Hence it follows that if F is a *Hecke form* (i.e. F is an eigenfunction of every T_n with $(n,q) = 1$), then so is $F_{|\omega}$ (of course, with different Hecke eigenvalues). By the multiplicity-one property it follows that both F and $F_{|\omega}$ are *primitive* (i.e. the eigenfunctions of all T_n). Therefore for any primitive form $F \in S_k(\Gamma,\chi)$ there exists a unique primitive form $G \in S_k(\Gamma,\chi)$ and a complex number $\eta_F(\omega)$ such that

$$F_{|\omega} = \eta_F(\omega)G. \tag{2.27}$$

As in [AL] we call $\eta_F(\omega)$ the *pseudo-eigenvalue* of $|_\omega$ at F. By (2.25) we find that $\eta_F\eta_G = \chi_r(-1)\chi_s(r) = \pm 1$. One can show that the Hecke eigenvalues of F and G satisfy

$$\lambda_G(n) = \chi_r(n)\lambda_F(n) \quad \text{if } (n,r) = 1, \tag{2.28}$$

$$\lambda_G(n) = \chi_s(n)\overline{\lambda}_F(n) \quad \text{if } (n,s) = 1. \tag{2.29}$$

These formulas are consistent by the property $\overline{\lambda}_F(n) = \chi(n)\lambda_F(n)$ if $(n,q) = 1$, and they determine G in terms of F. In particular we have $|\lambda_F(n)| = |\lambda_G(n)|$ for all $n \geq 1$. Hence one derives that $\langle G,G\rangle = \langle F,F\rangle$ and $\langle F,F\rangle =$

$|\eta_F(\omega)|^2\langle G,G\rangle$, so

$$|\eta_F(\omega)|=1. \tag{2.30}$$

Note that G depends only on r,s (G is a hybrid twist of F by the characters χ_r,χ_s), but not on $\alpha,\beta,\gamma,\delta$ in ω. If ω and ω' are given by (2.24) with the same r,s then

$$\varrho=\omega'\omega^{-1}=\begin{pmatrix}\alpha'\delta r-\beta'\gamma s & \beta'\alpha-\alpha'\beta\\ (\gamma'\delta-\delta'\gamma)q & \delta'\alpha r-\gamma'\beta s\end{pmatrix},$$
$$F_{|\omega'}=F_{|\varrho\omega}=(F_{|\varrho})_{|\omega}=\chi(\varrho)F_{|\omega}=\chi(\varrho)\eta_F(\omega)G,$$
$$\chi(\varrho)=\chi(\delta'\alpha r-\gamma'\beta s)=\chi_r(-\gamma'\beta s)\chi_s(\delta'\alpha r)=\chi_r(\beta'/\beta)\chi_s(\alpha'/\alpha)$$

by the determinant equation $\alpha'\delta' r-\beta'\gamma' s=1$. Hence we get the relation $\eta_F(\omega')=\chi_r(\beta'\beta)\chi_s(\alpha'\alpha)\eta_F(\omega)$. This relation shows that the pseudo-eigenvalue $\eta_F(\omega)$ of ω given by (2.24) factors into

$$\eta_F(\omega)=\chi_r(\beta)\chi_s(\alpha)\eta_F(r,s)=\chi_r(-\gamma s)\chi_s(\delta r)\eta_F(r,s) \tag{2.31}$$

where $\eta_F(r,s)$ depends only on r,s and F.

The case $r=q$ and $s=1$ is special. We can choose

$$\omega=\begin{pmatrix}0 & -1/\sqrt{q}\\ \sqrt{q} & 0\end{pmatrix} \tag{2.32}$$

getting $F_{|\omega}=\eta_F\overline{F}$, where $\overline{F}$ is obtained from F by complex conjugating the coefficients in the Fourier expansion (2.8). Moreover in this case one shows that (see Theorem 6.29 of [I])

$$\eta_F=\varepsilon_q\overline{\lambda}_F(q). \tag{2.33}$$

The modified Eisenstein series $y^{-k/2}E_{\mathfrak{a}}(z,s)$ is also a pseudo-eigenfunction of the ω-stroke operator. We shall verify this fact by explicit computations rather than by going through the theory of Hecke operators. Although we are only interested in $E_{\mathfrak{a}}(z)=y^{-1/2}E_{\mathfrak{a}}(z,1/2)$ we present the computations in a general case (i.e. for any $k\ge 1$) for record. Note that for any $\gamma,\omega\in\mathrm{SL}_2(\mathbb{R})$,

$$y^{-k/2}J_\gamma(z,s)_{|\omega}=y^{-k/2}J_{\gamma\omega}(z,s).$$

Hence, by (2.6), for $\operatorname{Re} s>1$ we get

$$y^{-k/2}E_{\mathfrak{a}}(z,s)_{|\omega}=\tfrac12 w^{-s}y^{-k/2}\sum\sum_{(c,d)=1}\chi_r(d)\chi_w(-c)J_\tau(z,s)$$

where

$$\tau=\begin{pmatrix} * & *\\ cv & d\end{pmatrix}\omega=\begin{pmatrix} * & *\\ (\alpha cv+\gamma ds)\sqrt{r} & (\beta cv+\delta dr)/\sqrt{r}\end{pmatrix}.$$

Put $r'=(r,v)$, $r_1=r/(r,v)$, $r_2=v/(r,v)$ and $s'=(s,v)$, $s_1=s/(s,v)$, $s_2=v/(s,v)$. Since $q=vw=rs$ is squarefree we have $r_1s_1=w$, $r_2s_2=v$,

$r_1 s_2 = r$, $r_2 s_1 = s$. In the lower row of τ we extract the factors r', s' getting

$$\tau = \begin{pmatrix} * & * \\ Cs'\sqrt{r} & Dr'/\sqrt{r} \end{pmatrix}$$

where $C = \alpha c s_2 + \gamma d s_1$ and $D = \beta c r_2 + \delta d r_1$. Solving this system of linear equations of determinant $\alpha\delta s_2 r_1 - \beta\gamma r_2 s_1 = \alpha\delta r - \beta\gamma s = 1$ we find $-c = \gamma s_1 D - \delta r_1 C$ and $d = \alpha s_2 D - \beta r_2 C$. Hence the condition $(c,d) = 1$ is equivalent to $(C,D) = 1$. Next we factor the characters $\chi_v = \chi_{r_2}\chi_{s_2}$ and $\chi_w = \chi_{r_1}\chi_{s_1}$ to compute $\chi_v(d) = \chi_{r_2}(\alpha s_2 D)\chi_{s_2}(-\beta r_2 C)$ and $\chi_w(-c) = \chi_{r_1}(\gamma s_1 D)\chi_{s_1}(-\delta r_1 C)$. Hence $\chi_v(d)\chi_w(-c) = \chi_{r_1 r_2}(D)\chi_{s_1 s_2}(-C)\eta$, where

$$\eta = \chi_{r_2}(\alpha s_2)\chi_{s_2}(\beta r_2)\chi_{r_1}(\gamma s_1)\chi_{s_1}(\delta r_1). \tag{2.34}$$

Extracting from τ the factor $u = r'/\sqrt{r}$ by the property $J_{(uc,ud)}(z,s) = u^{-2s}J_{(c,d)}(z,s)$, and using $s'\sqrt{r}/u = r_1 r_2$, $u^2 w r_1 r_2 = q$, we conclude from the above computations that

$$y^{-k/2}E_{\mathfrak{a}}(z,s)_{|\omega} = \eta y^{-k/2}E_{\mathfrak{a}^*}(z,s) \tag{2.35}$$

where $E_{\mathfrak{a}^*}(z,s)$ is the Eisenstein series for the cusp $\mathfrak{a}^* \sim 1/(r_1 r_2)$, i.e.

$$\mathfrak{a}^* \sim \frac{(r,v)^2}{rv}. \tag{2.36}$$

By the determinant equation $\alpha\delta s_2 r_1 - \beta\gamma r_2 s_1 = 1$ we eliminate α, β in (2.34) getting

$$\eta = \chi_r\left(\frac{\gamma s}{(s,v)}\right)\chi_s\left(\frac{\delta r}{(r,v)}\right)\chi_{v/(s,v)}(-1). \tag{2.37}$$

In particular for $k = 1$ and $s = 1/2$ we obtain from (2.35) (by analytic continuation)

PROPOSITION 2.1. *Let $q = vw = rs > 1$ be squarefree and odd. Then the holomorphic Eisenstein series $E_{\mathfrak{a}}(z) = y^{-1/2}E_{\mathfrak{a}}(z,1/2)$ for cusp $\mathfrak{a} \sim 1/v$ is a pseudo-eigenfunction of the ω-stroke operator* (*with ω given by* (2.24)), *specifically*

$$E_{\mathfrak{a}|\omega} = \eta E_{\mathfrak{a}^*} \tag{2.38}$$

where $\mathfrak{a}^*$ is given by (2.36) *and η by* (2.37).

In the special case (2.32) the formula (2.38) becomes

$$(z\sqrt{q})^{-1}E_{1/v}\left(\frac{-1}{qz}\right) = \chi_v(-1)E_{1/w}(z). \tag{2.39}$$

3. Summation formulas. Suppose we have two functions $A(z), B(z)$ on $\mathbb{H}$ given by Fourier series

$$A(z) = \sum_{n=0}^{\infty} a_n e(nz), \tag{3.1}$$

$$B(z)=\sum_{n=0}^{\infty} b_n e(nz), \tag{3.2}$$

with $a_n, b_n \ll n^{k-1+\varepsilon}$. Suppose that $A(z), B(z)$ are connected by the ω-stroke operator, say

$$A_{|\omega}(z)=\eta B(z) \tag{3.3}$$

for some $\omega=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{R})$ with $c>0$ and some complex number $\eta\neq 0$. In particular for $z=(-d+iy)/c$ we have

$$(iy)^{-k} A\Big(\frac{a}{c}+\frac{i}{cy}\Big)=\eta B\Big(\frac{-d}{c}+\frac{iy}{c}\Big) \tag{3.4}$$

for any $y>0$. Following Hecke this formula can be expressed as a functional equation for the L-functions

$$L_A\Big(s,\frac{a}{c}\Big)=\sum_{n=1}^{\infty} a_n e\Big(\frac{an}{c}\Big) n^{-s}, \tag{3.5}$$

$$L_B\Big(s,\frac{-d}{c}\Big)=\sum_{n=1}^{\infty} b_n e\Big(\frac{-dn}{c}\Big) n^{-s}. \tag{3.6}$$

Put

$$\Lambda_A\Big(s,\frac{a}{c}\Big)=\Big(\frac{c}{2\pi}\Big)^s \Gamma(s) L_A\Big(s,\frac{a}{c}\Big), \tag{3.7}$$

$$\Lambda_B\Big(s,\frac{-d}{c}\Big)=\Big(\frac{c}{2\pi}\Big)^s \Gamma(s) L_B\Big(s,\frac{a}{c}\Big). \tag{3.8}$$

First, by integrating (3.4) we establish the following formula:

$$\Lambda_A\Big(s,\frac{a}{c}\Big)+\frac{a_0}{s}+i^k\eta\frac{b_0}{k-s}=\int_1^\infty\Big[A\Big(\frac{a}{c}+\frac{iy}{c}\Big)-a_0\Big]y^{s-1}\,dy$$
$$+i^k\eta\int_1^\infty\Big[B\Big(\frac{-d}{c}+\frac{iy}{c}\Big)-b_0\Big]y^{k-s-1}\,dy$$

for $\operatorname{Re} s>k$. Since $A(z)-a_0$ and $B(z)-b_0$ have exponential decay as $y=\operatorname{Im} z\to\infty$, the above integrals converge absolutely and they are entire functions bounded on vertical strips. Similarly we have (because $B_{|\omega^{-1}}(z)=\eta^{-1}A(z)$ and $\omega^{-1}=-\left(\begin{smallmatrix} -d & * \\ c & -a\end{smallmatrix}\right)$)

$$\Lambda_B\Big(s,\frac{-d}{c}\Big)+\frac{b_0}{s}+(i^k\eta)^{-1}\frac{a_0}{k-s}=\int_1^\infty\Big[B\Big(\frac{-d}{c}+\frac{iy}{c}\Big)-b_0\Big]y^{s-1}dy$$
$$+(i^k\eta)^{-1}\int_1^\infty\Big[A\Big(\frac{a}{c}+\frac{iy}{c}\Big)-a_0\Big]y^{k-s-1}dy.$$

Combining both formulas we obtain the following functional equation:

$$\Lambda_A\left(s, \frac{a}{c}\right) = i^k \eta \Lambda_B\left(k - s, \frac{-d}{c}\right). \tag{3.9}$$

For notational convenience we put

$$a_n = n^{(k-1)/2} a(n), \quad b_n = n^{(k-1)/2} b(n) \tag{3.10}$$

so the corresponding L-functions are shifted from s to $s - (k-1)/2$, and the resulting functional equation connects values at s and $1 - s$.

Next we derive from (3.9) a formula for sums of type

$$S = \sum_{n=1}^{\infty} a(n) e\left(\frac{an}{c}\right) g(n)$$

where $g(x)$ is a nice test function.

Proposition 3.1. *Suppose $A(z), B(z)$ satisfy* (3.3) *for some $\omega = \left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ with $c > 0$ and some complex number $\eta \neq 0$. Then for any $g(x)$ smooth and compactly supported on $\mathbb{R}^+$ we have*

$$\sum_{n=1}^{\infty} a(n) e\left(\frac{an}{c}\right) g(n) = 2\pi i^k \frac{\eta}{c} \left\{ \frac{b_0}{\Gamma(k)} \int_0^{\infty} g(x) \left(\frac{2\pi\sqrt{x}}{c}\right)^{k-1} dx + \sum_{n=1}^{\infty} b(n) e\left(\frac{-dn}{c}\right) \int_0^{\infty} g(x) J_{k-1}\left(\frac{4\pi}{c}\sqrt{nx}\right) dx \right\} \tag{3.11}$$

where $J_{k-1}(x)$ is the Bessel function of order $k-1$.

Proof. The left side of (3.11) is given by the contour integral

$$S = \frac{1}{2\pi i} \int_{(\sigma)} L_A\left(s, \frac{a}{c}\right) G\left(s - \frac{k-1}{2}\right) ds$$

where $G(s)$ denotes the Mellin transform of $g(x)$ and $\sigma > k$. We move to $\operatorname{Re} s = k - \sigma$ passing a simple pole at $s = k$ with residue $i^k \eta b_0 (2\pi/c)^k \Gamma(k)^{-1} \times G((k+1)/2)$ (the point $s = 0$ is not a pole of $L_A(s, a/c)$). Then we apply the functional equation (3.9) getting

$$S = i^k \eta b_0 \left(\frac{2\pi}{c}\right)^k \Gamma(k)^{-1} G\left(\frac{k+1}{2}\right) + \frac{i^k \eta}{2\pi i} \int_{(\sigma)} L_B\left(s, \frac{-d}{c}\right) \left(\frac{c}{2\pi}\right)^{2s-k} \frac{\Gamma(s)}{\Gamma(k-s)} G\left(\frac{k+1}{2} - s\right) ds.$$

Expanding $L_B(s, -d/c)$ into the Dirichlet series and integrating termwise we get

$$\frac{1}{2\pi i}\int_{(\sigma)} = \frac{2\pi}{c}\sum_{n=1}^{\infty} b(n)e\Big(\frac{-dn}{c}\Big)H\Big(\frac{2\pi\sqrt{n}}{c}\Big)$$

where

$$H(y) = \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(s)}{\Gamma(k-s)}G\Big(\frac{k+1}{2}-s\Big)y^{k-1-2s}\,ds.$$

Here we can take for σ any positive number. If $\sigma < k/2$ we can open the Mellin transform

$$G\Big(\frac{k+1}{2}-s\Big) = \int_0^\infty g(x)x^{(k-1)/2-s}\,dx$$

and change the order of integration getting

$$\begin{aligned} H(y) &= \int_0^\infty g(x)\Big(\frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(s)}{\Gamma(k-s)}(\sqrt{x}\,y)^{k-1-2s}\,ds\Big)\,dx \\ &= \int_0^\infty g(x)J_{k-1}(2\sqrt{x}\,y)\,dx \end{aligned}$$

by (6.422.9) of [GR]. This yields (3.11).

Changing variables one can write (3.11) as follows:

$$\begin{aligned} (3.12)\qquad &\sum_{n=1}^{\infty} a(n)e\Big(\frac{an}{c}\Big)g\Big(\frac{2\pi n}{c}\Big) \\ &= i^k\eta\Big\{\frac{b_0}{\Gamma(k)}\int_0^\infty g(x)\Big(\frac{2\pi x}{c}\Big)^{(k-1)/2}\,dx + \sum_{n=1}^{\infty} b(n)e\Big(\frac{-dn}{c}\Big)h\Big(\frac{2\pi n}{c}\Big)\Big\} \end{aligned}$$

where $h(y)$ is a Hankel-type transform

$$h(y) = \int_0^\infty g(x)J_{k-1}(2\sqrt{xy})\,dx. \tag{3.13}$$

Now we specialize Proposition 3.1 for automorphic forms. First we treat the cusp forms.

PROPOSITION 3.2. *Let $F \in S_k(\Gamma,\chi)$ be a Hecke cusp form with eigenvalues $\lambda_F(n)$. Let $c \ge 1$ and $(a,c)=1$. Then for any function $g(x)$ smooth and compactly supported on $\mathbb{R}^+$ we have*

$$\begin{aligned} (3.14)\qquad &\sum_{n=1}^{\infty} \lambda_F(n)e\Big(\frac{an}{c}\Big)g(n) \\ &= 2\pi i^k\frac{\eta}{c\sqrt{r}}\sum_{n=1}^{\infty}\lambda_G(n)e\Big(\frac{-\overline{ar}n}{c}\Big)\int_0^\infty g(x)J_{k-1}\Big(\frac{4\pi}{c}\sqrt{\frac{nx}{r}}\Big)\,dx \end{aligned}$$

where $\lambda_G(n)$ *are given by* (2.28), (2.29) *with* $r = q/(c,q)$, $s = (c,q)$ *and*

$$\eta = \chi_s(a)\chi_r(-c)\eta_F(r,s). \tag{3.15}$$

Here $\eta_F(r,s)$ *depends only on* r, s, F *and* $|\eta_F(r,s)| = 1$.

Proof. The result follows by applying (3.11) for $A(z) = F(z)$ and $B(z) = F_{|\omega}(z)$ with

$$\omega = \begin{pmatrix} a\sqrt{r} & b/\sqrt{r} \\ c\sqrt{r} & d\sqrt{r} \end{pmatrix} \tag{3.16}$$

where b, d are integers such that $adr - bc = 1$. Note that $d \equiv \overline{ar} \pmod{c}$ so the corresponding pseudo-eigenvalue (2.31) is equal to (3.15).

Next we apply (3.11) for the Eisenstein series $E_{\mathfrak{a}}(z)$ with cusp $\mathfrak{a} \sim 1/v$ and the ω given by (3.16). In this case (2.38) holds with $\mathfrak{a}^* \sim 1/v^* = (r,v)^2/(rv)$, where $r = q/(c,q)$, $s = (c,q)$. The corresponding pseudo-eigenvalue (2.37) becomes

$$\eta = \chi_r\Big(\frac{c}{(c,v)}\Big)\chi_s\Big(\frac{av}{(c,v)}\Big)\chi_{v/(c,v)}(-1) \tag{3.17}$$

because $(s,v) = (c,v)$ and $(r,v) = v/(c,v)$. We have $q = vw = v^*w^*$ with

$$v^* = (c,v)w/(c,w), \qquad w^* = (c,w)v/(c,v). \tag{3.18}$$

We introduce the twisted divisor function

$$\tau(n;\chi_v,\chi_w) = \sum_{n_1n_2=n} \chi_v(n_1)\chi_w(n_2) \tag{3.19}$$

for any $n \geq 1$. Therefore $\tau(n;\chi_v,\chi_w)$ and $\tau(n;\chi_{v^*},\chi_{w^*})$ are the Hecke eigenvalues $\lambda_{\mathfrak{a}}(n)$ and $\lambda_{\mathfrak{a}^*}(n)$ for $E_{\mathfrak{a}}(z)$ and $E_{\mathfrak{a}^*}(z)$, respectively. These are proportional to the Fourier coefficients of $E_{\mathfrak{a}}(z)$ and $E_{\mathfrak{a}^*}(z)$ with the factor $\overline{\varepsilon}_v 2i/h$ and $\overline{\varepsilon}_{v^*} 2i/h$, respectively (see (2.11)). By (3.11) with $k = 1$ and $c\sqrt{r}$ in place of c we obtain

$$\sum_{n=1}^{\infty} \tau(n;\chi_v,\chi_w) e\Big(\frac{an}{c}\Big) g(n) = \varepsilon_v \overline{\varepsilon}_{v^*} \frac{2\pi i\eta}{c\sqrt{r}} \Big\{ \frac{h}{2}\int_0^\infty g(x)\,dx + \sum_{n=1}^{\infty} \tau(n;\chi_{v^*},\chi_{w^*}) e\Big(\frac{-\overline{ar}n}{c}\Big) \int_0^\infty g(x) J_0\Big(\frac{4\pi}{c}\sqrt{\frac{nx}{r}}\Big)\,dx \Big\}$$

where the leading term $\frac{1}{2}h\widehat{g}(0)$ appears only if $\mathfrak{a}^* \sim \infty$, or $\mathfrak{a}^* \sim 0$. Note that $\overline{\varepsilon}_{v^*} = \overline{\varepsilon}_{vr} = \overline{\varepsilon}_{-vs} = -i\varepsilon_{vs}$ so the factor $\sigma = i\eta\varepsilon_v\overline{\varepsilon}_{v^*}$ becomes

$$\sigma = \varepsilon_v\varepsilon_{vs}\chi_s(av/(c,v))\chi_r(c/(c,v))\chi_{(r,v)}(-1). \tag{3.20}$$

Before stating the final result we simplify the leading term. We have $\mathfrak{a}^* \sim \infty \Leftrightarrow (c,q) = v$, in which case $r = w$ and $s = v$ so that

$$\sigma = \varepsilon_v\chi_v(a)\chi_w(c/v) = \chi_v(a)\chi_w(c/v)\tau(\chi_v)/\sqrt{v}.$$

Similarly $\mathfrak{a}^* \sim 0 \Leftrightarrow (c,q) = w$, in which case $r = v$ and $s = w$. Thus, $\sigma = \varepsilon_v \varepsilon_q \chi_w(av)\chi_v(-c)$. By the reciprocity law $\chi_w(v)\chi_v(w) = 1$, because $vw = q \equiv -1 \pmod 4$. Moreover $\varepsilon_v\varepsilon_q\chi_v(-1) = i\overline{\varepsilon}_v = \varepsilon_{-v} = \varepsilon_w$ so that

$$\sigma = \varepsilon_w \chi_w(a)\chi_v(c/w) = \chi_w(a)\chi_v(c/w)\tau(\chi_w)/\sqrt{w}.$$

Hence, by the class number formula $\pi h = \sqrt{q}\,L(1,\chi)$, we conclude the following result.

Proposition 3.3. *Let q be squarefree, $q \equiv -1 \pmod 4$. Let $q = vw$, and let $\chi_q = \chi_v\chi_w$ be the corresponding real characters. Let $c \geq 1$ and $(a,c) = 1$. Let $q = v^*w^*$ be given by* (3.18), *and let $\chi_q = \chi_{v^*}\chi_{w^*}$ be the corresponding real characters. Then for any smooth function $g(x)$ compactly supported on $\mathbb{R}^+$ we have*

$$(3.21)\qquad \sum_{n=1}^{\infty} \tau(n;\chi_v,\chi_w) e\Big(\frac{an}{c}\Big) g(n)$$
$$= \Big\{\chi_v(a)\chi_w\Big(\frac{c}{v}\Big)\tau(\chi_v) + \chi_w(a)\chi_v\Big(\frac{c}{w}\Big)\tau(\chi_w)\Big\}\frac{L(1,\chi)}{c}\int_0^\infty g(x)\,dx$$
$$+ \frac{2\pi\sigma}{c\sqrt{r}}\sum_{n=1}^{\infty}\tau(n;\chi_{v^*},\chi_{w^*})e\Big(\frac{-\overline{ar}n}{c}\Big)\int_0^\infty g(x)J_0\Big(\frac{4\pi}{c}\sqrt{\frac{nx}{r}}\Big)\,dx$$

where σ is given by (3.20) *with $r = q/(c,q)$ and $s = (c,q)$.*

Remark. In the leading term of (3.21) we use the popular convention that $\chi(z)$ is zero if z is not an integer. Therefore the leading term vanishes unless $v \mid c$ and $(c,w) = 1$, or $w \mid c$ and $(c,v) = 1$.

4. Convolution sums. In this section we shall evaluate asymptotically sums of type

$$(4.1)\qquad \mathcal{B}(h) = \sum\sum_{m-n=h} \lambda(m)\overline{\lambda}(n)g(m)\overline{g}(n)$$

where $\lambda(n)$ are eigenvalues of the Hecke operators T_n for one of the holomorphic automorphic forms of level q and character χ_q which were considered in the last two sections. Here $h \neq 0$ is a fixed integer and $g(x)$ is a cut-off function which is smooth and compactly supported on $\mathbb{R}^+$. Naturally one can treat the convolution sum (4.1) by spectral methods using an appropriate Poincaré series. Indeed this method (the Rankin–Selberg method) has been applied by various authors in many cases of cusp forms, but we did not find satisfactory results ([1]). Moreover the spectral methods can also be applied to the Eisenstein series, which case is important for us. However,

([1]) See the note added at the end of Section 1.

the spectral methods are rather complicated and one has to provide a lot of background material. Therefore in this section we use arguments from the circle method of Kloosterman, because they yield the results faster, more general and of great uniformity with respect to the shift h.

For clarity we present the arguments in an axiomatic setting. Literally speaking we do not assert that the coefficients $\lambda(n)$ come from an automorphic form. All we need is an appropriate summation formula for

$$S(\alpha) = \sum_{n=1}^{\infty} \lambda(n) e(\alpha n) g(n) \tag{4.2}$$

at rational points. We assume that for any $c \geq 1$ and $(a, c) = 1$ one has the expansion

$$S\left(\frac{a}{c}\right) = \sum_{m=0}^{\infty} \psi_m(a) e\left(\frac{\overline{a}}{c} l_m\right) \int_0^{\infty} g(x) k_m(x)\, dx \tag{4.3}$$

where $a\overline{a} \equiv 1 \pmod{c}$, $\psi_m(a)$ are periodic in a, say of a fixed period q, l_m are integers, and $k_m(x)$ are smooth functions. We do allow $\psi_m(a)$, l_m and $k_m(x)$ to depend on c. However the frequencies l_m and the kernels $k_m(x)$ cannot depend on a. If $m > 0$ we require the coefficients $\psi_m(a)$ to satisfy

$$|\psi_m(a)| \leq A\tau(m)c^{-1} \tag{4.4}$$

where A is a constant, $A \geq 1$. Next we assume that the Fourier transform of $g_m(x) = g(x)k_m(x)$ satisfies

$$|\widehat{g}_m(\alpha)| \leq BcCm^{-5/4} \quad \text{if } |\alpha| \leq (cC)^{-1} \tag{4.5}$$

for all $1 \leq c \leq C$, where $B \geq 1$ is a constant and $C \geq 2$ is a fixed number (a quite large number which will be chosen optimally in applications). For $m = 0$ we need more precise conditions. We assume that

$$l_0 = 0, \qquad k_0(x) = 1 \tag{4.6}$$

and the absolute value of $\psi_0(a)$ does not depend on a, say

$$|\psi_0(a)| = p(c) \leq Ac^{-1}. \tag{4.7}$$

Finally we assume that

$$\int_{-\infty}^{\infty} |\widehat{g}(\alpha)|\, d\alpha \leq B, \tag{4.8}$$

$$\int_{-\infty}^{\infty} |\alpha| \cdot |\widehat{g}(\alpha)|^2\, d\alpha \leq B^2. \tag{4.9}$$

Now we are ready to estimate the convolution sum (4.1). We begin by the following formula for the zero detector in $\mathbb{Z}$:

$$(4.10)\qquad 2\sum\sum_{\substack{c\le C<d\le c+C\\(c,d)=1}}\int_0^{1/(cd)}\cos(2\pi n(a/c-\alpha))\,d\alpha=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{if } n\ne 0,\end{cases}$$

where $ad\equiv 1 \pmod c$ and $C\ge 2$ is at our disposal (see Proposition 11.1 of [I]). Hence we get

$$\mathcal{B}(h)=\sum\sum_{\substack{c\le C<d\le c+C\\(c,d)=1}}\int_{-1/(cd)}^{1/(cd)}\left|S\left(\frac{a'}{c}-\alpha\right)\right|^2 e\left(h\left(\alpha-\frac{a'}{c}\right)\right)d\alpha$$

where $a' \pmod c$ is determined by $a'd\equiv \operatorname{sign}\alpha \pmod c$. We rearrange this sum of integrals as follows:

$$(4.11)\qquad \mathcal{B}(h)=\sum_{c\le C}\int_{-1/(cC)}^{1/(cC)} e(\alpha h)V_c(\alpha)\,d\alpha$$

where

$$(4.12)\qquad V_c(\alpha)=\sum_{d\in I}{}^{*}\, e\left(-\frac{a'h}{c}\right)\left|S\left(\frac{a'}{c}-\alpha\right)\right|^2$$

and d runs over integers prime to c in the interval

$$(4.13)\qquad I=(C,\min\{c+C,1/(|\alpha|c)\}].$$

Note that $I=(C,c+C]$ has length exactly c if $|\alpha|\le c^{-1}(c+C)^{-1}$, and $I=(C,1/(|\alpha|c)]$ is shorter than c if $c^{-1}(c+C)^{-1}<|\alpha|\le c^{-1}C^{-1}$.

Suppose $\alpha>0$; the case $\alpha<0$ is similar. By the summation formula (4.3) we have

$$S\left(\frac{a}{c}-\alpha\right)=\sum_{m=0}^{\infty}\psi_m(a)e\left(\frac{d}{c}l_m\right)\widehat{g}_m(\alpha).$$

Inserting this into (4.12) and changing the order of summation we get

$$V_c(\alpha)=\sum_{m_1}\sum_{m_2}\widehat{g}_{m_1}(\alpha)\overline{\widehat{g}}_{m_2}(\alpha)\sum_{d\in I}{}^{*}\psi_{m_1}(a)\overline{\psi}_{m_2}(a)e\left(\frac{d}{c}(l_{m_1}-l_{m_2})-\frac{ah}{c}\right).$$

Recall that $\psi_m(a)$ are periodic of period q. Splitting into residue classes $d\equiv\delta \pmod q$ we obtain incomplete Kloosterman sums for which Weil's bound yields

$$(4.14)\qquad \sum_{d\in I,\, d\equiv\delta(q)}{}^{*}\, e\left(\frac{dl-ah}{c}\right)\ll (h,c)^{1/2}c^{1/2}\tau(c)\log C.$$

We apply this result with $l=l_{m_1}-l_{m_2}$ for all terms, except for $m_1=m_2=0$ in the range $|\alpha|<c^{-1}(c+C)^{-1}$, i.e. when the interval (4.13) has length c.

We derive, for any α,

$$V_c(\alpha) = |\widehat{g}(\alpha)p(c)|^2\Big\{\sideset{}{^*}\sum_{C<d\le c+C} e\Big(-\frac{ah}{c}\Big) + O(|\alpha|cC(h,c)^{1/2}c^{1/2}\tau(c)\log C)\Big\}$$
$$+ O\Big(A^2q(h,c)^{1/2}c^{3/2}\tau(c)(\log C)$$
$$\times \Big(|\widehat{g}(\alpha)| + \sum_{m=1}^{\infty}\tau(m)|\widehat{g}_m(\alpha)|\Big)\Big(\sum_{m=1}^{\infty}\tau(m)|\widehat{g}_m(\alpha)|\Big)\Big)$$

by (4.4), (4.7) and (4.14). In the leading term we get the exact Ramanujan sum

$$r_c(h) = \sideset{}{^*}\sum_{d \,(\mathrm{mod}\, c)} e\Big(\frac{dh}{c}\Big). \tag{4.15}$$

The same estimates hold for $\alpha < 0$. Adding these results we get, by (4.11),

$$\mathcal{B}(h) = \sum_{c\le C} r_c(h)p(c)^2 \int_{-1/(cC)}^{1/(cC)} e(\alpha h)|\widehat{g}(\alpha)|^2\,d\alpha + R$$

where

$$R \ll A^2\Big(\int|\alpha|\cdot|\widehat{g}(\alpha)|^2 d\alpha\Big)\Big(\sum_{c\le C}(h,c)^{1/2}c^{-1/2}\tau(c)\Big)C\log C$$
$$+ A^2q\sum_{c\le C}(h,c)^{1/2}c^{-3/2}\tau(c)\Big(\int|\widehat{g}(\alpha)|\,d\alpha + B\Big)BcC\log C.$$

To estimate R we apply (4.8) and (4.9) getting

$$\mathcal{B}(h) = \sum_{c\le C} r_c(h)p(c)^2 \int_{-1/(cC)}^{1/(cC)} e(\alpha h)|\widehat{g}(\alpha)|^2\,d\alpha + O(\tau(h)qA^2B^2C^{3/2}(\log C)^2).$$

In the leading term we extend the integration to all $\alpha \in \mathbb{R}$ at the cost of an error term which is already present. Then we derive by the Plancherel theorem that

$$\int_{-\infty}^{\infty} e(\alpha h)|\widehat{g}(\alpha)|^2\,d\alpha = \int_0^{\infty} g(x+h)\overline{g}(x)\,dx.$$

Finally we extend the summation over $c \le C$ to all c, getting

$$\sum_{c\le C} r_c(h)p(c)^2 = \sigma(h) + O(\tau(h)A^2C^{-1})$$

where $\sigma(h)$ is the infinite series

$$\sigma(h) = \sum_{c=1}^{\infty} r_c(h)p(c)^2. \tag{4.16}$$

We have established the following

Theorem 4.1. *Suppose the conditions* (4.3)–(4.9) *hold. Then for any integer* $h \neq 0$ *the sum* (4.1) *satisfies*

$$\mathcal{B}(h) = \{\sigma(h) + O(\tau(h)A^2C^{-1})\}\int g(x+h)\overline{g}(x)\,dx + O(\tau(h)qA^2B^2C^{3/2}(\log C)^2) \tag{4.17}$$

where $\sigma(h)$ *is given by* (4.16) *and the implied constant is absolute.*

Remarks. In the proof of Theorem 4.1 we assumed tacitly (just to simplify notation) that the cut-off functions $g(x), \overline{g}(x)$ are complex conjugate. However the formula (4.17) holds (by obvious alterations in the arguments) for any pair $g(x), \overline{g}(x)$, provided both functions satisfy the same relevant conditions. In forthcoming applications we shall have two functions $g_1(x), g_2(x)$ supported in $[X, 2X]$ with $X \geq 1/2$ such that

$$x^\nu |g_j^{(\nu)}(x)| \leq 1 \quad \text{if } \nu = 0, 1, 2, \tag{4.18}$$

for $j = 1, 2$. For such functions (4.8) and (4.9) hold with $B = 1$. Moreover we shall be able to verify (4.5) with $C = 2\sqrt{qX}$ and some constant $B \geq 1$. Therefore we state the following

Corollary 4.2. *Suppose the conditions* (4.3)–(4.7) *hold for the arithmetic function* $\lambda(n)$ *and for the cut-off functions* $g_1(x)$, $g_2(x)$ *supported in* $[X, 2X]$ *with derivatives satisfying* (4.18). *Precisely let* (4.5) *hold with* $C = 2\sqrt{qX}$. *Then for any integer* $h \neq 0$,

$$\begin{aligned}\mathcal{B}(h) &= \sum_{m-n=h} \lambda(m)\overline{\lambda}(n)g_1(m)g_2(n) \\ &= \sigma(h)\int g_1(x+h)g_2(x)\,dx + O(\tau(h)(qAB)^2X^{3/4}(\log 3X)^2)\end{aligned} \tag{4.19}$$

where $\sigma(h)$ *is given by* (4.16) *and the implied constant is absolute.*

Remarks. It would be convenient for applications to have a formula for $\mathcal{B}(h)$ with the cut-off functions $g(x)$ smooth on $\mathbb{R}^+$ such that

$$x^\nu |g^{(\nu)}(x)| \leq (1 + x/X)^{-4} \quad \text{if } \nu = 0, 1, 2, \tag{4.20}$$

rather than being supported in the dyadic segment $[X, 2X]$. Unfortunately for such functions the condition (4.5) may not be easily verifiable with a reasonable value of C (the optimal C should be of the order of $\sqrt{X}$). Nevertheless we shall be able to derive results for functions satisfying (4.20) by applying a smooth partition of unity, but not at the current position (see how we justify (6.27)).

In principle our analysis (the Kloosterman circle method) works also for $h = 0$, but, of course, giving a somewhat different main term. In fact the resulting error term is better, because the estimate for the incomplete Kloosterman sum (4.14) is replaced by a stronger bound for a Ramanujan sum. Rather than repeating and modifying the former arguments we shall derive an asymptotic formula for $\mathcal{B}(0)$ directly using the Rankin–Selberg zeta function (see (6.43)).

Now we apply Corollary 4.2 for the $\lambda(n)$'s which are Hecke eigenvalues of a holomorphic automorphic form of weight $k \geq 1$, level q and the real character χ_q of conductor q. As in Section 3 we assume that q is odd, so q is squarefree and $q \equiv 2k+1 \pmod 4$ by the consistency condition $\chi_q(-1) = (-1)^k$. This form is either a primitive cusp form, or the holomorphic Eisenstein series $E_{\mathfrak{a}}(z) = y^{-1/2}E_{\mathfrak{a}}(z, 1/2)$ of weight $k = 1$ for a cusp $\mathfrak{a} \sim 1/v$ with $vw = q$. In the latter case the Hecke eigenvalues are (see (3.19))

$$\tau(n; \chi_v, \chi_w) = \sum_{n_1 n_2 = n} \chi_v(n_1)\chi_w(n_2).$$

The summation formula (4.3) holds by Proposition 3.2 for cusp forms, or Proposition 3.3 for the Eisenstein series. In either case we have

$$|\psi_m(a)| \leq 2\pi\tau(m)c^{-1} \quad \text{if } m \geq 1,$$

and $|\psi_0(a)| = p(c)$ does not depend on a. In fact $p(c) = 0$, except for the Eisenstein series $E_{\mathfrak{a}}(z)$ with $\mathfrak{a} \sim 1/v$, in which case we have

$$p(c) = \frac{L(1,\chi)}{c} \begin{cases} \sqrt{v} & \text{if } (c,q) = v, \\ \sqrt{w} & \text{if } (c,q) = w, \end{cases} \tag{4.21}$$

and $p(c) = 0$ otherwise. In every case (cusp forms or Eisenstein series) the summation formula holds with the kernel

$$k_m(x) = J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mx}{r}}\right)$$

where $r = q/(c,q)$. Note that for $k = 1$ we have $k_0(x) = 1$ as required by (4.6). Hence the Fourier transform of $g_m(x)$ is

$$\widehat{g}_m(\alpha) = \int g(x)e(\alpha x)J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{mx}{r}}\right)dx. \tag{4.22}$$

Note that the Bessel function can be written as

$$J_{k-1}(2\pi y) = W(y)e(y) + \overline{W}(y)e(-y)$$

where $W(y)$ is a smooth non-oscillatory function whose derivatives satisfy

$$y^\nu W^{(\nu)}(y) \ll k^2 y^{-1/2} \quad \text{if } \nu = 0, 1, 2.$$

Let $g(x)$ be a smooth function supported on $[X, 2X]$ with $X \geq 1/2$ such that

$$x^\nu |g^{(\nu)}(x)| \leq 1 \quad \text{if } \nu = 0, 1, 2. \tag{4.23}$$

We choose $C = 2\sqrt{qX}$ so there is no stationary point in the Fourier integral (4.22) if $1 \leq c \leq C$ and $|\alpha|cC \leq 1$. Therefore integrating by parts two times we derive

$$\widehat{g}_m(\alpha) \ll k^2 X (c^2 r/(mX))^{5/4} \ll k^2 q^{3/2} cCm^{-5/4}$$

for $|\alpha| \leq (cC)^{-1}$, where the implied constant is absolute. Next we derive from (4.23) by partial integration that

$$\widehat{g}(\alpha) = \int g(x) e(-\alpha x)\, dx \ll X(1+|\alpha|X)^{-2}. \tag{4.24}$$

Hence

$$\int |\widehat{g}(\alpha)|\, d\alpha \ll 1, \qquad \int |\alpha| \cdot |\widehat{g}(\alpha)|^2\, d\alpha \ll 1.$$

The above estimates verify the conditions of Corollary 4.2 with $A = 2\sqrt{q}\, L(1,\chi), B = k^2 q^{3/2}$ and $C = 2\sqrt{qX}$. Hence we obtain the following two theorems.

Theorem 4.3. *Let $\lambda_F(n)$ be the eigenvalues of a primitive cusp form $F \in S_k(\Gamma_0(q), \chi_q)$ (recall that $k \geq 1$ and q is squarefree, $q \equiv 2k+1 \pmod 4$). Then for any integer $h \neq 0$ and for any smooth functions $g_1(x)$, $g_2(x)$ supported in $[X, 2X]$, $X \geq 1/2$, with derivatives satisfying* (4.23) *we have*

$$\sum\sum_{m-n=h} \lambda_F(m)\overline{\lambda}_F(n) g_1(m) g_2(n) \ll \tau(h) q^6 k^4 X^{3/4} (\log 3X)^2 \tag{4.25}$$

where the implied constant is absolute.

Theorem 4.4. *Let q be squarefree, $q \equiv -1 \pmod 4$. Let $uw = q$ and $\tau(n; \chi_v, \chi_w)$ be the twisted divisor function by the corresponding characters $\chi_v \chi_w = \chi_q$* (see (3.19))*. Then for any integer $h \neq 0$ and for any smooth functions $g_1(x)$, $g_2(x)$ supported in $[X, 2X]$, $X \geq 1/2$, with derivatives satisfying* (4.23) *we have*

$$\sum\sum_{m-n=h} \tau(m; \chi_v, \chi_w) \tau(n; \chi_v, \chi_w) g_1(m) g_2(n) = \sigma(h) \int g_1(x+h) g_2(x)\, dx + O(\tau(h) q^6 X^{3/4} (\log 3X)^2) \tag{4.26}$$

where $\sigma(h)$ is the infinite series (4.16) *with $p(c)$ given by* (4.21)*, and the implied constant is absolute.*

Remark. We emphasize that the estimates in the above theorems are uniform in every parameter.

We conclude this section by computing $\sigma(h)$. We have

$$\sigma(h) = \Big\{ \sum_{(c,q)=v} \frac{v}{c^2}\, r_c(h) + \sum_{(c,q)=w} \frac{w}{c^2}\, r_c(h) \Big\} L(1,\chi)^2 \tag{4.27}$$

where $r_c(h)$ is the Ramanujan sum. Since $r_c(h)$ is multiplicative in c we get

$$\sigma(h)=\Big\{\sum_{c|v^\infty}\frac{r_{cv}(h)}{c^2v}+\sum_{c|w^\infty}\frac{r_{cw}(h)}{c^2w}\Big\}\Big(\sum_{(c,q)=1}\frac{r_c(h)}{c^2}\Big)L(1,\chi)^2 \tag{4.28}$$

where

$$\sum_{(c,q)=1}\frac{r_c(h)}{c^2}=\frac{\zeta_q(2)}{\zeta(2)}\sum_{\substack{d|h\\(d,q)=1}}d^{-1}. \tag{4.29}$$

Moreover using the formula

$$r_c(h)=\sum_{d|(c,h)}d\mu(c/d) \tag{4.30}$$

one can show that

$$\sum_{c|v^\infty}\frac{r_{cv}(h)}{c^2v}=\mu\Big(\frac{v}{(h,v)}\Big)\frac{(h,v)}{v}\prod_{p|(h,v)}\Big(1-\frac{1}{p^\alpha}-\frac{1}{p^{\alpha+1}}\Big) \tag{4.31}$$

where $p^\alpha\,\|\,h$. Gathering the above results we arrive at

$$\begin{aligned}\sigma(h)=\Big\{\mu\Big(\frac{v}{(h,v)}\Big)\frac{(h,v)}{v}\prod_{p|(h,v)}\Big(1-\frac{1}{p^\alpha}-\frac{1}{p^{\alpha+1}}\Big)+(v\to w)\Big\}\\ \times\frac{\zeta_q(2)}{\zeta(2)}\Big(\sum_{\substack{d|h\\(d,q)=1}}\frac{1}{d}\Big)L(1,\chi)^2.\end{aligned} \tag{4.32}$$

In applications we shall appeal to the zeta function of the $\sigma(h)$,

$$Z(s)=\sum_{h=1}^{\infty}\sigma(h)h^{-s} \tag{4.33}$$

(note that $\sigma(h)=\sigma(-h)$ because the Ramanujan sums are even in h). Using (4.27) and (4.30) one derives

$$\begin{aligned}Z(s)=\Big\{\frac{1}{v}\prod_{p|v}\Big(1-\frac{1}{p^{s-1}}\Big)\prod_{p|w}\Big(1-\frac{1}{p^{s+1}}\Big)\\ +\frac{1}{w}\prod_{p|w}\Big(1-\frac{1}{p^{s-1}}\Big)\prod_{p|v}\Big(1-\frac{1}{p^{s+1}}\Big)\Big\}\\ \times\frac{\zeta_q(2)}{\zeta(2)}\zeta(s)\zeta(s+1)L(1,\chi)^2.\end{aligned} \tag{4.34}$$

Note that $Z(s)$ has no pole at $s=1$ except for $v=1$ or $w=1$, i.e. if the cusp is at ∞ or 0. In these cases the residue is

$$\operatorname*{res}_{s=1}Z(s)=L(1,\chi)^2\quad\text{ if } v=1 \text{ or } w=1. \tag{4.35}$$

The only other pole of $Z(s)$ is at $s = 0$ with residue

$$\operatorname*{res}_{s=0} Z(s) = (\mu(v) + \mu(w)) \frac{q}{\nu(q)} \cdot \frac{\zeta(0)}{\zeta(2)} L(1,\chi)^2 \tag{4.36}$$

where $\nu(q)$ is the multiplicative function with $\nu(p) = p + 1$ (see (2.1)). For curiosity we note that this residue vanishes if $\nu(q) = -1$, for example if q is prime.

5. Point to integral mean values of Dirichlet's series. Our objective is to estimate a Dirichlet series

$$A(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{5.1}$$

on average with respect to well-spaced points s. In this section we transform the problem to that for a corresponding integral in s. The procedure is well known and there are a variety of tools in the literature, just to mention the original one by P. X. Gallagher [G]. However the published results when applied directly to our series do not always produce the desired effects. What we need are integrals which can be treated further by quite delicate analysis in the off-diagonal range. For this reason we cannot afford to contaminate the coefficients a_n by wild test functions nor by sharp cuts. Therefore, rather than modifying the existing results, we shall develop the desired transformations from scratch.

Lemma 5.1. *Let a_n be any sequence of complex numbers such that*

$$\sum_n |a_n| < \infty. \tag{5.2}$$

Let $f(x)$ be a $\mathcal{C}^1$ function on $[1,\infty)$ such that

$$c_f = \int_1^{\infty} (x^{-1}|f(x)|^2 + x|f'(x)|^2)\, dx < \infty. \tag{5.3}$$

Then

$$\Big|\sum_n a_n f(n)\Big|^2 \le \frac{c_f}{\pi} \int_{-\infty}^{\infty} |A(it)|^2 \frac{dt}{t^2+1}. \tag{5.4}$$

Proof. First we extend $f(x)$ to the segment $[0,1)$ by setting $f(x) = xf(1)$. Then we write

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) x^{-it}\, dt \quad \text{where} \quad h(t) = \int_0^{\infty} f(x) x^{it-1}\, dx$$

by Mellin (or Fourier) inversion. This gives us

$$\sum_n a_n f(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t) A(it)\, dt.$$

Hence by the Cauchy–Schwarz inequality

$$\Big|\sum_n a_n f(n)\Big|^2 \le \frac{1}{4\pi^2}\Big(\int_{-\infty}^{\infty} |h(t)|^2(t^2+1)\, dt\Big) \int_{-\infty}^{\infty} |A(it)|^2 \frac{dt}{t^2+1}.$$

By Plancherel's theorem

$$\int_{-\infty}^{\infty} |h(t)|^2\, dt = 2\pi \int_0^{\infty} x^{-1}|f(x)|^2\, dx,$$

$$\int_{-\infty}^{\infty} |h(t)|^2 t^2\, dt = 2\pi \int_0^{\infty} x|f'(x)|^2\, dx.$$

Hence we obtain (5.4) with the constant

$$c_f^* = \frac{1}{2}\int_0^{\infty} (x^{-1}|f(x)|^2 + x|f'(x)|^2)\, dx$$

in place of c_f. Here the integral over the segment $[0,1)$ equals $|f(1)|^2$. Moreover we have

$$f(1)^2 = -\int_1^{\infty} (f(x)^2)'\, dx = -2\int_1^{\infty} f(x)f'(x)\, dx$$

$$\le \int_1^{\infty} (x^{-1}|f(x)|^2 + x|f'(x)|^2)\, dx.$$

Hence $c_f^* \le c_f$, proving (5.4).

COROLLARY 5.2. *Let the conditions be as in Lemma* 5.1. *Then for* $\varrho = \beta + i\gamma$ *with* $0 \le \beta \le 1/2$ *we have*

$$\Big|\sum_n a_n n^{-\varrho} f(n)\Big|^2 \le \frac{2c_f}{\pi} \int_{-\infty}^{\infty} |A(it)|^2 \frac{dt}{(t-\gamma)^2+1}. \tag{5.5}$$

Proof. Apply (5.4) for $a_n n^{-i\gamma}$ and $n^{-\beta} f(n)$ in place of a_n and $f(n)$.

Let $\mathcal{R}$ be a set of points $\varrho_r = \beta_r + i\gamma_r$ for $r = 1, \ldots, R$ such that

$$0 \le \beta_r \le 1/2, \tag{5.6}$$

$$T \le \gamma_r \le 2T, \tag{5.7}$$

$$|\gamma_r - \gamma_{r'}| \ge \delta \quad \text{if } r \ne r'. \tag{5.8}$$

Here δ, T are fixed numbers with $0 < \delta \le 1$ and $T \ge 2$. Note that $R \le 1 + T/\delta \le 3T/(2\delta)$.

Suppose for every $r = 1, \dots, R$ we have a $\mathcal{C}^1$ function $f_r(x)$ on $[1, \infty)$ such that the corresponding integrals (5.3) are bounded. Put

$$c = \max_r c_{f_r}, \tag{5.9}$$

$$G(t) = \sum_r ((t - \gamma_r)^2 + 1)^{-1}. \tag{5.10}$$

From (5.5) we get immediately

$$\sum_r \Big| \sum_n a_n n^{-\varrho_r} f_r(n) \Big|^2 \le \frac{2c}{\pi} \int_{-\infty}^{\infty} |A(it)|^2 G(t)\, dt. \tag{5.11}$$

Now we are going to estimate $G(t)$. By the spacing condition (5.8) we derive that

$$\begin{aligned} G(t) &\le 1 + \sum_{n=1}^{\infty} (\delta^2 n^2 + 1)^{-1} \\ &\le 1 + \int_0^{\infty} (\delta^2 t^2 + 1)^{-1}\, dt = 1 + \frac{\pi}{2\delta} < \frac{\pi}{\delta}. \end{aligned}$$

If t is far beyond the segment (5.7) we can do better. Indeed if $t < T/2$ then $(t - \gamma_r)^2 \ge (t - T)^2 \ge \frac{1}{5}(t^2 + T^2)$ and if $t > 3T$ then $(t - \gamma_r)^2 \ge (t - 2T)^2 \ge \frac{1}{10}(t^2 + T^2)$. Hence in these ranges

$$G(t) \le \frac{10R}{t^2 + T^2} \le \frac{15T}{\delta(t^2 + T^2)}.$$

Inserting these estimates into (5.11) we get

$$\begin{aligned} \sum_r \Big| \sum_n a_n n^{-\varrho_r} f_r(n) \Big|^2 \le{}& \frac{2c}{\delta} \int_{T/2}^{3T} |A(it)|^2\, dt \\ &+ \frac{10c}{\delta T} \int_{-\infty}^{\infty} |A(it)|^2 \Big(1 + \frac{t^2}{T^2}\Big)^{-1} dt. \end{aligned} \tag{5.12}$$

The first integral is no larger than ten times the second one, so we have

$$\sum_r \Big| \sum_n a_n n^{-\varrho_r} f_r(n) \Big|^2 \le 30 \frac{c}{\delta} \int_{-\infty}^{\infty} |A(it)|^2 \Big(1 + \frac{t^2}{T^2}\Big)^{-1} dt \tag{5.13}$$

(later we shall do better with the first integral). The last integral is exactly equal to

$$\int_{-\infty}^{\infty} |A(it)|^2 \Big(1 + \frac{t^2}{T^2}\Big)^{-1} dt = \pi T \sum_m \sum_n a_m \overline{a}_n \min\Big(\frac{m}{n}, \frac{n}{m}\Big)^T.$$

Now assuming that

$$G_1 = \sum_{n=1}^{\infty} n|a_n|^2 < \infty \tag{5.14}$$

we estimate as follows:

$$\sum_m \sum_n a_m \overline{a}_n \min\left(\frac{m}{n}, \frac{n}{m}\right)^T \leq \sum_n |a_n|^2 \sum_{m\geq n} \left(\frac{n}{m}\right)^T,$$

$$\sum_{m\geq n} \left(\frac{n}{m}\right)^T \leq 1 + \int_n^{\infty} \left(\frac{n}{x}\right)^T dx = 1 + \frac{n}{T-1} \leq 2\left(1 + \frac{n}{T}\right).$$

Hence

$$\int_{-\infty}^{\infty} |A(it)|^2 \left(1 + \frac{t^2}{T^2}\right)^{-1} dt \leq 2\pi(TG + G_1) \tag{5.15}$$

where

$$G = \sum_n |a_n|^2. \tag{5.16}$$

Inserting (5.15) into (5.13) we get

Lemma 5.3. *Let ϱ_r and $f_r(x)$ be as above. Suppose the complex numbers a_n satisfy* (5.2) *and* (5.14). *Then*

$$\sum_r \Big| \sum_n a_n n^{-\varrho_r} f_r(n) \Big|^2 \leq 189 \frac{c}{\delta} (TG + G_1) \tag{5.17}$$

where G, G_1 are defined by (5.16) *and* (5.14).

The estimate (5.17) (nevermind the constant 189) is not sufficiently strong when the range of coefficients a_n exceeds T. Having this case in mind we retain the first integral in (5.12) and apply (5.15) only to the second one. Actually we enlarge the first integral slightly while smoothing the integration. Precisely we set

$$\mathcal{A}(T) = \int_{-\infty}^{\infty} K(t/T) |A(it)|^2 \, dt \tag{5.18}$$

where $K(u)$ is a non-negative function on $\mathbb{R}$ such that $K(u) \geq 1$ for $1/2 \leq u \leq 3$. We obtain

Proposition 5.4. *Let $\varrho_r, f_r(x)$ and a_n be as above. Then*

$$\sum_r \Big| \sum_n n^{-\varrho_r} f_r(n) \Big|^2 \leq \frac{2c}{\delta} \mathcal{A}(T) + \frac{63c}{\delta T} (TG + G_1). \tag{5.19}$$

6. Evaluation of $\mathcal{A}(T)$. By (5.15) one gets the bound $\mathcal{A}(T) \ll TG+G_1$, which is essentially best possible in general. In this section we evaluate $\mathcal{A}(T)$ more precisely for special sequences $\mathcal{A} = (a_n)$. We assume that the cut-off function $K(u)$ in the integral (5.18) is continuous and symmetric on $\mathbb{R}$ with

$$K(0) = 0. \tag{6.1}$$

Moreover we assume that the cosine-Fourier transform

$$L(v) = 2\int_0^\infty K(u)\cos(uv)\,du \tag{6.2}$$

has fast decaying derivatives, specifically

$$|L^{(j)}(v)| \le (1+|v|)^{-4}, \qquad 0 \le j \le 5. \tag{6.3}$$

Clearly any smooth, symmetric and compactly supported function on $\mathbb{R}\backslash\{0\}$ does satisfy the above conditions up to a constant factor. We get

$$\mathcal{A}(T) = T\sum_m\sum_n a_m\overline{a}_n L\Big(T\log\frac{m}{n}\Big).$$

Here $L\big(T\log\frac{m}{n}\big)$ localizes the terms close to the diagonal. Therefore, we arrange this double sum according to the difference $m-n=h$ with the intention to treat every partial sum

$$S(h) = \sum_{m-n=h} a_m\overline{a}_n L\Big(T\log\frac{m}{n}\Big) \tag{6.4}$$

separately. Note that $S(-h) = \overline{S(h)}$ so we have

$$\mathcal{A}(T) = L(0)TG + 2T\,\mathrm{Re}\sum_{h>0} S(h). \tag{6.5}$$

Recall that G is given by (5.16). Here the zero term comes from the diagonal $m=n$; we have $S(0) = L(0)G$ and

$$L(0) = 2\int_0^\infty K(u)\,du. \tag{6.6}$$

Let $h>0$. Thinking of h as being relatively small we use the approximation

$$\log\frac{m}{n} = \log\Big(1+\frac{h}{n}\Big) = \frac{h}{n} + O\Big(\frac{h^2}{n^2}\Big)$$

to modify $L\big(T\log\frac{m}{n}\big)$ as follows:

$$L\Big(T\log\frac{m}{n}\Big) = L\Big(\frac{hT}{n}\Big) + O\Big(\frac{1}{T}\Big(1+\frac{hT}{n}\Big)^{-2}\Big).$$

The contribution of the error term to $S(h)$, say $S'(h)$, satisfies

$$S'(h) \ll \frac{1}{T} \sum_{m-n=h} |a_m a_n| \left(1 + \frac{hT}{n}\right)^{-2}$$
$$\ll \frac{1}{T} \sum_{m-n=h} (|a_m|^2 + |a_n|^2) \left(1 + \frac{hT}{n}\right)^{-2}.$$

Hence the contribution of the error terms to $\mathcal{A}(T)$, say $\mathcal{A}'(T)$, satisfies

$$\mathcal{A}'(T) \ll \sum\sum_{m>n} (|a_m|^2 + |a_n|^2) \left(1 + \frac{m-n}{n} T\right)^{-2}.$$

Hence it follows that

$$\mathcal{A}'(T) \ll G + T^{-1} G_1. \tag{6.7}$$

We are left with

$$\mathcal{A}(T) = L(0)TG + 2T \operatorname{Re} \sum_{h>0} S^*(h) + O(G + T^{-1}G_1) \tag{6.8}$$

where $S^*(h)$ is the modified sum

$$S^*(h) = \sum_n a_{n+h} \overline{a}_n L\left(\frac{hT}{n}\right). \tag{6.9}$$

We may estimate $S^*(h)$ trivially as follows:

$$S^*(h) \ll \sum_n (|a_{n+h}|^2 + |a_n|^2) \left(\frac{n}{hT}\right)^2 \le 2(hT)^{-2} G_2 \tag{6.10}$$

subject to the condition

$$G_2 = \sum_n n^2 |a_n|^2 < \infty. \tag{6.11}$$

This estimate is quite useful for large h, say $h \ge H$, where H will be defined later. Inserting (6.10) into (6.8) we get

$$\mathcal{A}(T) = L(0)TG + 2T \operatorname{Re} \sum_{0<h\le H} S^*(h) + O(G + T^{-1}G_1 + T^{-1}H^{-1}G_2). \tag{6.12}$$

Observe that the terms of (6.9) with $n \le 2h$ contribute less than

$$\sum_{n\le 2h} |a_{n+h} a_n L(hT/n)| \ll (hT)^{-2} \sum_{n \le 3h} n^2 |a_n|^2.$$

Summing over h we find that these small terms contribute to $\mathcal{A}(T)$ less than

$$T^{-1} \sum_n n^2 |a_n|^2 \sum_{3h \ge n} h^{-2} \ll T^{-1} G_1,$$

which is absorbed by the error term already present in (6.12).

Now we require that the coefficients a_n are given by

$$a_n = \lambda(n)a(n) \tag{6.13}$$

where $\lambda(n)$ is a nice arithmetic function and $a(y)$ is a smooth cut-off function. We do not restrict $a(y)$ to a dyadic segment, but for practical needs we require only that $a(y)$ is a $\mathcal{C}^2$ function on $\mathbb{R}^+$ such that

$$y^\nu |a^{(\nu)}(y)| \le (1 + y/Y)^{-4} \quad \text{if } \nu = 0, 1, 2, \tag{6.14}$$

where $Y \ge 2$. Concerning $\lambda(n)$ we assume that it is bounded by the divisor function

$$|\lambda(n)| \le \tau(n). \tag{6.15}$$

Therefore our coefficients are almost bounded, precisely

$$|a_n| \le \tau(n)(1 + n/Y)^{-4}. \tag{6.16}$$

Hence the series of $|a_n|^2$, $n|a_n|^2$, $n^2|a_n|^2$ converge and satisfy

$$G \ll Y(\log Y)^3, \quad G_1 \ll Y^2(\log Y)^3, \quad G_2 \ll Y^3(\log Y)^3. \tag{6.17}$$

Moreover about $\lambda(n)$ we postulate that for any two smooth functions $g_1(x)$, $g_2(x)$ supported in $[X, 2X]$ with $X \ge 1/2$ such that

$$x^\nu |g_j^{(\nu)}(x)| \le 1 \quad \text{if } \nu = 0, 1, 2, \tag{6.18}$$

and for any $h \ge 1$ we have

$$\sum_{m-n=h} \lambda(m)\overline{\lambda}(n)g_1(m)g_2(n) = \sigma(h)\int g_1(x+h)g_2(x)\,dx + O(B\tau(h)X^{3/4}(\log 3X)^2). \tag{6.19}$$

Here $\sigma(h)$ is another nice arithmetic function depending on λ, B is a positive constant depending on λ, and the implied constant in the error term is absolute. We assume that

$$|\sigma(h)| \le C\sum_{d|h} d^{-1}. \tag{6.20}$$

In other words the generating series

$$Z(s) = \sum_{h=1}^{\infty} \sigma(h)h^{-s} \tag{6.21}$$

is majorized by $C\zeta(s)\zeta(s+1)$. More precisely we assume that

$$Z(s) = \zeta(s)z(s) \tag{6.22}$$

where $z(s)$ is holomorphic in $\operatorname{Re} s \ge 0$, except for a simple pole at $s = 0$. Suppose

$$|z(s) - Z/s| \le C(|s| + 1) \tag{6.23}$$

in the strip $0 \le \operatorname{Re} s \le 3/2$. We do not exclude the residue $Z = 0$, and we assume

$$(6.24)\qquad |Z| \le C.$$

For our primary example we let the $\lambda(n) = \lambda_F(n)$ be the Hecke eigenvalues of a primitive cusp form $F \in S_k(\Gamma_0(q), \chi_q)$. In this case (6.15) is proved by P. Deligne (the Ramanujan conjecture) and the formula (6.19) is established in our Theorem 4.3 with $\sigma(h) = 0$ and $B = k^4 q^6$. Therefore $Z = 0$ and $C = 0$.

Our second example is the Hecke eigenvalue $\lambda(n) = \tau(n; \chi_v, \chi_w)$ of a holomorphic Eisenstein series of weight $k = 1$ and level q. In this case (6.15) is obvious by (3.19) and the formula (6.19) is established in our Theorem 4.4 with $\sigma(h)$ given by (4.32) and $B = q^6$. The generating series $Z(s)$ is computed in (4.34) and the residue of $z(s)$ at $s = 0$ is

$$(6.25)\qquad Z = -\frac{3}{\pi^2}(\mu(v) + \mu(w))\frac{q}{\nu(q)} L(1, \chi_q)^2$$

(see (4.36)). In this case the estimates (6.20), (6.23), (6.24) hold with

$$(6.26)\qquad C \ll \frac{\nu(q)}{q} L(1, \chi_q)^2 \log q.$$

Now we are ready to evaluate $S^*(h)$. By (6.19) we derive

$$(6.27)\qquad S^*(h) = \sigma(h) \int a(y + h)\overline{a}(y) L(hT/y)\, dy \\ + O(B\tau(h) Y^{3/4} (\log Y)^4 + T^{-2} h |\sigma(h)| + T^{-2} h (\log 3h)^3).$$

Well, not immediately because $a(y)$ is not supported in a dyadic segment. However, using a smooth partition of unity with constituents in m, n supported in segments of type $[Y_1, \sqrt{2}\, Y_1]$, $[Y_2, \sqrt{2}\, Y_2]$ respectively one can justify the applicability of (6.19) as follows. Indeed there is no question when the two segments $[Y_1, \sqrt{2}\, Y_1]$, $[Y_2, \sqrt{2}\, Y_2]$ are equal or adjacent. If these segments are separated then they produce nothing from the sum nor from the integral in (6.27) unless $Y_1, Y_2 \le \sqrt{2}\, h$. In this case we estimate trivially by

$$\sum_{n \le 2h} |a_{n+h} a_n| \left(\frac{n}{hT}\right)^2 \le \sum_{n \le 3h} |a_n|^2 \left(\frac{n}{hT}\right)^2 \ll T^{-2} h (\log 3h)^3,$$

which yields the third error term in (6.27). Moreover, the integral over $x \le 2h$ is estimated similarly by

$$\int_0^{2h} |a(x + h) a(x)| \left(\frac{x}{hT}\right)^2 dx \le \int_0^{3h} |a(y)|^2 \left(\frac{y}{hT}\right)^2 dy \ll T^{-2} h,$$

which yields the second error term in (6.27).

Next we replace $a(y+h)$ in (6.27) by $a(y)$ with the difference $O(Y^2T^{-2} \times h^{-1}|\sigma(h)|)$. We obtain

$$S^*(h) = \sigma(h)\int |a(y)|^2 L(hT/y)\,dy + O(B\tau(h)(Y^{3/4} + Y^2T^{-2}h^{-1} + T^{-2}h)(\log hY)^4) \tag{6.28}$$

where the implied constant is absolute. This is true for all $h \geq 1$, but we only use this for $1 \leq h \leq H$, where H will be chosen later. Introducing (6.28) into (6.12) we derive

$$\begin{aligned}\mathcal{A}(T) = L(0)TG + 2T\int |a(y)|^2 \Big(\sum_{h=1}^{H} \sigma(h)L(hT/y)\Big)\,dy \\ + O(B(THY^{3/4} + T^{-1}Y^2 + T^{-1}H^2)(\log HY)^5) \\ + O((Y + T^{-1}Y^2 + T^{-1}H^{-1}Y^3)(\log Y)^3).\end{aligned}$$

Note that we can extend the sum over $1 \leq h \leq H$ to the infinite series

$$D(v) = \sum_{h=1}^{\infty} \sigma(h)L(hv) \tag{6.29}$$

with $v = Ty^{-1}$, up to the error term $O(T^{-1}H^{-1}Y^3 \log H)$ which is already present (the last one). Having done this we choose

$$H = B^{-1/2}T^{-1}Y^{9/8}(\log Y)^{-1} \tag{6.30}$$

(this choice equalizes the first and the last error terms), getting

$$\mathcal{A}(T) = L(0)TG + 2T\int |a(y)|^2 D(T/y)\,dy + O(B^{1/2}Y^{15/8}(\log Y)^4 + T^{-3}Y^{9/4}(\log Y)^3). \tag{6.31}$$

Next we evaluate the series $D(v)$. Let $M(s)$ be the Mellin transform of $L(v)$,

$$M(s) = \int_0^\infty L(v)v^{s-1}\,dv.$$

Integrating by parts we derive, by (6.2),

$$sM(s) \ll (|s| + 1)^{-4}. \tag{6.32}$$

Note that

$$M(1) = \int_0^\infty L(v)\,dv = 2\pi K(0) = 0$$

by our assumption (6.1). Therefore the product $M(s)Z(s)$ is holomorphic in the strip $0 < \sigma \leq 3/2$ (no pole at $s = 1$) and

$$M(s)Z(s) \ll C|s|^{-2}$$

by (6.22)–(6.24) and (6.26). However for s near zero we need a more precise expansion. To this end we use

$$\zeta(s) = \zeta(0) + O(|s|), \qquad z(s) = Z/s + O(C),$$

and we derive an expansion for $M(s)$ as follows:

$$\begin{aligned}M(s) &= L(0)/s + \int_0^1 (L(v)-L(0))v^{s-1}\,dv + \int_1^\infty L(v)v^{s-1}\,dv \\ &= L(0)/s + O(1).\end{aligned}$$

From these expansions we get

$$M(s)Z(s) = \frac{a}{s^2} + \frac{b}{s} + O(C)$$

where $a = \zeta(0)L(0)Z = -\frac{1}{2}L(0)Z \ll C$ and $b \ll C$. Combining both estimates we get

$$M(s)Z(s) = \frac{a}{s^2} + \frac{b}{s(s+1)} + O\left(\frac{C}{|s|^2+1}\right) \tag{6.33}$$

uniformly in $0 < \sigma \le 3/2$. Now we are ready to evaluate $D(v)$. We have

$$\begin{aligned}D(v) &= \frac{1}{2\pi i}\int_{(\sigma)} M(s)Z(s)v^{-s}\,ds \\ &= a\log^+\frac{1}{v} + b\max(0, 1-v) + O(Cv^{-\sigma}).\end{aligned}$$

Hence we write

$$D(v) = -\frac{1}{2}L(0)Z\log^+\frac{1}{v} + D_0(v) \tag{6.34}$$

with $D_0(v)$ a bounded function, specifically

$$D_0(v) \ll C \tag{6.35}$$

by letting $\sigma \to 0$ (recall the uniformity in σ). Inserting (6.34) into (6.31) we summarize the above considerations in the following

THEOREM 6.1. *Let $K(u)$ be a continuous and symmetric function on $\mathbb{R}$ with $K(0) = 0$ such that* (6.2) *holds. Let $\lambda(n)$ be an arithmetic function with $|\lambda(n)| \le \tau(n)$ which satisfies the formula* (6.19) *with the surrounding conditions* (6.18)–(6.24). *Let $a(y)$ be a $\mathcal{C}^2$ function on $\mathbb{R}^+$ such that* (6.14) *holds. Then*

$$\begin{aligned}&\int_{-\infty}^{\infty} K\left(\frac{t}{T}\right)\Big|\sum_n a(n)\lambda(n)n^{-it}\Big|^2\,dt \\ &\quad= \widehat{K}(0)T\left\{G - Z\int_T^\infty |a(y)|^2\left(\log\frac{y}{T}\right)dy\right\} \\ &\qquad+ 2T\int_0^\infty |a(y)|^2 D_0\left(\frac{T}{y}\right)dy + O(B^{1/2}Y^{15/8}(1+T^{-3}Y^{3/8})(\log Y)^4)\end{aligned} \tag{6.36}$$

where

$$G = \sum_n |a(n)\lambda(n)|^2 \tag{6.37}$$

and B, C, Z are the constants depending on $\lambda(n)$ given by the postulated properties (6.18)–(6.24). *Moreover $D_0(v)$ is defined by* (6.34), *so $D_0(v) \ll C$, the implied constants being absolute.*

From (6.36) one can derive a mean-value theorem for $A(s)$ on the line $\mathrm{Re}\, s = 1/2$, however not without some loss in the error term. We do it for $A(1/2+it)$ localized between T and Y. Precisely we get

Corollary 6.2. *Let the conditions be as in Theorem* 6.1, *except for* (6.14) *which is now replaced by*

$$y^\nu |a^{(\nu)}(y)| \le \left(1 + \frac{y}{Y} + \frac{T}{y}\right)^{-4} \qquad \text{if } \nu = 0, 1, 2, \tag{6.38}$$

where $T \le Y \le T^8$. Then

$$\begin{aligned} &\int_{-\infty}^{\infty} K\left(\frac{t}{T}\right) \Big| \sum_n a(n)\lambda(n) n^{1/2-it} \Big|^2 \, dt \\ &= \widehat{K}(0) T \left\{ G - Z \int_T^\infty |a(y)|^2 \left(\log \frac{y}{T}\right) \frac{dy}{y} \right\} \\ &\quad + 2T \int_0^\infty |a(y)|^2 D_0\left(\frac{T}{y}\right) \frac{dy}{y} + O(B^{1/2} T^{-1} Y^{15/8} (\log Y)^4) \end{aligned} \tag{6.39}$$

where Z, $D_0(v)$ and B are as before, but

$$G = \sum_n |a(n)\lambda(n)|^2 n^{-1}. \tag{6.40}$$

Proof. Apply (6.36) for the function $a(y)\sqrt{T/y}$ in place of $a(y)$. This modified function satisfies (6.14) apart of an absolute constant factor by virtue of (6.38). Then divide the resulting formula throughout by T.

Estimating all but the first term on the right side of (6.39) we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} K\left(\frac{t}{T}\right) |A(1/2+it)|^2 \, dt \\ &= \widehat{K}(0) T G + O\left(T\left(|Z| \log \frac{Y}{T} + C\right) \log \frac{Y}{T} + B^{1/2} T^{15/16} (\log T)^4\right) \end{aligned} \tag{6.41}$$

if $2T \le Y \le T^{31/30}$. Moreover $G \ll (\log Y)^3 \log(Y/T)$ by (6.15). But we are looking for a better estimate of G; besides reducing by logarithms we want to see the implied constant.

We are most interested in Hecke eigenvalues $\lambda(n)$ of automorphic forms associated with the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-q})$. For these the Ramanujan bound (6.15) can be improved to

$$|\lambda(n)| \le \tau(n, \chi). \tag{6.42}$$

The extremal case $\lambda(n) = \tau(n,\chi)$ comes from the Eisenstein series $E_{\mathfrak{a}}(z) = y^{-1/2}E_{\mathfrak{a}}(z,1/2)$ for cusps $\mathfrak{a} = 0$ or $\mathfrak{a} = \infty$. The zeta function of $\tau(n,\chi)$ is the L-function of $K = \mathbb{Q}(\sqrt{-q})$,

$$L_K(s) = \sum_{0\neq\mathfrak{a}\subset\mathcal{O}_K} (N\mathfrak{a})^{-s} = \sum_{n=1}^{\infty} \tau(n,\chi)n^{-s} = \zeta(s)L(s,\chi).$$

The Rankin–Selberg L-function is

$$(6.43)\quad R_K(s) = \sum_{n=1}^{\infty} \tau(n,\chi)^2 n^{-s} = \zeta(s)^2 L(s,\chi)^2 \zeta(2s)^{-1} \prod_{p|q}\Big(1+\frac{1}{p^s}\Big)^{-1}.$$

This has the Taylor expansion

$$R_K(s) = \frac{\alpha}{(s-1)^2} + \frac{\beta}{s-1} + \gamma + \dots$$

with the polar coefficients given by

$$(6.44)\qquad \alpha = \frac{q}{\nu(q)}\cdot\frac{L(1,\chi)^2}{\zeta(2)},$$

$$(6.45)\qquad \beta = \frac{q}{\nu(q)}\cdot\frac{L(1,\chi)^2}{\zeta(2)}\Big[2\frac{L'}{L}(1,\chi) + \gamma_1 + \sum_{p|q}\frac{\log p}{p+1}\Big].$$

Moreover

$$R_K(s) \ll q^{1/2}|s|^{5/6} \qquad \text{if } \operatorname{Re} s \geq 1/2.$$

Let $g(y)$ be a $\mathcal{C}^2$ function on $\mathbb{R}^+$ such that

$$(6.46)\qquad y^\nu |g^{(\nu)}(y)| \leq \Big(1+\frac{y}{Y}+\frac{X}{y}\Big)^{-1} \qquad \text{if } \nu = 0,1,2,$$

where $Y \geq X > 0$. Let $\check{g}(s)$ be the Mellin transform of $g(y)$,

$$\check{g}(s) = \int_0^\infty g(y)y^{s-1}\,dy = \int_0^\infty g(y)(1+s\log y+\dots)\,\frac{dy}{y}.$$

By partial integration we get

$$\check{g}(s) \ll |s|^{-2}(X^\sigma + Y^\sigma) \qquad \text{if } \sigma = \operatorname{Re} s = \pm 1/2.$$

By contour integration the sum

$$(6.47)\qquad G = \sum_n \tau(n,\chi)^2\,\frac{g(n)}{n}$$

is equal to

$$G = \frac{1}{2\pi i} \int_{(1/2)} \check{g}(s) R_K(s+1)\, ds$$
$$= \operatorname*{res}_{s=0} \check{g}(s) R_K(s+1) + \frac{1}{2\pi} \int_{(-1/2)} \check{g}(s) R_K(s+1)\, ds.$$

Hence using the above estimates we get

$$G = \int_0^\infty g(y)(\alpha \log y + \beta)\, \frac{dy}{y} + O((q/X)^{1/2}). \tag{6.48}$$

Corollary 6.3. *For $Y \ge 2X \ge 2$ we have*

$$\sum_{X \le n \le Y} \tau(n,\chi)^2 n^{-1} \ll \mathcal{L}(Y) \log(Y/X) + (q/X)^{1/2} \tag{6.49}$$

where

$$\mathcal{L}(Y) = L(1,\chi)(L(1,\chi)\log Y + |L'(1,\chi)|). \tag{6.50}$$

For G given by (6.40) the formula (6.48) becomes

$$G = \int_0^\infty |a(y)|^2 (\alpha \log y + \beta)\, \frac{dy}{y} + O((q/T)^{1/2}). \tag{6.51}$$

Hence, if $2T \le Y \le T^{31/30}$ we get

$$G \ll \mathcal{L}(T) \log(Y/T) + (q/T)^{1/2}.$$

Introducing this into (6.35) we end up with the following

Proposition 6.4. *Let $\lambda(n)$ be the coefficients of an automorphic form given by Hecke characters of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-q})$ of discriminant $-q$. Let $a(y)$ be a function satisfying* (6.38) *with $Y = qT$ and $T \ge K^{32} q^{65}$. Then*

$$\int_T^{2T} \Big| \sum_n a(n)\lambda(n) n^{-1/2-it} \Big|^2 dt \ll T\mathcal{L}(T) \log q \tag{6.52}$$

where $\mathcal{L}(T)$ is defined by (6.50) *and the implied constant is absolute.*

Remark. The above bound comes from the diagonal terms. The other terms contribute slightly less, namely $T\mathcal{L}(q)\log q$.

7. Approximate functional equation. We restrict our attention to L-functions for class group characters of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-q})$ where $-q$ is the discriminant. We assume that q is odd and $q > 4$, so $q \equiv 3 \pmod 4$ and q is squarefree. Fix $\psi \in \widehat{\mathcal{C}\ell}(K)$ and put

$$\lambda(n) = \sum_{N\mathfrak{a}=n} \psi(\mathfrak{a}). \tag{7.1}$$

These are Hecke eigenvalues of an automorphic form (a theta series) of weight $k = 1$, level q and character

$$\chi(n) = \left(\frac{n}{q}\right), \tag{7.2}$$

the Jacobi symbol. Let

$$L(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s} \tag{7.3}$$

be the corresponding Hecke L-function. For example, if ψ is a genus character then

$$L(s) = L(s, \chi_v) L(s, \chi_w) \tag{7.4}$$

where χ_v, χ_w are the real characters of conductor v, w respectively with $vw = q$, i.e.

$$\chi_v(n) = \left(\frac{n}{v}\right), \quad \chi_w(n) = \left(\frac{n}{w}\right) \tag{7.5}$$

are the corresponding Jacobi symbols. Observe that χ_v, χ_w are characters for real and imaginary quadratic fields. If $\psi \in \widehat{\mathcal{C}\ell}(K)$ is not a genus character (i.e. ψ is not real) then the corresponding L-function does not factor into Dirichlet L-functions. However, in any case the complete product

$$\Lambda(s) = Q^s \Gamma(s) L(s) \quad \text{with} \quad Q = \frac{\sqrt{q}}{2\pi} \tag{7.6}$$

has an analytic continuation to the whole complex s-plane, except for a simple pole at $s = 1$ if ψ is the trivial character, in which case

$$L(s) = \zeta_K(s) = \zeta(s) L(s, \chi) \tag{7.7}$$

is the zeta function of K. Moreover for any $\psi \in \widehat{\mathcal{C}\ell}(K)$ we have the functional equation (which is due to Hecke, see also (3.9))

$$\Lambda(s) = \Lambda(1 - s). \tag{7.8}$$

In this section we derive a Dirichlet series representation of $L(s)$ tempered by a test function which makes the series rapidly convergent. Formulas of this type are known in the literature as "approximate functional equations". In our context, this is a somewhat misleading name, because we need exact expressions to be able to differentiate. We rather think of these as a kind of Poisson's summation formulas.

Let $G(u)$ be a holomorphic function in the strip $|\mathrm{Re}\, u| \le 1$ such that

$$G(u) = G(-u), \tag{7.9}$$

$$G(0) = 1, \tag{7.10}$$

$$G(u) \ll 1. \tag{7.11}$$

Consider the integral

$$I(s) = \frac{1}{2\pi i} \int_{(1)} \Lambda(s+u) G(u) u^{-1}\, du$$

for $0 < \mathrm{Re}\, s < 1$. Moving the path of integration to the line $\mathrm{Re}\, u = -1$ and applying (7.8) we get

$$\Lambda(s) = I(s) + I(1-s) - \frac{G(s-1)}{s-1} \operatorname*{res}_{u=1} \Lambda(u).$$

On the other hand, introducing the Dirichlet series (7.3) and integrating termwise we obtain

$$I(s) = \sum_n \lambda(n) \frac{1}{2\pi i} \int_{(1)} \left(\frac{Q}{n}\right)^{s+u} \Gamma(s+u) G(u) u^{-1}\, du.$$

From both expressions we obtain (after dividing by $\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s)$)

Proposition 7.1. *For s with $0 < \mathrm{Re}\, s < 1$ we have*

$$L(s) = \sum_n \lambda(n) n^{-s} V_s\left(\frac{n}{Q}\right) + X(s) \sum_n \lambda(n) n^{s-1} V_{1-s}\left(\frac{n}{Q}\right) - \frac{G(s-1)}{(s-1)\Gamma(s)} Q^{1-s} L(1,\chi), \tag{7.12}$$

where

$$X(s) = Q^{1-2s} \Gamma(1-s)/\Gamma(s), \tag{7.13}$$

$$V_s(y) = \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma(s+u)}{\Gamma(s)} \cdot \frac{G(u)}{u}\, y^{-u}\, du \tag{7.14}$$

and the last (*residual*) *term in* (7.12) *exists only if ψ is the trivial character of $\mathcal{C}\ell(K)$.*

We shall apply (7.12) for points on the critical line $\mathrm{Re}\, s = 1/2$. Choosing

$$G(u) = \left(\cos \frac{\pi u}{A}\right)^{-A} \tag{7.15}$$

where $A \ge 4$ is a fixed integer we derive

LEMMA 7.2. *If* $\operatorname{Re} s = 1/2$ *then*

$$y^a V_s^{(a)}(y) = \delta(a) - \frac{G(s)}{\Gamma(s+1-a)}\, y^s + O\Big(\frac{y}{|s|}\Big), \tag{7.16}$$

$$y^a V_s^{(a)}(y) \ll \Big(1 + \frac{y}{|s|}\Big)^{-A} \tag{7.17}$$

for any $a \ge 0$, *the implied constant depending only on* a *and* A (*here* $\delta(0) = 1$ *and* $\delta(a) = 0$ *if* $a > 0$).

REMARK. We have $G(s) \ll e^{-\pi|s|}$ and $\Gamma(s+1-a)^{-1} \ll |s|^{a-1} e^{\frac{\pi}{2}|s|}$; hence, Lemma 7.2 yields

$$y^a V_s^{(a)}(y) = \delta(a) + O(\sqrt{y/|s|}).$$

Proof. Differentiating (7.14) a times we get

$$y^a V_s^{(a)}(y) = \frac{1}{2\pi i} \int_{(1)} \frac{\Gamma(s+u)}{\Gamma(s)} \cdot \frac{G(u)}{u} \prod_{0 \le b < a} (-u-b) y^{-u}\, du. \tag{7.18}$$

For the proof of (7.16) we move the integration to the line $\operatorname{Re} u = -1$ getting the first two terms as residues at $u = 0$ and $u = -s$ respectively. By Stirling's formula the resulting integral on $\operatorname{Re} u = -1$ is estimated by

$$\int_{(-1)} |s+u|^{-1} e^{-\frac{\pi}{2}|s+u| + \frac{\pi}{2}|s| - \pi|u|} |u|^{a-1} y\, |du| \ll \int_{(-1)} |s+u|^{-1} e^{-\frac{\pi}{2}|u|} |u|^{a-1} y\, |du| \ll y/|s|.$$

For the proof of (7.17) we move the integration to the line $\operatorname{Re} u = A$. By Stirling's formula the resulting integral is estimated by

$$\int_{(A)} |s+u|^A e^{-\frac{\pi}{2}|u|} |u|^{a-1} y^{-A}\, |du| \ll (|s|/y)^A.$$

This yields (7.17) if $y > |s|$. In the case $y \le |s|$ we get (7.17) from (7.16).

Actually we shall apply (7.12) to estimate the quotients

$$\ell(s) = \frac{L(s) - L(s')}{s - s'} \tag{7.19}$$

for points s, s' on the critical line (if $s = s'$, then $\ell(s) = L'(s)$ is the derivative of $L(s)$). Here we do not display the dependence of $\ell(s)$ on the second point s' for notational simplicity. This abbreviated notation (also used for other forthcoming quotients) will be justified when we fix s' in terms of s. Put

$$x(s) = \frac{X(s) - X(s')}{s - s'}, \tag{7.20}$$

$$v_s(y) = \frac{V_s(y) - V_{s'}(y)}{s - s'}, \tag{7.21}$$

$$w_s(y) = \frac{1-y^{s-s'}}{s-s'}. \tag{7.22}$$

From (7.12) we derive (by adding and subtracting terms)

$$\begin{aligned}\ell(s) = &\Big(\sum_n - X(s)\overline{\sum_n}\Big)\lambda(n)n^{-s}w_s(n)V_s(n/Q)\\ &+\Big(\sum_n - X(s)\overline{\sum_n}\Big)\lambda(n)n^{-s}v_s(n/Q)\\ &+ x(s)\overline{\sum_n}\lambda(n)n^{-s}V_s(n/Q) + O(Q/|s|),\end{aligned} \tag{7.23}$$

where $\overline{\sum_n}$ stands for the complex conjugate of $\sum_n$.

Now we need estimates for $x(s)$ and for derivatives of $v_s(y), w_s(y)$.

LEMMA 7.3. *For s, s' on the critical line we have*

$$|w_s(y)| \le |\log y|, \qquad |w_s'(y)| = y^{-1}. \tag{7.24}$$

LEMMA 7.4. *For s, s' on the critical line we have*

$$y^a v_s^{(a)}(y) \ll (y/|s|)^{1/4}(1+y/|s|)^{-A} \tag{7.25}$$

if $a \ge 0$, and the implied constant depends only on a and A.

Proof. If $|s-s'| > 1$ then Lemma 7.4 follows from Lemma 7.2 by subtracting the estimates. Let $|s-s'| \le 1$. Subtract (7.18) for s' from that for s and divide by $s-s'$ to obtain a corresponding expression for derivatives of $v_s(y)$. Then move the integration from $\operatorname{Re} u = 1$ to $\operatorname{Re} u = \alpha$ with $-1/4 \le \alpha \le A$. Note that there is no pole at $u = 0$. Then estimate as follows:

$$\begin{aligned}\frac{1}{|s-s'|}\left|\frac{\Gamma(s+u)}{\Gamma(s)} - \frac{\Gamma(s'+u)}{\Gamma(s')}\right| &\le \left|\frac{\partial}{\partial s}\frac{\Gamma(s+u)}{\Gamma(s)}\right|_{s=s_0}\\ &= \left|\frac{\Gamma(s_0+u)}{\Gamma(s_0)}\right| |\psi(s_0+u) - \psi(s_0)|\\ &\ll |s_0+u|^{\alpha} e^{\frac{\pi}{2}|u|}|u|\end{aligned}$$

where s_0 is a point on the critical line between s and s'. Moreover $G(u) \ll e^{-\pi|u|}$. Hence

$$y^a v_s^{(a)}(y) \ll y^{-\alpha}\int_{(\alpha)} |s_0+u|^{\alpha}|u|^a e^{-\frac{\pi}{2}|u|}\,|du| \ll (|s|/y)^{\alpha}.$$

This implies (7.25) by taking $\alpha = -1/4$ if $y \le |s|$, or $\alpha = A$ if $y > |s|$.

Before estimating $x(s)$ note that $|X(s)| = 1$ for $\operatorname{Re} s = 1/2$; more precisely

$$X(1/2+it) = Q^{-2it}\frac{\Gamma(1/2-it)}{\Gamma(1/2+it)} = \left(\frac{e}{tQ}\right)^{2it}\{1+\varepsilon(t)\} \tag{7.26}$$

if $t \ge 1$, where $\varepsilon(t) \ll t^{-1}$ and $\varepsilon'(t) \ll t^{-2}$. Hence we derive

LEMMA 7.5. *For $s = 1/2 + it$ and $s = 1/2 + it'$ with $t, t' \ge 1$ we have*

$$x(s) = -2\left(\frac{e}{tQ}\right)^{it}\left(\frac{e}{t'Q}\right)^{it'}\frac{\sin(t-t')\log tQ}{t-t'} + O\left(\frac{1}{t}\right); \tag{7.27}$$

consequently,

$$|x(s)| = 2\left|\frac{\sin(t-t')\log tQ}{t-t'}\right| + O\left(\frac{1}{t}\right). \tag{7.28}$$

Proof. If $|t-t'| < t/2$ then (7.27) follows from (7.26); otherwise (7.27) is trivial.

Applying the inequality $\sin x \ge \alpha x$ if $0 \le x \le \pi(1-\alpha)$ we get

COROLLARY 7.6. *Let $0 \le \alpha \le 1$. If $|t-t'|\log tQ \le \pi(1-\alpha)$, then*

$$|x(s)| \ge 2\alpha \log tQ + O(1/t). \tag{7.29}$$

8. Evaluation of $\ell(s)$ on average. Our goal is to eliminate most of the terms in (7.23) by estimating them on average with respect to a well-spaced set of points s on the critical line. We start with any set, say $S(T)$, of points

$$s_r = 1/2 + it_r, \qquad r = 1, \ldots, R, \tag{8.1}$$

such that for $T \ge 2$,

$$T < t_1 < \ldots < t_R \le 2T, \tag{8.2}$$

$$t_{r+1} - t_r \ge 1 \quad \text{if } 1 \le r < R. \tag{8.3}$$

To each point s_r we associate a point

$$s_r' = 1/2 + it_r'. \tag{8.4}$$

REMARKS. The companion s_r' to s_r may not be in $S(T)$. Actually our main interest will be to choose s_r' very close to s_r. For example s_r, s_r' can be consecutive zeros of $L(s)$ on the critical line. We may have $s_r = s_r'$ if this is a double zero. For the time being we assume that $T \ge q^{65}$ to comply with the condition of Proposition 6.4, but after shaping the basic estimates this assumption can be dispensed with because the results hold true trivially if $T < q^{65}$.

First we estimate the sum

$$A_1(s) = \sum_n \lambda(n) n^{-s} w_s(n) V_s(n/Q) \tag{8.5}$$

on average with respect to the points $s \in S(T)$. Recall that $w_s(y)$ satisfies (7.24) and $V_s(y)$ satisfies (7.17). We partition this sum smoothly into three sums, say $A_1(s) = A_{11}(s) + A_{12}(s) + A_{13}(s)$, where the partial sums are supported on the segments $n_1 \ll q^4 \ll n_2 \ll T \ll n_3$, respectively. For estimation of $A_{11}(s)$ we apply Lemma 5.3 with $c \ll \log q$ and $a_n \ll |\lambda(n)| n^{-1/2} \log n$ and obtain

$$\sum_s |A_{11}(s)|^2 \ll T\Big(\sum_{n \ll q^4} \tau(n)^2 n^{-1} (\log n)^2\Big) \log q \ll T(\log q)^7$$

by the trivial estimate (6.15). For estimation of $A_{12}(s)$ we apply Lemma 5.3 with $c \ll \log T$ and $a_n \ll |\lambda(n)| n^{-1/2} \log n$; now, however, we take advantage of the better bound given by (6.42) to see that

$$\sum_s |A_{12}(s)|^2 \ll T\Big(\sum_{q^4 \ll n \ll T} \tau(n,\chi)^2 n^{-1} (\log n)^2\Big) \log T \ll T\mathcal{L}(T)(\log T)^4$$

by (6.50). For estimation of $A_{13}(s)$ we apply Proposition 5.4 with $c \ll \log q$ and $a_n \ll |\lambda(n)| n^{-1/2} \log n$, together with Proposition 6.4, getting

$$\sum_s |A_{13}(s)|^2 \ll T\mathcal{L}(T)(\log T)^2 (\log q)^2.$$

Next we estimate the sum

$$A_2(s) = \sum_n \lambda(n) n^{-s} v_s(n/Q). \tag{8.6}$$

The arguments are the same as those applied for $A_1(s)$ above, and the corresponding estimates are sharper by two logarithms because $v_s(y) \ll 1$ and $w_s(y) \ll \log y$. Precisely, we get $A_2(s) = A_{21}(s) + A_{22}(s) + A_{23}(s)$ with

$$\sum_s |A_{21}(s)|^2 \ll T(\log q)^5,$$
$$\sum_s |A_{22}(s)|^2 \ll T\mathcal{L}(T)(\log T)^2,$$
$$\sum_s |A_{23}(s)|^2 \ll T\mathcal{L}(T)(\log q)^2.$$

Remark. If we used the more precise bound for $v_s(y)$ given in Lemma 7.4, then the above estimates could be improved further, but that leads to no advantage here.

It remains to estimate the sum

$$A_3(s) = \sum_n \lambda(n) n^{-s} V_s(n/Q). \tag{8.7}$$

Similarly we partition this sum smoothly into $A_3(s) = A_{31}(s) + A_{32}(s) +$

$A_{33}(s)$, and apply the same arguments as those for $A_1(s)$, getting

$$\sum_s |A_{31}(s)|^2 \ll T(\log q)^5,$$
$$\sum_s |A_{32}(s)|^2 \ll T\mathcal{L}(T)(\log T)^2,$$
$$\sum_s |A_{33}(s)|^2 \ll T\mathcal{L}(T)(\log q)^2.$$

However, we are not satisfied with the above bound for $A_{31}(s)$. First we clear from $A_{31}(s)$ the factor

$$V_s\left(\frac{n}{Q}\right) = 1 + O\left(\sqrt{\frac{n}{QT}}\right)$$

(see Lemma 7.2) and replace the smooth cut-off function (from the relevant partition) in the range $n \asymp q^4$. We get

$$A_{31}(s) = N(s) + \widetilde{N}(s) + O(q^4 T^{-1/2})$$

where

$$N(s) = \sum_{n \le q^4} \lambda(n) n^{-s}, \tag{8.8}$$

$$\widetilde{N}(s) = \sum_{x < n \le y} \lambda(n)\alpha(n) n^{-s}, \tag{8.9}$$

for some x, y with $q^4 \ll x < y \ll q^4$ and $\alpha(n) \ll 1$. By Lemma 5.3 and Corollary 6.3 we derive

$$\sum_s |\widetilde{N}(s)|^2 \ll T \sum_{x<n\le y} \tau(n,\chi)^2 n^{-1} \ll T\mathcal{L}(q)\log q.$$

Moreover the error term $O(q^4T^{-1/2})$ contributes at most $R(q^4T^{-1/2})^2 \ll q^8$, which is absorbed by $T\mathcal{L}(q)\log q$. Therefore we have

$$\sum_s |A_{31}(s) - N(s)|^2 \ll T\mathcal{L}(q)\log q.$$

Gathering the above estimates together with (7.23) we obtain

PROPOSITION 8.1. *Let $S(T)$ be a set of points satisfying* (8.1)–(8.3) *with $T \ge 2$. Put*

$$D(T) = \sum_s |\ell(s) - x(s)\overline{N}(s)|^2$$

where s runs over $S(T)$ (recall the settings (7.19), (7.20), (8.8)*). We have*

$$D(T) \ll T(\log q)^7 + T\mathcal{L}(T)(\log T)^4 \tag{8.10}$$

where $\mathcal{L}(T)$ is defined by (6.50)*, the implied constant being absolute.*

Assuming that $L(1,\chi)$ is small relatively to $\log T$ (so is $\mathcal{L}(T)$) we can interpret the bound (8.10) as saying that $x(s)\overline{N}(s)$ approximates $\ell(s)$ at almost all points s in any well-spaced set $S(T)$.

9. Estimation of $x(s)$ on average. Recall that the Hecke L-function for a character $\psi \in \widehat{\mathcal{C}\ell}(K)$ has the Euler product

$$L(s) = \prod_{\mathfrak{p}} (1 - \psi(\mathfrak{p})(N\mathfrak{p})^{-s})^{-1} = \sum_n \lambda(n) n^{-s}; \tag{9.1}$$

similarly the inverse satisfies

$$L(s)^{-1} = \prod_{\mathfrak{p}} (1 - \psi(\mathfrak{p})(N\mathfrak{p})^{-s}) = \sum_m \lambda^*(m) m^{-s}, \tag{9.2}$$

say, where

$$\lambda^*(m) = \sum_{N\mathfrak{a}=m} \mu(\mathfrak{a})\psi(\mathfrak{a}). \tag{9.3}$$

Note that $\lambda^*(m)$, like $\lambda(m)$, often vanishes if the class number is small. We have

$$|\lambda^*(m)| \le \tau(m,\chi). \tag{9.4}$$

Hence we have a reason to believe that the partial sum of $L(s)^{-1}$,

$$M(s) = \sum_{m \le q^4} \lambda^*(m) m^{-s}, \tag{9.5}$$

approximates $N(s)^{-1}$ at almost all points s on the critical line. Our goal is to estimate the sum

$$E(T) = \sum_s |\ell(s)\overline{M}(s) - x(s)|. \tag{9.6}$$

We begin by writing $M(s)N(s) = 1 + B(s)$, where

$$B(s) = \sum_{q^4 < l \le q^8} b(l) l^{-s} \quad \text{with} \quad b(l) = \sum_{\substack{mn=l \\ m,n \le q^4}} \lambda^*(m)\lambda(n).$$

Then we split $E(T)$ as follows:

$$\begin{aligned} E(T) &= \sum_s |(\ell(s) - x(s)\overline{N}(s))\overline{M}(s) + x(s)\overline{B}(s)| \\ &\ll D(T)^{1/2}\Big(\sum_s |M(s)|^2\Big)^{1/2} + (\log T)\sum_s |B(s)|. \end{aligned}$$

Here we have

$$\sum_s |M(s)|^2 \ll T(\log q) \sum_{m \le q^4} \tau(m)^2 m^{-1} \ll T(\log q)^5$$

and

$$\sum_s |B(s)| = \sum_s \Big| \sum_{\substack{m,n\le q^4\\ mn>q^4}} \lambda^*(m)\lambda(n)(mn)^{-s}\Big|.$$

Note that the condition $mn > q^4$ implies that either m or n is larger than q^2. Having this information recorded we relax the condition $mn > q^4$ by any method of separation of variables, for example by applying Lemma 9 of [DFI1]. This separation costs us a factor $\log q$. It follows that

$$\sum_s |B(s)| \ll T(\log q)^2\Big(\sum_{q^2<m\le q^4} \tau(m,\chi)^2 m^{-1}\Big)^{1/2}\Big(\sum_{n\le q^4} \tau(n)^2 n^{-1}\Big)^{1/2}.$$

By (6.49) we derive

$$\sum_s |B(s)| \ll T\mathcal{L}(q)^{1/2}(\log q)^{9/2}.$$

These estimates yield

Proposition 9.1. *Let $S(T)$ be a set of points satisfying* (8.1)–(8.3) *with $T \ge 2$. Then*

$$E(T) \ll T(\log q)^6 + T\mathcal{L}(T)^{1/2}(\log T)^2(\log q)^{5/2} \tag{9.7}$$

where the implied constant is absolute.

Assuming $L(1,\chi)$ is relatively small, Proposition 9.1 asserts that $\ell(s)\overline{M}(s)$ approximates $x(s)$ at almost all points s in any well-spaced set $S(T)$. This assertion is particularly interesting if $\ell(s) = (L(s) - L(s'))(s - s')^{-1}$ is very small, because it implies that $x(s) = (X(s) - X(s'))(s - s')^{-1}$ is also quite small. Put

$$\varDelta(T) = \sum_s |\ell(s)|^2 = \sum_s \Big|\frac{L(s) - L(s')}{s - s'}\Big|^2. \tag{9.8}$$

By Cauchy's inequality we get

$$\sum_s |\ell(s)M(s)| \ll (T\varDelta(T)(\log q)^5)^{1/2}.$$

Applying this to $E(T)$ in (9.6) we derive by (9.7) the following estimate:

$$\sum_s |x(s)| \ll T(\log q)^6 + T\mathcal{L}(T)^{1/2}(\log T)^2(\log q)^{5/2} + (T\varDelta(T)(\log q)^5)^{1/2}. \tag{9.9}$$

We shall make the estimate (9.9) more explicit by cosmetic preparations. First on the left-hand side we use (see (7.28))

$$|x(s)| = 2\Big|\frac{\sin(t-t')\log t}{t-t'}\Big| + O(\log q). \tag{9.10}$$

Note that the error term $O(\log q)$ contributes in total $O(R\log q)$, which is absorbed by the first term $T(\log q)^6$ on the right-hand side of (9.9). Next we replace $\mathcal{L}(T)$ in (9.9) by $L(1,\chi)\log q$. This can be justified, because the modified inequality is trivial unless

$$(\log T)L(1,\chi)^{1/2}(\log q)^3 \le 1. \tag{9.11}$$

Moreover, if (9.11) holds then we find that $\mathcal{L}(T) \ll L(1,\chi)\log q$. Finally, we no longer restrict the points $s = 1/2 + it$ to a dyadic segment $T < t \le 2T$. The extension to the segment $1 \le t \le T$ can be now derived by adding the new inequalities (9.9) for sets of points in the segments $2^\nu \le t \le 2^{\nu+1}$ with $1 \le 2^\nu \le T$. We state the result in a self-contained format.

PROPOSITION 9.2. *Let s run over a set of points on the critical line $s = 1/2 + it$ with $2 \le t \le T$ which are spaced by at least one. To every s in the set we associate a point $s' = 1/2 + it'$. Then*

$$\sum_s \left|\frac{\sin(t-t')\log t}{(t-t')\log t}\right| \ll \frac{T}{\log T}(\log q)^6 + T(\log T)L(1,\chi)^{1/2}(\log q)^3 + \frac{(\log q)^{5/2}}{\log T}\left(T\sum_s\left|\frac{L(s)-L(s')}{s-s'}\right|^2\right)^{1/2}. \tag{9.12}$$

This is our principal estimate from which one can deduce numerous attractive propositions. But first we wish to emphasize that (9.12) has no permanent value; it has some quality only in the absence of the Riemann hypothesis. Indeed, assuming only the lower bound

$$L(1,\chi) \gg (\log q)^{-6} \tag{9.13}$$

(recall that the Riemann hypothesis for $L(s,\chi)$ yields (1.5)) we find that the middle term on the right side of (9.12) is bounded below by $T\log T$. On the other hand the left side of (9.12) is trivially bounded by $R \le T$. Therefore our principal estimate (9.12) is insignificant if (9.13) is true. We certainly believe in the truth of (9.13), nevertheless as long as $L(1,\chi)$ is not proved to be relatively large (the best known unconditional estimate being $L(1,\chi) \gg q^{-\varepsilon}$, which is not effective), there are some valuable features of (9.12).

10. Applications. In this section we derive a few consequences of the principal estimate (9.12). We begin by eliminating the last term $(T\Delta(T))^{1/2} \times (\log q)^{5/2}(\log T)^{-1}$.

If all the points s and their companions s' are zeros of $L(s)$ (double zeros if $s = s'$) on the critical line, then

$$\ell(s) = \frac{L(s)-L(s')}{s-s'} = 0, \tag{10.1}$$

and consequently $\Delta(T) = 0$. Actually we do not require s and s' to be zeros of $L(s)$; the condition (10.1) means that s and its companion s' are on the same level curve of $L(s)$ (and $L'(s) = 0$ if $s = s'$). We can still assume less than (10.1). For example if s and its companion s' satisfy

$$\left|\frac{L(s)-L(s')}{s-s'}\right| \le (\log q)^{7/2} \tag{10.2}$$

then $\Delta(T) \le T(\log q)^7$, so on the right side of (9.12) the last term is absorbed by the first one. From now on we assume that the points s and their companions s' satisfy (10.2). For the points so chosen the estimate (9.12) reduces to

$$\sum_s \left|\frac{\sin(t-t')\log t}{(t-t')\log t}\right| \ll \frac{T}{\log T}(\log q)^6 + T(\log T)L(1,\chi)^{1/2}(\log q)^3. \tag{10.3}$$

Choose any T with

$$(\log q)^{A+6} \le \log T \le L(1,\chi)^{-1/2}(\log q)^{-A-3} \tag{10.4}$$

where A is a positive constant; then (10.3) implies that

$$\sum_s \left|\frac{\sin(t-t')\log t}{(t-t')\log t}\right| \ll T(\log q)^{-A}. \tag{10.5}$$

We tacitly assumed that $L(1,\chi)$ is small to be sure that the interval (10.4) is not void; precisely for this reason we require

$$L(1,\chi) \le (\log q)^{-4A-18}. \tag{10.6}$$

Let $R = R(\alpha, T)$ be the number of points s in the set with the companions s' satisfying (10.2) and

$$|t-t'| \le \frac{\pi(1-\alpha)}{\log t} \tag{10.7}$$

where $0 < \alpha \le 1$. For such points we have

$$\frac{\sin(t-t')\log t}{(t-t')\log t} \ge \alpha. \tag{10.8}$$

Hence (10.5) gives the following bound for the number of points in question:

$$R \ll \alpha^{-1}T(\log q)^{-A} \tag{10.9}$$

where the implied constant is absolute.

In conclusion we rephrase the results obtained in a positive mode.

Proposition 10.1. *Let $A \ge 0$ and $\log T \ge (\log q)^{A+6}$. Suppose there is a set of points*

$$S(T) = \{s_r = 1/2 + it_r;\ 2 \le t_1 < \ldots < t_R \le T,\ t_{r+1} - t_r \ge 1\}$$

and a set of companions $S'(T) = \{s'_r = 1/2 + it'_r;\ r = 1, \ldots, R\}$ *such that*

$$|t_r - t'_r| \leq \frac{\pi(1-\alpha)}{\log t_r} \quad \text{with } 0 < \alpha \leq 1, \tag{10.10}$$

$$\left|\frac{L(s_r) - L(s'_r)}{s_r - s'_r}\right| \leq (\log q)^{7/2}. \tag{10.11}$$

Suppose the number of points in the set $S(T)$ *satisfies*

$$R = |S(T)| \geq \frac{cT}{\alpha(\log q)^A} \tag{10.12}$$

where c *is a large absolute constant, effectively computable. Then*

$$L(1,\chi) \geq (\log T)^{-2}(\log q)^{-2A-6}. \tag{10.13}$$

We certainly believe that any Hecke L-function satisfies the conditions of Proposition 10.1, provided q is large. We recommend the points s_r to be zeros of $L(s)$ and s'_r to be the nearest zero to s_r on the critical line (if s_r has order two or more, then $s'_r = s_r$). For this choice (10.11) holds automatically, while (10.10) asserts that the gaps between chosen pairs of zeros is smaller than the normal average spacing. We need a considerable number of such small gaps between consecutive zeros, but less than the true order of magnitude. In particular taking $A = 6$ and $\log T = (\log q)^{12}$ we get $L(1,\chi) \geq (\log q)^{-42}$, provided the number of well-spaced zeros of sub-normal gaps and height up to T is at least $T(\log T)^{-1/2}$.

Remark. If the points of $S(T)$ are zeros of $L(s)$ then the condition that they are spaced by at least one can be dropped at the expense of an extra factor $\log T$ in (10.12) (see how this is justified in the proof of Corollary 10.2). Hence we derive Theorem 1.1.

An interesting case is the trivial class group character $\psi = 1$. In this case the Riemann zeta function appears as a factor of the Hecke L-function, $L(s) = \zeta_K(s) = \zeta(s)L(s,\chi)$, so we can choose the zeros of $L(s)$ from those of $\zeta(s)$ and state the conditions without ever mentioning the exceptional conductor q. Note we have precise control over α. Taking $A = 12$ and $\alpha = (\log T)^{-1/2}$ we derive from Proposition 10.1 the following

Corollary 10.2. *Let* $\varrho = 1/2 + i\gamma$ *denote the zeros of* $\zeta(s)$ *on the critical line and* $\varrho' = 1/2 + i\gamma'$ *denote the nearest zero to* ϱ *on the critical line* ($\varrho' = \varrho$ *if it is a multiple zero*). *Suppose that*

$$\#\left\{\varrho;\ 0 < \gamma \leq T, |\gamma - \gamma'| \leq \frac{\pi}{\log\gamma}\left(1 - \frac{1}{\sqrt{\log\gamma}}\right)\right\} \gg T(\log T)^{4/5} \tag{10.14}$$

for any $T \geq 2001$. *Then*

$$L(1,\chi) \gg (\log q)^{-90} \tag{10.15}$$

where the implied constant is effectively computable in terms of that in (10.14).

Proof. The number of zeros $\varrho = 1/2 + i\gamma$ with $t < \gamma \le t + 1$ is bounded by $O(\log t)$. Therefore one can select from the set of zeros in (10.14) a subset of well-spaced points of cardinality $R \gg T(\log T)^{-1/5}$. This subset satisfies the conditions of Proposition 10.1 with $A = 14, \log T = (\log q)^{20}$ and $\alpha = (\log T)^{-1/2} = (\log q)^{-10}$, provided q is sufficiently large, giving $L(1,\chi) \ge (\log q)^{-90}$. For small q the lower bound (10.15) is obtained by adjusting the implied constant.

REMARK. The normal average gap between consecutive zeros $\varrho = 1/2 + i\gamma$, $\varrho' = 1/2 + i\gamma'$ of $\zeta(s)$ is $2\pi(\log \gamma)^{-1}$. Hence the condition (10.14) refers to gaps slightly smaller than half the average.

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