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On limit points of subsequences of uniformly distributed sequences

by

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1. Introduction. If a sequence $(x_n)_{n=1}^{\infty}$ is dense in the interval $[0,1]$, it is a standard exercise to prove that for any nonempty closed set $C \subset [0,1]$ there exists an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that the set of all limit points of the subsequence $(x_{n_k})_{k=1}^{\infty}$ is equal to C. The question becomes more complicated if one imposes additional conditions on the growth of (n_k). On the one hand, this growth can be arbitrarily rapid, on the other hand, it cannot be too slow in general. Recently Bugeaud [B], extending the previous result by Dubickas [D], proved the following theorem.

Theorem 1.1 ([B]). *Let ξ be an irrational real number. Let S be a finite, nonempty set of distinct real numbers in $[0,1]$. Let $(g_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $g_n \geq 1$ for $n \geq 1$ and $\lim_{n\to\infty} g_n = +\infty$. Then there exists an increasing sequence $(a_n)_{n=1}^{\infty}$ of positive integers satisfying $a_n \leq n g_n$ for $n \geq 1$ and such that the set of limit points of the sequence of fractional parts of $(a_n \xi)_{n=1}^{\infty}$ is equal to S.*

The purpose of this paper is to generalize this theorem in two directions. Firstly, the finite set S is replaced by an arbitrary nonempty closed subset of $[0,1]$. Secondly, the special sequence $(n\xi)$ is replaced by an arbitrary sequence uniformly distributed modulo 1.

Recall that for a set $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$, denoting by $A(n)$ the cardinality of $A \cap \{1,\ldots,n\}$, the *lower* and *upper asymptotic densities* of A are defined respectively by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n},$$

or equivalently by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{n}{a_n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{n}{a_n}.$$

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When the above values are equal, we speak about the *asymptotic density* of A and use the notation $d(A)$. For the Lebesgue measure of a set X of real numbers we use the symbol $\lambda(X)$, and $\{x\}$ stands for the fractional part of a real number x. A sequence (x_n) of real numbers is *uniformly distributed (u.d.) mod. 1* if for every interval $J \subset [0,1]$,

$$d(\{n \in \mathbb{N};\ \{x_n\} \in J\}) = \lambda(J). \tag{1.1}$$

Note that the above equality then also holds for J being any finite union of disjoint intervals.

For a sequence (x_n) and a set $A \subset \mathbb{N}$ we denote by $L(x_A)$ the set of all limit points of the subsequence $(\{x_n\})_{n\in A}$. For simplicity, we will not distinguish infinite subsets of $\mathbb{N}$ from the increasing sequences of all their elements, i.e. $A = (a_n)$ means that the set A is formed by all terms of the increasing sequence of positive integers $a_1 < a_2 < \cdots$.

For more references on this topic see [KN] or [SP].

2. Results

THEOREM 2.1. *Let (x_n) be a u.d. sequence mod. 1, let C be a nonempty closed subset of $[0,1]$ and let $h_n \to \lambda(C)$ be a sequence of positive real numbers with $h_n \le 1$ for $n \ge 1$. Then there exists an increasing sequence $A = (a_n)_{n=1}^{\infty}$ of positive integers satisfying $a_n \le n/h_n$ for $n \ge 1$ and such that $L(x_A) = C$.*

Proof. The conclusion definitely holds if $\lambda(C) = 1$, i.e. $C = [0,1]$, thus we assume $\lambda(C) < 1$. Set $h_0 = 1$ and denote by k_0 the smallest positive integer such that $h_k < 1$ for all $k \ge k_0$. First we show that we can assume that $h_n > h_{n+1}$ for all $n \ge k_0$. If this is not the case, set $h'_i = 1$ for all $i = 1, \dots, k_0 - 1$ and for $\varepsilon = 1 - \sup\{h_k;\ k \ge k_0\}$ define

$$h'_n = \sup\{h_k;\ k \ge n\} + \varepsilon/2^n$$

for $n = k_0, k_0 + 1, \dots,$ and note that

(i) $h'_n \to \lambda(C)$,
(ii) $h'_n > h'_{n+1}$ for all $n \ge k_0$,
(iii) $h'_n \ge h_n$, consequently $a_n \le n/h'_n$ implies $a_n \le n/h_n$ for every $n \in \mathbb{N}$.

This shows that it is sufficient to prove the theorem for the sequence (h'_n), thus we can assume from the outset that the original sequence is decreasing for $n \ge k_0$.

Our proof will involve an inductive construction of an increasing sequence (n_k) of positive integers as well as a nonincreasing sequence (C_k) of closed sets, each being a union of finitely many disjoint closed intervals, such that $\bigcap_{k=1}^{\infty} C_k = C$.

We start from the construction of (C_k). Set $C_k = [0,1]$ for each $k = 0, \dots, k_0$ and for each $k > k_0$ let $C_k \subset C_{k-1}$ be a finite union of disjoint closed intervals, each intersecting C, and such that $C_k \supset C$ and

$$\lambda(C_k) = h_{k-1}. \tag{2.1}$$

The existence of such a set is guaranteed by elementary topological properties of the real line and closedness of C. Also notice that $\bigcap_{k=0}^{\infty} C_k = C$ by the convergence of (h_k) to $\lambda(C)$.

Now we construct (n_k). First, set $n_k = k$ for all nonnegative integers $k < k_0$. Further, assume that n_k has already been defined for all $k < m$ where $m \geq k_0$. We choose $n_m > n_{m-1}$ satisfying the following two conditions: for all $n \geq n_m$ we have

$$\frac{\#\{j \leq n;\ \{x_j\} \in C_{m+1}\}}{n} > h_{m+1} \tag{2.2}$$

and for each $c \in C$ there exists an $i \in [n_{m-1}, n_m)$ such that

$$\{x_i\} \in C_{m+1} \quad \text{and} \quad |\{x_i\} - c| < \frac{1}{m+1}. \tag{2.3}$$

The existence of such an n_m is guaranteed by (2.1), (1.1) and the note following it, and uniform distribution of (x_n).

For each $k \in \mathbb{N}$ denote $I_k = \{i \in \mathbb{N};\ \{x_i\} \in C_k\} = \{i_1^k < i_2^k < \cdots\}$. As each C_k is a finite union of disjoint closed intervals, (2.1) shows that

$$\lim_{n\to\infty} n/i_n^k = h_{k-1} \quad \text{for each } k \in \mathbb{N}, \tag{2.4}$$

and moreover

$$n/i_n^k > h_k \quad \text{for every } i_n^k \geq n_{k-1}. \tag{2.5}$$

which follows from (2.2) by setting $m = k-1$ and noting that

$$\#\{j \leq i_n^k;\ \{x_j\} \in C_k\} = n.$$

Define

$$A = \{1, \dots, n_{k_0}\} \cup \Big(\bigcup_{k=k_0+1}^{\infty} (I_k \cap (n_{k-1}, n_k]) \Big) = \{a_1 < a_2 < \cdots\}. \tag{2.6}$$

We are going to show that

$$a_n \leq n/h_n$$

for every $n \in \mathbb{N}$. This is definitely true for every $n \leq n_{k_0}$, as $a_n = n$. Now let $n > n_{k_0}$. There are unique $k, j \in \mathbb{N}$ such that $a_n = i_j^k$. As all $I_k \cap (n_{k-1}, n_k]$ are nonempty and $I_k \subset I_{k-1}$ for all $k \in \mathbb{N}$, by (2.6) we have $k \leq n$ and $j \leq n$. Using (2.5) and monotonicity of (h_n) we obtain

$$a_n = i_j^k < j/h_k \leq n/h_n.$$

Now we are going to show that $L(x_A) = C$. The inclusion $L(x_A) \subset C$ is evident since $L(x_A) \subset C_k$ for all $k \in \mathbb{N}$ and $\bigcap_{k=0}^{\infty} C_k = C$. Conversely, $C \subset L(x_A)$ follows easily from (2.3). ■

Remark 2.2. Note that Theorem 1.1 can be deduced from the previous one directly taking $S = C$ a finite set and setting $g_n = 1/h_n$ and $x_n = \xi n$.

The following theorem says that the bound on the growth of a_n in the previous theorem cannot be relaxed.

Theorem 2.3. *Let (x_n) be a u.d. sequence mod. 1 and $A \subset \mathbb{N}$. Then $\overline{d}(A) \leq \lambda(L(x_A))$.*

Proof. Suppose $\overline{d}(A) > \lambda(L(x_A))$. Then $L(x_A)$, being compact, can be covered by finitely many mutually disjoint open intervals whose union D fulfils $\lambda(D) < \overline{d}(A)$. Denote $I = \{n \in \mathbb{N};\ \{x_n\} \in D\}$. Then $d(I) = \lambda(D)$, and consequently A contains infinitely many elements outside of I, contradicting $L(x_A) \subset D$. ■

The last theorem of this paper says that there are no other bounds on the asymptotic density except that stated in the previous theorem.

Theorem 2.4. *Let (x_n) be a u.d. sequence mod. 1, let C be a nonempty closed subset of $[0,1]$ and $0 \leq \alpha \leq \beta \leq \lambda(C)$. Then there exists $A \subset \mathbb{N}$ with $\underline{d}(A) = \alpha$, $\overline{d}(A) = \beta$ and $L(x_A) = C$.*

The proof will be based on three simple lemmas. By $\lfloor x \rfloor$ we denote the integer part of a real x.

Lemma 2.5. *Let $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$ be such that $d(A) = d$. For every $\eta \in (0,1)$ let $A_\eta = \{a_{\lfloor n/\eta \rfloor};\ n \in \mathbb{N}\} \subset A$. Then $d(A_\eta) = \eta d$.*

Proof. We calculate

$$d(A_\eta) = \lim_{n\to\infty} \frac{n}{a_{\lfloor n/\eta \rfloor}} = \lim_{n\to\infty} \frac{n}{\lfloor n/\eta \rfloor} \lim_{n\to\infty} \frac{\lfloor n/\eta \rfloor}{a_{\lfloor n/\eta \rfloor}} = \eta d. \ ■$$

Lemma 2.6. *Let $\alpha < \beta$ and $A, B \subset \mathbb{N}$ be such that $d(A) = \alpha$ and $d(B) = \beta$. Define*

$$C = \Big(\bigcup_{k=1}^{\infty} ((2n-1)!, (2n)!] \cap A\Big) \cup \Big(\bigcup_{k=1}^{\infty} ((2n)!, (2n+1)!] \cap B\Big).$$

Then $\underline{d}(C) = \alpha$ and $\overline{d}(C) = \beta$.

Proof. Obviously, for every sufficiently large $n \in \mathbb{N}$,

$$\frac{A((2n)!)}{(2n)!} \leq \frac{C((2n)!)}{(2n)!} \leq \frac{A((2n)!) - A((2n-1)!) + (2n-1)!}{(2n)!}.$$

As the limits of both the leftmost and rightmost terms are equal to α, also $\lim_{n\to\infty} C((2n)!)/(2n)! = \alpha$. In a similar way one can easily show that $\lim_{n\to\infty} C((2n+1)!)/(2n+1)! = \beta$. Finally, for every sufficiently large $k \in \mathbb{N}$ we have

$$\frac{A(k)}{k} \le \frac{C(k)}{k} \le \frac{B(k)}{k},$$

thus $\lim_{n\to\infty} C(k_n)/k_n \in [\alpha,\beta]$ provided the limit exists, which proves the lemma. ∎

LEMMA 2.7. *Let (t_n) be a dense sequence in $[0,1]$ and let C be a nonempty closed subset of $[0,1]$. Then there is a set $A \subset \mathbb{N}$ such that $d(A) = 0$ and $L(t_A) = C$.*

Proof. For every $n \in \mathbb{N}$ there exist unique nonnegative integers $k(n)$ and $l(n)$ such that $l(n) < 2^{k(n)}$ and $n = 2^{k(n)} + l(n)$. Define

$$J_n = \left[\frac{l(n)}{2^{k(n)}}, \frac{l(n)+1}{2^{k(n)}}\right], \qquad n = 2,3,\ldots.$$

Denote by (I_n) the subsequence of (J_k) consisting of all J_k intersecting C, arranged in the same order as they appear in (J_k). Note that this sequence is infinite, as C is nonempty and for each $k \in \mathbb{N}$ the finite sequence $J_{2^k}, J_{2^k+1}, \ldots, J_{2^{k+1}-1}$ covers $[0,1]$. Let $A = \{a_1 < a_2 < \cdots\} \subset \mathbb{N}$ be any sequence such that

$$t_{a_n} \in I_n \qquad \text{for all } n \in \mathbb{N}, \tag{2.7}$$

$$a_{n+1} \ge 2a_n \qquad \text{for all } n \in \mathbb{N}. \tag{2.8}$$

Denseness of (t_n) guarantees that there are many such sets A.

First we show that $d(A) = 0$. Indeed, it follows immediately from (2.8) that $a_n \ge 2^{n-1}$, and consequently

$$d(A) = \lim_{n\to\infty} \frac{n}{a_n} \le \lim_{n\to\infty} \frac{n}{2^{n-1}} = 0.$$

Now we show $L(t_A) = C$. For each $x \in C$ there are infinitely many n_k such that $x \in I_{n_k}$. As $\lim_{k\to\infty} |I_{n_k}| = 0$, the relation (2.7) implies that $\lim_{k\to\infty} t_{a_{n_k}} = x$, thus $C \subset L(t_A)$. On the other hand, as C is closed, for every $x \notin C$ there is a $k \in \mathbb{N}$ such that $(x - 1/2^k, x + 1/2^k) \cap C = \emptyset$. Consequently, $t_n \notin (x - 1/2^k, x + 1/2^k)$ for all $n > 2^{k+1}$, thus $x \notin L(t_A)$ and $L(t_A) \subset C$. ∎

Proof of Theorem 2.4. The case $\lambda(C) = 0$ being straightforward, we assume $\lambda(C) > 0$. The proof will consist of several short steps, each using some previous statement.

In the first step we find a set $A' \subset \mathbb{N}$ guaranteed by Theorem 2.1, i.e. $d(A') = \lambda(C)$ and $L(x_{A'}) = C$.

If $\alpha = \lambda(C)$, the proof is complete, otherwise use Lemma 2.5 twice, with $\eta_1 = \alpha/\lambda(C)$ and $\eta_2 = \beta/\lambda(C)$ respectively, to produce subsets $A_1 \subset A'$ and $A_2 \subset A'$ such that $d(A_1) = \eta_1\lambda(C) = \alpha$ and $d(A_2) = \eta_2\lambda(C) = \beta$.

In the next step we use Lemma 2.6 to find that the set

$$B = \Big(\bigcup_{k=1}^{\infty}((2n-1)!, (2n)!] \cap A_1\Big) \cup \Big(\bigcup_{k=1}^{\infty}((2n)!, (2n+1)!] \cap A_2\Big)$$

has $\underline{d}(B) = \alpha$ and $\overline{d}(B) = \beta$. As $B \subset A'$, the relation $L(x_B) \subset C$ also holds.

In the last step we use Lemma 2.7 to find $D \subset \mathbb{N}$ such that $d(D) = 0$ and $L(x_D) = C$.

To finish the proof, set $A = B \cup D$. Obviously $\underline{d}(A) = \underline{d}(B) = \alpha$, $\overline{d}(A) = \overline{d}(B) = \beta$ and $L(x_A) = C$, so the proof is complete. ∎

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