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On the convergence to 0 of $m_n\xi \bmod 1$

by

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1. Introduction. We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $x \in \mathbb{R}$ we define $\|x\| := \inf_{n\in\mathbb{N}} |x-n|$. We denote by $x[1]$ the fractional part of x.

In this paper, we prove the following results.

Theorem 1. *For any $\alpha \in \mathbb{R} - \mathbb{Q}$ and a sequence $\{m_l\}_{l\in\mathbb{N}}$ of integers such that $\lim_{l\to\infty} \|m_l\alpha\| = 0$, there exists a measure μ on $\mathbb{T}$ which has no atoms and is such that $\lim_{l\to\infty} \int_{\mathbb{T}} \|m_l\theta\|\, d\mu(\theta) = 0$.*

Theorem 2. *For any $\alpha \in \mathbb{R} - \mathbb{Q}$, there exists a sequence $\{m_l\}_{l\in\mathbb{N}}$ of integers such that $\|m_l\alpha\| \to 0$ and such that $m_l\theta[1]$ is dense in $\mathbb{T}$ if and only if $\theta \notin \mathbb{Q}\alpha + \mathbb{Q}$.*

Due to the Gaussian measure space construction (see [4], or for example [2, Proposition 2.30]), Theorem 1 has a direct consequence for rigidity sequences of weakly mixing dynamical systems. The following statement is in fact equivalent to Theorem 1.

Corollary 1. *For any $\alpha \in \mathbb{R} - \mathbb{Q}$ and a sequence of integers $\{m_l\}_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \|m_l\alpha\| = 0$, there exists a weak mixing dynamical system (T, M, m) such that $\{m_l\}_{l\in\mathbb{N}}$ is a rigidity sequence for (T, M, m).*

A consequence of Corollary 1 is a positive answer to a question raised in [2], namely whether *a rigidity sequence of any ergodic transformation* (*on a probability space without atoms*) *with discrete spectrum is a rigidity sequence for some weakly mixing dynamical system.* Indeed, Corollary 1 deals with the case of a pure point spectrum with an irrational rotation of the circle as a factor. The case of a purely rational spectrum was treated in [2, Proposition 3.27]. In case the spectrum is purely rational, our proof of Theorem 1 given below applies with only one modification: instead of working with the orbit of 0 under the rotation R_α ($\alpha \notin \mathbb{Q}/\mathbb{Z}$), one considers the union of the orbits

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of 0 under the actions of all the finite groups which appear in the (necessarily dense) support of the spectral measure.

A completely different solution to the same question was given by Adams [1], who proved directly Corollary 1 based on a sophisticated and involved cut and stack construction.

In contrast, our proof is much simpler and is based on the straightforward characterization of rigidity as a spectral property, which reduces answering the question to the construction of a continuous probability measure on the circle with Fourier transform converging to 1 along the rigidity subsequence as stated in Theorem 1. This possible approach to the question was discussed in detail in [2].

The second result, Theorem 2, asserts that it is not possible to expect more than what is obtained in Theorem 1, namely, strong convergence of $\|m_n\theta\|$ to 1 on an uncountable set K is not possible in general for a sequence $\{m_n\}_{n\in\mathbb{N}}$ such that $\lim_{l\to\infty}\|m_l\alpha\| = 0$, $\alpha \in \mathbb{R}-\mathbb{Q}$. Constructing such a set K was a possible strategy for proving Corollary 1 (see for example [2, Proposition 3.3]), and Theorem 2 shows that this approach cannot be adopted in general.

Given an increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ of integers, the study of the accumulation points of the sequence $\{m_n\xi\}$, for ξ irrational, on the circle has a long history and a rich literature (see for example [3, 5] and references therein). Weyl [6] proved, for any increasing sequence $\{m_n\}_{n\in\mathbb{N}}$, that for almost every ξ, $\{m_n\xi\}$ is dense on the circle. The set of irrationals ξ such that $\{m_n\xi\}$ is not dense in $\mathbb{T}$ is called the *set of exceptional points* for the sequence $\{m_n\}_{n\in\mathbb{N}}$. Our result asserts the existence for any $\alpha \in \mathbb{R}-\mathbb{Q}$ of a sequence $\{m_n\}_{n\in\mathbb{N}}$ for which the *set of exceptional points* is reduced to $\mathbb{Q}\alpha+\mathbb{Q}$. To our knowledge, no other examples of increasing sequences $\{m_n\}_{n\in\mathbb{N}}$ with a countable exceptional set are known in the literature.

2. Proof of Theorem 1. Fix $\alpha \in \mathbb{R}-\mathbb{Q}$ and a sequence $\{m_l\}_{l\in\mathbb{N}}$ of integers such that

$$(\star)\qquad \lim_{l\to\infty}\|m_l\alpha\| = 0.$$

For a probability measure μ on $\mathbb{T}$ we write $\mu^n = \left|\int_{\mathbb{T}}\|m_n\theta\|\,d\mu(\theta)\right|$.

We will construct a sequence μ_p, $p \ge 0$, of probability measures on $\mathbb{T}$ of the form $2^{-p}\sum_{i=1}^{2^p}\delta_{x_i}$ with $x_i = k_i\alpha$ such that there exists an increasing sequence $\{N_p\}$ for which

(1) $\mu_p^n < 1/2^j$ for every $p \ge 1$, every $j \in [0, p-1]$, and all $n \in [N_j, N_{j+1}]$;
(2) for every $p_0 \in \mathbb{N}^*$, if we let

$$\eta_{p_0} = \frac{1}{4}\inf_{1\le i<i'\le 2^{p_0}}\|k_i\alpha - k_{i'}\alpha\|$$

then $\|k_{l2^{p_0}+r}\alpha - k_r\alpha\| < \eta_{p_0}$ for every $l \in \mathbb{N}$ and every $r \in [1, 2^{p_0}]$;

(3) $\mu_p^n < 1/2^{p+1}$ for $n \geq N_p$.

When going from the measure μ_p to μ_{p+1} we will *add* 2^p masses at points selected near $x_1, \ldots, x_{2^p}$ that are already chosen for μ_p.

Theorem 1 clearly follows from the above construction. Indeed, property (1) will imply that any weak limit μ_∞ of μ_p satisfies $\mu_\infty^n \to 0$, while by (2) we deduce that for each p_0 the intervals $(k_r\alpha - \eta_{p_0}, k_r\alpha + \eta_{p_0})$, $r \in [1, 2^{p_0}]$, on the circle are disjoint and have mass $1/2^{p_0}$ each for all μ_p, $p \geq p_0$, and hence for μ_∞, which therefore has no atoms.

Property (3) is not necessary in the proof of the theorem, but it is useful to fulfill the inductive hypotheses (1) and (2) of the construction.

For $p = 0$, we let $k_1 = 0$ and μ_0 is thus the Dirac measure at 0. We let $N_0 = 0$. For $p = 1$, we let $k_2 = 1$ so μ_1 is the average of the Dirac measures at 0 and at α. Observe that for any n, we have $\mu_1^n < 1/2$, which fulfills (1) for $p = 1$. We also choose N_1 sufficiently large so that $\mu_1^n < 1/2^2$ for $n \geq N_1$, the latter being possible due to ($\star$).

We now assume that we have selected k_i for $i \leq 2^p$ and N_l for $l \leq p$ so that (1) and (3) are satisfied up to p, and (2) is satisfied for every $p_0 \leq p$ and every $0 \leq l \leq 2^{p-p_0} - 1$.

We choose k_{2^p+1} such that $k_{2^p+1}\alpha$ is sufficiently close to $k_1\alpha$ so that

$$\nu_{p,1} = \frac{1}{2^{p+1}} \sum_{i=1}^{2^{p+1}} \delta_{k_i'\alpha},$$

where $k_i' = k_i$ for $i \leq 2^p$ and $k_{2^p+1}' = k_{2^p+1}$ while $k_{2^p+r}' = k_r$ for $r \in [2, 2^p]$, satisfies $\nu_{p,1}^n < 1/2^j$ for every $n \in [N_j, N_{j+1}]$ and $j \in [0, p-1]$.

Since for every n we have

$$|\nu_{p,1}^n - \mu_p^n| < \frac{1}{2^{p+1}} \| m_n k_{2^p+1}\alpha - m_n k_1 \alpha \| < \frac{1}{2^{p+1}},$$

we deduce by (3) that $\nu_{p,1}^n < 1/2^{p+1} + 1/2^{p+1} = 1/2^p$ for every $n \geq N_p$. Next we choose $N_{p,1} > N_p$ sufficiently large so that $\nu_{p,1}^n < 1/2^{p+2}$ for $n \geq N_{p,1}$, which is possible by ($\star$). In this way, we select inductively k_{2^p+s}, then $N_{p,s}$ for $s = 1, \ldots, 2^p$, and set

$$\nu_{p,s} = \frac{1}{2^{p+1}} \sum_{i=1}^{2^{p+1}} \delta_{k_i'\alpha},$$

where $k_i' = k_i$ for $i \leq 2^p + s$ and $k_{2^p+t}' = k_t$ for $t \in [s+1, 2^p]$. Choosing, for each s, $k_{2^p+s}\alpha$ sufficiently close to $k_s\alpha$, and then $N_{p,s}$ sufficiently large, we can ensure that

- $\nu_{p,s}^n < 1/2^j$ for every $n \in [N_j, N_{j+1}]$ and $j \leq p-1$;
- $\nu_{p,s}^n < 1/2^p$ for every $n \geq N_p$;
- $\nu_{p,s}^n < 1/2^{p+2}$ for every $n \geq N_{p,s}$.

The first item can be established inductively due to the fact that if $k_{2^p+s}\alpha$ is chosen very close to $k_s\alpha$ then the measures $\nu_{p,s-1}$ and $\nu_{p,s}$ are very close.

The same argument gives the second item for $N_p \le n \le N_{p,s-1}$. As for $n \ge N_{p,s-1}$, we use the facts that $|\nu^n_{p,s} - \nu^n_{p,s-1}| < 1/2^{p+1}$ and $\nu^n_{p,s-1} < 1/2^{p+2}$ for every $n \ge N_{p,s-1}$ to conclude that $\nu^n_{p,s} < 1/2^p$. For the third item we just choose $N_{p,s}$ sufficiently large and use $(\star)$.

Finally, we let $N_{p+1} = N_{p,2^p}$ and $\mu_{p+1} = \nu_{p,2^p}$ and observe that the measure μ_{p+1} satisfies (1).

Also, since $k_{2^p+s}\alpha$ can be chosen arbitrarily close to $k_s\alpha$ for $s = 1, \ldots, 2^p$, we see that for every $p_0 \le p+1$, and every $l = 2^{p-p_0} + l' - 1$, $l' \le 2^{p-p_0}$, we have $\|k_{l2^{p_0}+r}\alpha\| \sim \|k_{l'2^{p_0}+r}\alpha\| \sim \|k_r\alpha\|$, from which (2) follows for $p+1$. The proof of Theorem 1 is thus complete. ∎

3. Proof of Theorem 2. In all this section $\alpha \in \mathbb{R} - \mathbb{Q}$ is fixed.

Definition 1. For an interval $I \subset \mathbb{T}$, $\varepsilon > 0$, and integers $N_1 < N_2$, we say that $\theta \in \mathcal{A}(N_1, N_2, I, \varepsilon, \alpha)$ if for every $m \in [N_1, N_2)$ such that $\|m\alpha\| < \varepsilon$ we have $\{m\theta\} \notin I$.

Lemma 1. *For every $l \ge 2$, there exists $L(l) \in \mathbb{N}$ such that for every $0 < \varepsilon \le 1/(2l^2)$ and all $\nu > 0$ and $N \in \mathbb{N}$, there exist $K(\varepsilon) > 0$ and $N' = N'(l, \varepsilon, \nu, N) \in \mathbb{N}$ such that if $\theta \in \mathcal{A}(N, N', I, \varepsilon, \alpha)$ for some interval I of size $1/l$ then $\|k\alpha - s\theta\| < \nu$ for some $|k| \le K(\varepsilon)$ and some $|s| \le L(l)$.*

Proof. For any $\varepsilon > 0$, consider an approximation $\phi_\varepsilon : \mathbb{T} \to \mathbb{R}$ of $2\chi_\varepsilon$ by trigonometric polynomials, where χ_ε is the characteristic function of the subset $[0, \varepsilon] \cup [1-\varepsilon, 1]$ of $\mathbb{T}$, such that:

- $\phi_\varepsilon(x) > 1$ for every $x \in [0, \varepsilon] \cup [1-\varepsilon, 1]$;
- $\phi_\varepsilon(x) > -\varepsilon^3$ for every $x \in \mathbb{T}$;
- there exists $K \in \mathbb{N}$ such that $\phi_\varepsilon(x) = \sum_{|k| \le K} \hat{\phi}_k e^{i2\pi kx}$.

Similarly, for $l \ge 2$, let $\varphi_l : \mathbb{T} \to \mathbb{R}$ be such that:

- $\varphi_l(y) > 1$ for every $y \notin [0, 1/l]$;
- $|\varphi_l(y)| < l^2$ for every $y \in \mathbb{T}$;
- there exists $L \in \mathbb{N}$ such that $\varphi_l(y) = \sum_{0 < |k| \le L} \hat{\varphi}_k e^{i2\pi ky}$.

Note that the second requirement includes the fact that $\int \varphi_l(y)\, dy = 0$.

For $\psi : \mathbb{T}^2 \to \mathbb{R}$ and $(\alpha, \theta) \in \mathbb{R}^2$ we define, for $k \in \mathbb{N}$,

$$S_k^{\alpha,\theta}\psi(x,y) = \sum_{i=0}^{k-1} \psi(x + i\alpha, y + i\theta).$$

Fix $I = [y_0, y_0 + 1/l]$ for some $y_0 \in \mathbb{T}$, $l \ge 2$.

Define $\psi_{\varepsilon,l} : \mathbb{T}^2 \to \mathbb{R}$ by $\psi_{\varepsilon,l}(x,y) = \phi_\varepsilon(x)\varphi_l(y-y_0)$. For $N' \in \mathbb{N}$ sufficiently large there exist more than $\varepsilon^2 N'$ integers $i \in [N, N')$ such that $\|i\alpha\| < \varepsilon$. If $\theta \in \mathcal{A}(N, N', I, \varepsilon, \alpha)$ then $S_{N'}^{\alpha,\theta}\psi_{\varepsilon,l}(0,0) > (\varepsilon^2 - l^2\varepsilon^3)N' \geq \frac{1}{2}\varepsilon^2 N'$.

On the other hand, we have

$$S_{N'}^{\alpha,\theta}\psi_{\varepsilon,l}(x,y) = \sum_{|k|\leq K,\, 0<|j|<L} \hat{\phi}_k \hat{\varphi}_j \frac{1 - e^{i2\pi N'(k\alpha + j\theta)}}{1 - e^{i2\pi(k\alpha+j\theta)}} e^{i2\pi(kx+jy)},$$

hence, if $\|k\alpha - j\theta\| \geq \nu$ for all $|k| \leq K$ and $0 < |j| \leq L$, then $S_{N'}^{\alpha,\theta}\psi_{\varepsilon,l}(x,y)$ is bounded independently of N', which contradicts $S_{N'}^{\alpha,\theta}\psi_{\varepsilon,l}(0,0) > \frac{1}{2}\varepsilon^2 N'$. ∎

Proof of Theorem 2. For $n \geq 1$, define $l_n = n+1$ and $L_n := L(l_n)$ as given by Lemma 1. Let $\varepsilon_n = 1/(2(n+1)^2)$ and $K_n = K(\varepsilon_n)$ as given by Lemma 1. Let $\nu_n = n^{-1} \inf_{0<|k|\leq(n+1)K_{n+1}} \|k\alpha\|$. Take $N_0 = 0$ and apply Lemma 1 with $l = l_1$, $\varepsilon = \varepsilon_1$, $N = N_0$ and $\nu = \nu_1$. Define $N_1 = N'(l_1, \varepsilon_1, \nu_1, N_0)$. We then apply inductively Lemma 1 with $l = l_n$, $\varepsilon = \varepsilon_n$, $N = N_n$ and $\nu = \nu_n$ and choose N_{n+1} arbitrarily large such that $N_{n+1} \geq N'(l_n, \varepsilon_n, \nu_n, N_n)$.

We define an increasing sequence m_l by taking successively, for every i, all the integers $m \in [N_i, N_{i+1})$ such that $\|m\alpha\| < \varepsilon_i$ (choosing N_{n+1} to be sufficiently large in our inductive construction guarantees that the sequence m_n is not empty).

Suppose now θ is such that $\{m_n\theta[1]\}$ is not dense on the circle. Then there exist k and an interval I of size l_k such that $m_n\theta[1] \notin I$ for every n. In other words, $\theta \in \mathcal{A}(N_n, N_{n+1}, I, \varepsilon_n, \alpha)$ for every $n \geq n_0$, for n_0 sufficiently large. Let $L = L_k$. By Lemma 1 we get $\|k_n\alpha - l\theta\| < \nu_n$ for some $|k_n| \leq K_n$ and some $0 < |l| \leq L$. Hence $\|k'_n\alpha - L!\theta\| < L!\nu_n$ for some $|k'_n| \leq L!K_n$. It follows that $\|(k'_{n+1} - k'_n)\alpha\| < 2L!\nu_n$. From the definition of ν_n this implies that $k'_{n+1} = k'_n$ for sufficiently large n, say $n \geq n_1$. Since $\nu_n \to 0$, we get $\|k'_{n_1}\alpha - L!\theta\| = 0$, which gives $\theta \in \mathbb{Q}\alpha + \mathbb{Q}$. Conversely, $\{m_n\theta\}$ for $\theta \in \mathbb{Q}\alpha + \mathbb{Q}$ is clearly not dense on the circle. Theorem 2 is proved. ∎

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