ACTA ARITHMETICA
138.2 (2009)

# Irrationality proof of a $q$-extension of $\zeta(2)$ using little $q$-Jacobi polynomials

by

Christophe Smet and Walter Van Assche (Leuven)

**1. Introduction.** The $\zeta$-function at integer points, $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ ($s \in \mathbb{N}$), has a $q$-analogue, defined by

$$\zeta_q(s) = \sum_{k=1}^{\infty} \frac{k^{s-1} q^k}{1-q^k}.$$

It makes sense to call this a $q$-analogue, since

$$\lim_{q \to 1^-} (1-q)^s \zeta_q(s) = (s-1)!\zeta(s).$$

One property this $\zeta_q(s)$ shares with $\zeta(s)$ is that a lot of questions concerning irrationality remain to be answered. So far, for $q = 1/p$ with $p \in \mathbb{N} \setminus \{0,1\}$, only $\zeta_q(1)$ and $\zeta_q(2)$ have been shown to be irrational. The former was done by Borwein [5, 6] in 1991–1992, and, using a different approach, by Bundschuh and Väänänen [7] in 1994. Note that a 1988 result by Bézivin [3] can be used to prove this irrationality. The irrationality of $\zeta_q(2)$ was proven by Duverney [9] in 1995, the transcendence of $\zeta_q(2)$ (and in fact of $\zeta_q(2s), s \in \mathbb{N}$) is a consequence of a general result by Nesterenko [12], [13]. Moreover, the three values $1, \zeta_q(1), \zeta_q(2)$ have been shown to be $\mathbb{Q}$-linearly independent by Bundschuh and Väänänen [8], by Zudilin [18] and by Postelmans and Van Assche [14].

In this paper we will prove the following result.

Theorem 1.1. *Let $q = 1/p$, with $p \in \mathbb{N}\setminus\{0,1\}$; let $\varrho = 10\pi^2/(5\pi^2 - 24)$. Then $\zeta_q(2)$ is irrational, and the inequality*

$$\left| \zeta_q(2) - \frac{a}{b} \right| \le |b|^{-\varrho}$$

*has at most finitely many integer solutions $(a, b)$.*

2000 *Mathematics Subject Classification*: 11J72, 11J82, 33D45, 41A21.
*Key words and phrases*: little $q$-Jacobi polynomials, irrationality, measure of irrationality, $q$-zeta function.

DOI: 10.4064/aa138-2-5  

Another way of putting this statement is by using the *irrationality measure* $\mu$. There are a number of equivalent definitions for this irrationality measure (Liouville–Roth constant, order of approximation, irrationality exponent). One of them is

$$\mu(x) = \inf\left\{\varrho : \left|x - \frac{a}{b}\right| > \frac{1}{b^{\varrho+\varepsilon}},\ \forall\varepsilon > 0,\ \forall a, b \in \mathbb{Z},\ b \,\text{sufficiently large}\right\}.$$

Notice that for a rational number $x$ we have $\mu(x) = 1$, whereas for an irrational number $x$ we have $\mu(x) \geq 2$ ([10, Theorem 187]). So the theorem implies that

$$2 \leq \mu(\zeta_q(2)) \leq \frac{10\pi^2}{5\pi^2 - 24} \approx 3.8936.$$

This is sharper than the upper bound 4.07869374 given by Zudilin [17].

The proof of the theorem is a $q$-adaptation of proofs of the irrationality of $\zeta(2)$, as given by Apéry [1], based on Hermite–Padé approximation [2, 15]. It uses type I Hermite–Padé approximation to two functions $f_1$ and $f_2$, with the property that $f_1(1) = \zeta_q(1)$ and $f_2(1) = \zeta_q(2)$. The polynomials that arise in this approximation can be found explicitly because they are closely related to a specific family of orthogonal polynomials, namely the little $q$-Jacobi polynomials.

We can prove the irrationality and give the upper bound for the measure of irrationality of $\zeta_q(2)$ using the following two lemmas:

Lemma 1.2. *Let $x$ be a real number and suppose there exist integer sequences $a_n, b_n$ $(n \in \mathbb{N})$ such that*

1. $b_n x - a_n \neq 0$ *for all* $n \in \mathbb{N}$;
2. $\lim_{n\to\infty}(b_n x - a_n) = 0$.

*Then $x$ is irrational.*

Lemma 1.3. *If the conditions of the previous lemma hold, with* $|b_n x - a_n| = \mathcal{O}(1/b_n^s)$ *and* $b_n < b_{n+1} < b_n^{1+o(1)}$, *then* $\mu(x) \leq 1 + 1/s$.

For the latter, see e.g. [4, Ex. 3, p. 376].

Remark 1.4. Since an easy calculation shows that for a natural number $r$,

$$\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} - \sum_{k=1}^{\infty} \frac{kq^{rk}}{1-q^k} = \sum_{i=1}^{r-1} \frac{q^i}{(1-q^i)^2} \in \mathbb{Q},$$

we also obtain the irrationality of the series

$$\sum_{k=1}^{\infty} \frac{kq^{rk}}{1-q^k}.$$

Moreover, these numbers obviously have the same irrationality measure as $\zeta_q(2)$.

**2. Some $q$-calculus.** The following elements of $q$-calculus will often be used:

- the *$q$-Pochhammer symbols* $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ and $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$;
- the *$q$-binomial factors*
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_p = \frac{(p;p)_n}{(p;p)_k(p;p)_{n-k}},$$
where a straightforward calculation shows that
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_p;$$
- the *$q$-derivative*
$$D_q f(x) = \begin{cases} \dfrac{f(x)-f(qx)}{x(1-q)} & \text{if } x \neq 0, \\ f'(0) & \text{if } x = 0; \end{cases}$$
- the *$q$-integral*
$$\int_0^{q^i} f(x)\,d_qx = \sum_{k=i}^{\infty} q^k f(q^k)$$
and
$$\int_{q^j}^{q^i} f(x)\,d_qx = \int_0^{q^i} f(x)\,d_qx - \int_0^{q^j} f(x)\,d_qx;$$
- the *$q$-Leibniz rule*
$$D_q^n[f(x)g(x)] = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k(f(x))D_q^{n-k}(g(q^kx)).$$

In the literature, the $q$-integral is often defined with an extra factor $1-q$, which makes the $q$-integration and the $q$-derivation inverse operations. Since we do not need this property, we drop the factor to prevent it from arising everywhere in the approximations and the analysis. We will need the little $q$-Jacobi polynomials. These are given by the explicit formula (see [11])

$$P_n(x;a,b|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\,\middle|\, q; qx\right) = \sum_{k=0}^{n}\frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(aq;q)_k}\,q^kx^k, \tag{2.1}$$

and there also exists a Rodrigues formula

$$\tag{2.2} \frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\, x^\alpha P_n(x;q^\alpha,q^\beta\,|\,q) = \frac{q^{n\alpha+n(n-1)/2}(1-q)^n}{(q^{\alpha+1};q)_n}\, D_p^n\left(\frac{(qx;q)_\infty}{(q^{\beta+n+1}x;q)_\infty}\, x^{\alpha+n}\right).$$

The orthogonality is given by

$$\tag{2.3} \sum_{k=0}^{\infty} \frac{(bq;q)_k}{(q;q)_k}\,(aq)^k P_n(q^k;a,b\,|\,q)q^{km} = 0, \qquad m = 0,\dots,n-1.$$

For $q$-integrals there exists an analogue to integration by parts, which is called summation by parts.

Lemma 2.1 (Summation by parts). *If $g(p) = 0$ or $f(1) = 0$ and if both series converge, then*

$$\sum_{k=0}^{\infty} q^k f(q^k) D_p g(x)|_{q^k} = -q \sum_{k=0}^{\infty} q^k g(q^k) D_q f(x)|_{q^k}.$$

## 3. Approximations to $\zeta_q(2)$

**3.1.** *The Hermite–Padé approximation problem.* The following Hermite–Padé approximation problem is considered: find polynomials $A_n$, $B_n$ of degree $\le n$ and $C_n$ of degree $\le n-1$ for which

$$\tag{3.1} F_n(z) = A_n(z) + B_n(z)\log_q(z) = 0 \quad \text{for } z = 1, p, p^2, \dots, p^n,$$

$$\tag{3.2} A_n(z)f_1(z) + B_n(z)f_2(z) - C_n(z) = \mathcal{O}(1/z^{n+1}), \qquad z\to\infty,$$

with $\log_q z = \log z/\log q$ the logarithm to base $q$ and

$$f_1(z) = \int_0^1 \frac{d_qx}{z-x}, \qquad f_2(z) = \int_0^1 \log_q x\,\frac{d_qx}{z-x}.$$

We suggest the following expression for $F_n(x)$, where $R_n$ is a yet unknown polynomial of degree $n$ and $x$ is any point on the grid $\{q^i : i\in\mathbb{Z}\}$:

$$\tag{3.3} F_n(x) = \int_x^1 R_n\left(\frac{x}{t}\right)(qt;q)_n\,\frac{d_qt}{t}.$$

This choice of $F_n$ satisfies the first condition (3.1): if we use the definition of the $q$-integral we get

$$\tag{3.4} F_n(p^l) = -\sum_{k=-l}^{-1} R_n(p^{l+k})(q^{k+1};q)_n = -\sum_{k=0}^{l-1} R_n(p^k)(q^{k+1-l};q)_n.$$

It is now obvious that for $1 \leq l \leq n$ we have $(q^{k+1-l};q)_n = (1-q^{k+1-l})\cdots(1-q^{k+n-l})$, and since the summation index $k$ runs from 0 to $l-1$, the factor $(1-q^0)$ is present in every term, hence $F_n(p^l)=0$. If $l=0$, then $F_n(p^l)$ is an empty sum, hence also zero.

The orthogonality conditions that follow from the second part of the Hermite–Padé approximation problem are given by

$$\int_0^1 F_n(x)x^m\,d_qx = 0, \qquad m=0,\dots,n-1, \tag{3.5}$$

and they will allow us to find what the polynomials $R_n$ really are:

$$\begin{aligned}0 = \int_0^1 F_n(x)x^m\,d_qx &= \int_0^1\int_x^1 R_n\left(\frac{x}{t}\right)(qt;q)_n\,\frac{d_qt}{t}\,x^m\,d_qx \\ &= q^{m+1}\int_0^1 (qt;q)_n t^m\,d_qt\int_0^1 R_n(qy)y^m\,d_qy.\end{aligned}\tag{3.6}$$

Remark 3.1. The diligent reader may think that there is a mistake in this last expression: by switching the order of integration and replacing $x$ by $yt$, one would expect a factor $R_n(y)$ in the last line. However, if one replaces the integrals by sums according to the definition of $q$-integration, and one then switches the order of summation, this factor turns out to be $R_n(qy)$. This shows that one has to be extremely careful when working with $q$-integrals.

The first integral of (3.6) is the integral over $[0,1]$ of a strictly positive integrand, so this factor is positive, which means that the other integral has to be zero. Comparing this to (2.3), we conclude that the polynomials $R_n$ should really be the little $q$-Jacobi polynomials $P_n(px;1,1\,|\,q)$ (up to a constant factor), which are in fact little $q$-Legendre polynomials [11, §3.12.1].

We use the explicit expression for the little $q$-Jacobi polynomials (2.1) and insert it in (3.3), to get

$$\begin{aligned}F_n(x) &= \int_x^1 (qt;q)_n \sum_{k=0}^n \frac{(q^{-n};q)_k(q^{n+1};q)_k}{(q;q)_k^2}\,\frac{x^k}{t^k}\,\frac{d_qt}{t} \\ &= \sum_{k=0}^n (-1)^k\,p^{k(k+1)/2-nk}\begin{bmatrix} n \\ k\end{bmatrix}_p\begin{bmatrix} n+k \\ k\end{bmatrix}_p x^k\int_x^1 \frac{(qt;q)_n}{t^{k+1}}\,d_qt.\end{aligned}$$

Since

$$(qt;q)_n = \sum_{k=0}^n p^{k(k-1)/2-nk}\begin{bmatrix} n \\ k\end{bmatrix}_p (-1)^k t^k,$$

we only need an expression for $\int_x^1 t^m\,d_qt$ with $m=-n-1,\dots,n-1$. It is easily shown that

$$\int_x^1 t^m \, d_q t = \begin{cases} \dfrac{1 - x^{m+1}}{1 - q^{m+1}} & \text{if } m \neq -1, \\ \log_q x & \text{if } m = -1. \end{cases}$$

Hence

$$\begin{aligned} F_n(x) = {} & \sum_{k=0}^{n} (-1)^k p^{k(k+1)/2 - nk} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n+k \\ k \end{bmatrix}_p \\ & \times \sum_{i=0,\, i \neq k}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_p p^{i(i+1)/2 - ni} (-1)^i \, \frac{x^k - x^i}{p^i - p^k} \\ & + \sum_{k=0}^{n} p^{-2kn + k^2} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_p x^k \log_q x. \end{aligned}$$

So, using the definition of $F_n$ in (3.1), we find that the polynomials $A_n$ and $B_n$ are given by

$$\begin{aligned} A_n(x) = {} & \sum_{k=0}^{n} (-1)^k p^{k(k+1)/2 - nk} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n+k \\ k \end{bmatrix}_p \\ & \times \sum_{i=0,\, i \neq k}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_p p^{i(i+1)/2 - ni} (-1)^i \, \frac{x^k - x^i}{p^i - p^k}, \end{aligned} \tag{3.7}$$

$$B_n(x) = \sum_{k=0}^{n} p^{-2kn + k^2} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_p x^k. \tag{3.8}$$

The Hermite–Padé approximation theory also gives us an expression for the third polynomial $C_n$. We have

$$C_n(x) = \sum_{l=0}^{\infty} q^l \, \frac{A_n(x) - A_n(q^l)}{x - q^l} + \sum_{l=0}^{\infty} l q^l \, \frac{B_n(x) - B_n(q^l)}{x - q^l}. \tag{3.9}$$

Plugging in the formulae (3.7)–(3.8) for $A_n$ and $B_n$ and changing the order of summation, we arrive at

$$\begin{aligned} C_n(x) = {} & \sum_{k=0}^{n} \sum_{i=0,\, i \neq k}^{n} (-1)^{k+i} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ i \end{bmatrix}_p \begin{bmatrix} n+k \\ k \end{bmatrix}_p \frac{p^{-nk - ni + k(k+1)/2 + i(i+1)/2}}{p^i - p^k} \\ & \times \sum_{l=0}^{\infty} q^l \, \frac{q^{li} - q^{lk} + x^k - x^i}{x - q^l} \\ & + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_p p^{-2kn + k^2} \sum_{l=0}^{\infty} \frac{l q^l}{x - q^l} \, (x^k - q^{lk}). \end{aligned}$$

Using three times the identity

$$\frac{A^n - B^n}{A - B} = \sum_{t=0}^{n-1} A^t B^{n-t-1}$$

we can further isolate the infinite sums, which can now be calculated. To that end we use the series

$$\sum_{l=0}^{\infty} l A^l = \frac{A}{(1-A)^2}, \qquad |A| < 1,$$

in the second term. This leads us to a closed formula for $C_n$:

$$\begin{aligned}
C_n(x) = {} & \sum_{k=0}^{n} \sum_{i=0, i \neq k}^{n} (-1)^{k+i} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ i \end{bmatrix}_p \begin{bmatrix} n+k \\ k \end{bmatrix}_p \\
& \times \frac{p^{-nk-ni+k(k+1)/2+i(i+1)/2}}{p^i - p^k} \left[ \sum_{t=0}^{k-1} \frac{p^{k-t} x^t}{p^{k-t} - 1} - \sum_{t=0}^{i-1} \frac{p^{i-t} x^t}{p^{i-t} - 1} \right] \\
& + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_p^2 \begin{bmatrix} n+k \\ k \end{bmatrix}_p p^{-2kn+k^2} \sum_{t=0}^{k-1} \frac{p^{k-t} x^t}{(p^{k-t} - 1)^2}.
\end{aligned} \tag{3.10}$$

Evaluating (3.2) at $p^n$ and using (3.1) shows that $a_n^*/b_n^*$ is an approximation for $\zeta_q(2)$, with

$$a_n^* = B_n(p^n) \sum_{k=1}^{n-1} \frac{k}{p^k - 1} + C_n(p^n), \tag{3.11}$$

$$b_n^* = B_n(p^n). \tag{3.12}$$

**3.2.** *Integer sequences.* To get some results regarding irrationality and the irrationality measure, the numerator and denominator of the approximant should be integers. So we will have to multiply them by an expression $e_n$, in such a way that the numbers

$$a_n = e_n a_n^* \quad \text{and} \quad b_n = e_n b_n^* \tag{3.13}$$

are integers. Looking at the explicit formulae (3.8)–(3.10) for $B_n(p^n)$ and $C_n(p^n)$, we can deduce the factors that are needed in $e_n$. Keep in mind that $p = 1/q$ is a natural number greater than 1.

It is well-known that the $p$-binomial factors are integers, hence only powers of $p$ can arise in the denominator of $B_n(p^n)$. There is a factor $p^{k^2-nk}$ in $B_n(p^n)$, with the summation index $k$ going from 0 to $n$. The minimum of this exponent is obviously $-\lfloor n^2/4 \rfloor$ (with $\lfloor\cdot\rfloor$ the floor function), so we conclude that

$$p^{\lfloor n^2/4 \rfloor} B_n(p^n) \in \mathbb{Z}.$$

The possible denominators that arise in the term

$$B_n(p^n)\sum_{k=1}^{n-1}\frac{k}{p^k-1}$$

are then clearly cancelled out by $p^{\lfloor n^2/4\rfloor}\,\mathrm{lcm}\{p^j-1 : 1\le j\le n-1\}$, where lcm denotes the least common multiple.

Finally, we need to find the denominator of $C_n(p^n)$. This denominator consists of factors $p^j$ and $p^j-1$. Looking at (3.10), we see that the latter are completely cancelled by the factor

$$(\mathrm{lcm}\{p^j-1 : 1\le j\le n\})^2.$$

It is well-known that, as a polynomial in $x$,

$$\mathrm{lcm}\{x^j-1 : 1\le j\le n\} = d_n(x),$$

with

$$d_n(x)=\prod_{d=1}^{n}\Phi_d(x), \tag{3.14}$$

where $\Phi_d$ are the cyclotomic polynomials defined by

$$\Phi_d(x)=\prod_{\substack{k=1\\ \gcd(k,d)=1}}^{d}(x-\omega_d^k)\quad\text{with}\quad \omega_d=e^{2\pi i/d}.$$

Hence, putting a factor $d_n^2(p)$ in $e_n$ will cancel all factors of type $p^j-1$. The only other denominators that can originate from $C_n(p^n)$ are powers of $p$.

At first glance, the factor needed to cancel these powers of $p$ would be $p^{n^2-n}$. However, calculations using `Maple` indicate that a factor $p^{\lfloor n^2/4\rfloor}$ is enough. This will indeed be proved in the next section. This leads us to the following

LEMMA 3.2. *If we choose*

$$e_n=p^{\lfloor n^2/4\rfloor}d_n^2(p), \tag{3.15}$$

*then $a_n$ and $b_n$, as defined in* (3.13), *are integers.*

REMARK 3.3. From (3.9) or (3.10), we see that $C_n(p^n)$ consists of two terms: one originating from $A_n$, the other originating from $B_n$. It is easy to show that both these terms have the predicted denominator $p^{n^2-n}$, but as mentioned before, putting these two terms together results in the disappearance of these high power denominators, making $\lfloor n^2/4\rfloor$ the largest remaining exponent of $p$ in the denominator of $C_n(p^n)$.

**3.3.** *Proof of Lemma 3.2.* We will work with the $q$-Mellin transform of the expression $F_n$. The *$q$-Mellin transform* of a measurable function $f$ on the $q$-exponential lattice on $(0,1]$ is given by

$$\widehat{f}(s) = \int_0^1 f(x) x^s \, d_q x.$$

The particular structure of $F_n$ as given in (3.1), and the orthogonality conditions as stated in (3.5), allow us to give an explicit expression for $\widehat{F}_n$:

$$\widehat{F}_n(s) = \frac{(p;p)_n}{p^{n^2+n+1}} \frac{q^s (q^{s-n+1};q)_n}{(q^{s+1};q)_{n+1}^2}.$$

The Hermite–Padé theory gives us an expression for the error term of the approximation. In this case we get

$$b_n^* \zeta_q(2) - a_n^* = \sum_{k=0}^{\infty} \frac{q^k}{p^n - q^k} F_n(q^k) = q^n \sum_{l=0}^{\infty} q^{nl} \widehat{F}_n(l).$$

Let us now introduce the rational function

$$R_n(T;q) = \frac{T^n (Tq^{-n+1};q)_n}{(qT;q)_{n+1}^2}$$

and the series

$$S_n(q) = \sum_{l=0}^{\infty} q^l R_n(q^l;q).$$

Then obviously

$$S_n(q) = \frac{p^{n^2+n+1}}{(p;p)_n} \sum_{l=0}^{\infty} q^{nl} \widehat{F}_n(l) = \frac{p^{n^2+2n+1}}{(p;p)_n} (b_n^* \zeta_q(2) - a_n^*). \tag{3.16}$$

A partial fraction decomposition gives

$$R_n(T;q) = \sum_{s=1}^{2} \sum_{j=1}^{n+1} \frac{d_{s,j,n}(q)}{(1-q^j T)^s}$$

with

$$d_{s,j,n}(q) = (-1)^s q^{js} \frac{d^{2-s}}{dT^{2-s}} \left( R_n(T;q)(T-q^{-j})^2 \right) \Big|_{T=q^{-j}}$$

for $s = 1, 2$. Isolation of the infinite sums allows us to recognize the expressions for $\zeta_q(1)$ and $\zeta_q(2)$, and we obtain

$$S_n(q) = \sum_{j=1}^{n+1} d_{1,j,n}(q) q^{-j} \zeta_q(1) + \sum_{j=1}^{n+1} d_{2,j,n}(q) q^{-j} \zeta_q(2) - D_1(n,q) - D_2(n,q)$$

with

$$D_1(n,q) = \sum_{j=1}^{n+1} d_{1,j,n}(q) q^{-j} \sum_{l=1}^{j-1} \frac{q^l}{1-q^l},$$

$$D_2(n,q) = \sum_{j=1}^{n+1} d_{2,j,n}(q) q^{-j} \sum_{l=1}^{j-1} \frac{q^l}{(1-q^l)^2}.$$

Since we already know from (3.16) that $S_n(q)$ is a $\mathbb{Q}$-linear combination of 1 and $\zeta_q(2)$, and since the three numbers 1, $\zeta_q(1)$, $\zeta_q(2)$ are $\mathbb{Q}$-linearly independent (see [14]), we see that the coefficient of $\zeta_q(1)$ has to be zero. Moreover, calculating $D_1$ and $D_2$ in terms of the integer $p = 1/q$, we obtain

$$D_1(n,q) = -p^{(n+1)(n+2)} \sum_{j=1}^{n+1} p^{j^2-nj-2j} (p;p)_{j-1}^{-2} (p;p)_{n-j+1}^{-2} \sum_{l=1}^{j-1} \frac{1}{p^l-1}$$
$$\times \left( n - \sum_{k=1}^{n} \frac{p^{n-k+j}}{1-p^{n-k+j}} - 2\sum_{k=1}^{j-1} \frac{p^{j-k}}{p^{j-k}-1} - 2\sum_{k=j+1}^{n+1} \frac{1}{1-p^{k-j}} \right),$$

$$D_2(n,q) = p^{(n+1)(n+2)} \sum_{j=1}^{n+1} p^{j^2-nj-2j} (p^j;p)_n (p;p)_{j-1}^{-2} (p;p)_{n-j+1}^{-2} \sum_{l=1}^{j-1} \frac{p^l}{(p^l-1)^2}.$$

Now it is an easy task to see that both these quantities contain a power $p^{\lceil 3n^2/4 \rceil + 2n+1}$ as a factor in their numerator. Together with (3.16) this allows us to conclude that the highest possible exponent of $p$ in the denominator of $a_n^*$ is $\lfloor n^2/4 \rfloor$, and hence that the factor $e_n$ as proposed in (3.15) indeed makes $a_n$ and $b_n$ integers.

Since we have an explicit expression for the coefficient of $\zeta_q(1)$, its vanishing yields a $q$-binomial identity:

Corollary 3.4. *The following identity holds:*

$$\sum_{j=0}^{n} q^{-2nj+j^2} \begin{bmatrix} n+j \\ n \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q^2$$
$$\times \left( n + \sum_{k=1}^{n+j} \frac{1}{1-q^k} - 3\sum_{k=1}^{j} \frac{1}{1-q^k} + 2\sum_{k=1}^{n-j} \frac{q^k}{1-q^k} \right) = 0.$$

*Multiplying by $1-q$ and letting $q$ tend to 1, we obtain the identity*

$$\sum_{j=0}^{n} \binom{n+j}{n} \binom{n}{j}^2 (H(n+j) + 2H(n-j) - 3H(j)) = 0,$$

*where $H(n) = \sum_{k=1}^{n} 1/k$ are harmonic numbers.*

## 4. Irrationality of $\zeta_q(2)$

**4.1.** *Estimate for the error term.* So far we know that $a_n$ and $b_n$ are integers, and that $a_n/b_n$ is an approximation of $\zeta_q(2)$. Now we want to estimate $|b_n\zeta_q(2) - a_n|$. To meet the conditions of Lemma 1.2, we need to prove that this quantity tends to zero as $n$ tends to infinity, and that it is never zero. Once again we use the expression for the error term of the approximation:

$$b_n\zeta_q(2) - a_n = e_n \sum_{k=0}^{\infty} \frac{q^k}{p^n - q^k} F_n(q^k). \tag{4.1}$$

In this last expression we need $F_n(q^k)$. This can be calculated using (3.3):

$$F_n(q^k) = \int_{q^k}^{1} R_n\left(\frac{q^k}{t}\right)(qt;q)_n \frac{d_q t}{t} = \sum_{l=0}^{k-1} P_n(q^{k-l-1};1,1\,|\,q)(q^{l+1};q)_n. \tag{4.2}$$

If we now use the Rodrigues formula for the little $q$-Jacobi polynomials (2.2) and plug this into (4.2), then after changing the order of summation we get

$$|b_n\zeta_q(2) - a_n| = e_n \frac{q^{n(n-1)/2+1}(1-q)^n}{(q;q)_n} \times \left|\sum_{l=0}^{\infty}(q^{l+1};q)_n q^l \sum_{k=0}^{\infty} \frac{q^k}{p^n - q^{k+l+1}} D_p^n[(qx;q)_n x^n]\bigg|_{x=q^k}\right|.$$

Applying $n$ times summation by parts (Lemma 2.1) we have

$$|b_n\zeta_q(2) - a_n| = e_n \frac{q^{n(n+1)/2+1}(1-q)^n}{(q;q)_n} \times \left|\sum_{l=0}^{\infty}(q^{l+1};q)_n q^l \sum_{k=0}^{\infty}(q^{k+1};q)_n q^k q^{nk} D_q^n\left(\frac{1}{p^n - q^{l+1}x}\right)\bigg|_{x=q^k}\right|.$$

Now it can be proven by induction that the $q$-derivative needed in this last expression is given by

$$D_q^n \frac{1}{p^n - q^{l+1}x} = \frac{q^{ln}(q;q)_n p^{n(n-1)/2}}{(1-q)^n \prod_{j=0}^{n}(p^{n+j} - xq^{l+1})}.$$

We recognize a double $q$-integral for $|b_n\zeta_q(2) - a_n|$:

$$|b_n\zeta_q(2) - a_n| = e_n q^{n+1}\left|\int_0^1\!\!\int_0^1 \frac{(qx;q)_n x^n (qy;q)_n y^n}{\prod_{j=0}^{n}(p^{n+j} - qxy)}\, d_qx\, d_qy\right|.$$

None of these factors is zero, and the integrand is strictly positive on $(0,1]^2$, so we see that the first condition of Lemma 1.2 is satisfied.

Obviously, $\prod_{j=0}^{n}(p^{n+j}-qxy)$ reaches its minimum at $(x,y)=(1,1)$. Moreover, the function $(qx;q)_n x^n$ reaches its maximum in $x=1$, as long as $0<q\le 1/2$, which is the case we are working with since $p=1/q$ is an integer. To see this, it is enough to show that $x(1-q^m x)/(1-q^m)\le 1$ for all $x\in[0,1]$ and for $m=1,\dots,n$. So we can make the estimate

$$\begin{aligned}|b_n\zeta_q(2)-a_n| &\le e_n\,\frac{(q;q)_n^2}{(1-q)^2}\,\frac{q^{n+1}}{\prod_{j=0}^{n}(p^{n+j}-q)}\\ &= e_n\,\frac{(q;q)_n^2 q^{n+1}}{(1-q)^2(q^{n+1};q)_{n+1}}\,q^{3n(n+1)/2}.\end{aligned}\tag{4.3}$$

**4.2.** *Asymptotic behaviour.* The asymptotic behaviour of the cyclotomic polynomials is known (see e.g. [16]) and is given in the following lemma.

LEMMA 4.1. *Suppose $p$ is an integer greater than one and let $d_n$ be given by* (3.14)*. Then*

$$\lim_{n\to\infty} d_n(p)^{1/n^2}=p^{3/\pi^2}.$$

Hence the expression (3.15) has the asymptotic behaviour

$$\lim_{n\to\infty} e_n^{1/n^2}=p^{6/\pi^2+1/4}\tag{4.4}$$

and (4.3) has the asymptotic behaviour

$$\lim_{n\to\infty}|b_n\zeta_q(2)-a_n|^{1/n^2}\le p^{6/\pi^2+1/4-3/2}\approx p^{-0.6421}.\tag{4.5}$$

So we conclude that also the second condition of Lemma 1.2 is satisfied, and hence that $\zeta_q(2)$ is irrational.

REMARK 4.2. One could try to use the same method to prove the irrationality of

$$\zeta_{q_1,q_2}(2)=\sum_{k=1}^{\infty}\frac{kq_1^k}{1-q_2^k}$$

with $q_2=1/p_2$, $q_1=1/p_1$ and integers $p_1,p_2$. Little $q$-Jacobi polynomials with different parameters are needed in this case. However, the $e_n$ which is needed to cancel the denominators turns out to be too large and

$$\lim_{n\to\infty}|b_n\zeta_{q_1,q_2}(2)-a_n|^{1/n^2}>1.$$

Hence we cannot deduce the irrationality for this family of numbers. The case where $p_1$ and $p_2$ are related in a certain way (they are both powers of the same integer $p$) gives asymptotically better results, but still not good enough to prove irrationality. So we only obtain the irrationality result for the family of numbers mentioned in Theorem 1.1 and Remark 1.4.

**4.3.** *The measure of irrationality.* To use Lemma 1.3, we need to get a value for $s$ in $|b_n\zeta_q(2) - a_n| = \mathcal{O}(1/b_n^s)$. We already know that

$$\lim_{n\to\infty} |b_n\zeta_q(2) - a_n|^{1/n^2} \le p^{6/\pi^2-5/4}.$$

If we can now find the asymptotic relation between $b_n = e_n B_n(p^n)$ and $p^{n^2}$, then we obtain the desired value for $s$. From the explicit formula (3.8) for $B_n(p^n)$ and the asymptotic behaviour of $e_n$ in (4.4), it is clear that

$$\lim_{n\to\infty} b_n^{1/n^2} = p^{6/\pi^2+1/4+1} = p^{(24+5\pi^2)/4\pi^2}.$$

Together with (4.5), this means that

$$|b_n\zeta_q(2) - a_n| = \mathcal{O}(1/b_n^{(5\pi^2-24)/(5\pi^2+24)}).$$

Hence Lemma 1.3 gives us an upper bound for the measure of irrationality:

$$\mu(\zeta_q(2)) \le 1 + \frac{5\pi^2+24}{5\pi^2-24} = \frac{10\pi^2}{5\pi^2-24},$$

which concludes the proof of Theorem 1.1.

**5. Concluding remark.** It is worth noticing that our sequence of rational approximants coincides with Zudilin's [17]. However, thanks to the treatment in Section 3.3, we were able to extract a better exponent of $p$: compare our

$$\lim_{n\to\infty} |b_n\zeta_q(2) - a_n|^{1/n^2} \le p^{6/\pi^2-5/4}$$

to formulas (70)–(72) in [17], which give Zudilin's estimate

$$\lim_{n\to\infty} |\widetilde{H}_n|^{1/n^2} \le p^{6/\pi^2-1},$$

where $\widetilde{H}_n$ is a $\mathbb{Z}$-linear combination of 1 and $\zeta_q(2)$.

## References


[1] R. Apéry, *Irrationalité de* $\zeta(2)$ *et* $\zeta(3)$, Astérisque 61 (1979), 11–13.

[2] F. Beukers, *Padé-approximations in number theory*, in: Padé Approximation and its Applications (Amsterdam, 1980), Lecture Notes in Math. 888, Springer, Berlin, 1981, 90–99.

[3] J.-P. Bézivin, *Indépendance linéaire des valeurs des solutions transcendantes de certaines équations fonctionnelles*, Manuscripta Math. 61 (1988), 103–129.

[4] J. M. Borwein and P. B. Borwein, *Pi and the AGM. A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1995.

[5] P. B. Borwein, *On the irrationality of* $\sum \frac{1}{q^n+r}$, J. Number Theory 37 (1991), 253–259.

[6] —, *On the irrationality of certain series*, Math. Proc. Cambridge Philos. Soc. 112 (1992), 141–146.

[7] P. Bundschuh and K. Väänänen, *Arithmetical investigations of a certain infinite product*, Compos. Math. 91 (1994), 175–199.

[8] —, —, *Linear independence of q-analogues of certain classical constants*, Results Math. 47 (2005), 33–44.

[9] D. Duverney, *Irrationalité d'un q-analogue de* $\zeta(2)$, C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), 1287–1289.

[10] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Univ. Press, Oxford, 1938 (5th ed., 1979).

[11] R. Koekoek and R. F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, reports of the Faculty of Technical Mathematics and Informatics, no. 98-17, Technische Universiteit Delft, 1998; math.CA/9602214 at arXiv.org.

[12] Yu. V. Nesterenko, *Modular functions and transcendence questions*, Mat. Sb. 187 (1996), no. 9, 65–96 (in Russian); English transl.: Sb. Math. 187 (1996), 1319–1348.

[13] —, *Modular functions and transcendence problems*, C. R. Acad. Sci. Paris Sér. I Math. 322 (1996), 909–914.

[14] K. Postelmans and W. Van Assche, *Irrationality of* $\zeta_q(1)$ *and* $\zeta_q(2)$, J. Number Theory 126 (2007), 119–154.

[15] W. Van Assche, *Multiple orthogonal polynomials, irrationality and transcendence*, in: Continued Fractions: From Analytic Number Theory to Constructive Approximation (Columbia, MO, 1998), Contemp. Math. 236, Amer. Math. Soc., Providence, RI, 1999, 325–342.

[16] —, *Little q-Legendre polynomials and irrationality of certain Lambert series*, Ramanujan J. 5 (2001), 295–310.

[17] W. Zudilin, *On the irrationality measure for a q-analogue of* $\zeta(2)$, Mat. Sb. 193 (2002), no. 8, 49–70 (in Russian); English transl.: Sb. Math. 193 (2002), 1151–1172.

[18] —, *Approximations to q-logarithms and q-dilogarithms, with applications to q-zeta values*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 322 (2005), 107–124.

Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200B–box 2400
BE-3001 Leuven, Belgium
E-mail: christophe@wis.kuleuven.be
walter@wis.kuleuven.be

*Received on 25.8.2008*
*and in revised form on 19.12.2008* (5784)