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On the limiting distribution of a generalized divisor problem for the case $-1/2 < a < 0$

by

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1. Introduction. Let $\sigma_a(n) = \sum_{d|n} d^a$. Define

$$\Delta_a(x) = {\sum_{n\le x}}' \sigma_a(n) - \zeta(1-a)x - \frac{\zeta(1+a)}{1+a}x^{1+a} + \frac{1}{2}\zeta(-a)$$

where $\sum'_{n\le x}$ means that the last term is halved when x is an integer. Taking $a \to 0^-$, we recover the classical error term of Dirichlet's divisor problem

$$\Delta(x) = {\sum_{n\le x}}' d(n) - x(\log x + 2\gamma - 1) - 1/4$$

with $d(n) = \sigma_0(n)$. The determination of the precise order of magnitude of $\Delta(x)$ remains an open problem. Nevertheless, there are numerous papers devoted to the study of its properties such as its power moments, $\Omega_\pm$-results, gaps between sign-changes. In particular, Heath-Brown [3] in 1992 showed that $x^{-1/4}\Delta(x)$ has a limiting distribution and explored its properties.

Unlike $\Delta(x)$ there are not many results about $\Delta_a(x)$. In this paper, we are concerned with the limiting distribution of $\Delta_a(x)$ with $-1/2 < a < 0$. It is worthwhile to note that from the available results, $\Delta_a(x)$ seems to behave like $\Delta(x)$ only in the range of $-1/2 < a < 0$ (or even $-1/2 \le a < 0$ perhaps). When $-1 \le a < -1/2$, the behavior of $\Delta_a(x)$ is rather different. Nonetheless the limiting distribution in this case also exists, shown in [7]. A further investigation will be carried out in the sequel paper.

Let us go back to the case $-1/2 < a < 0$. Analogously to the case $a = 0$, we can prove that for $-1/2 < a < 0$ and $1 \le M \le x$,

$$(1.1)\quad \Delta_a(x) = \frac{x^{1/4+a/2}}{\pi\sqrt{2}} \sum_{n\le M} \frac{\sigma_a(n)}{n^{3/4+a/2}} \cos(4\pi\sqrt{nx} - \pi/4) + O\left(\frac{x^{1/2+\varepsilon}}{\sqrt{M}}\right)$$

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where the O-constant depends on a and ε only. (A proof can be found in [6].) This is the so-called truncated Voronoi formula, which is the basic tool in our discussion.

A direct application of (1.1) and [1, Theorem 4.1] yields the following result.

THEOREM 1. *For* $-1/2 < a < 0$, $t^{-(1/4+a/2)}\Delta_a(t)$ *has a limiting distribution* $D_a(u)$ *which is also the distribution of the random series* $X = \sum_{n=1}^{\infty} a_n(t_n)$ *where*

$$a_n(t) = \frac{1}{\pi\sqrt{2}} \cdot \frac{\mu(n)^2}{n^{3/4+a/2}} \sum_{r=1}^{\infty} \frac{\sigma_a(nr^2)}{r^{3/2+a}} \cos(2\pi rt - \pi/4) \tag{1.2}$$

and $t_1, t_2, \ldots$ *are independent random variables uniformly distributed on* $[0,1]$. *Moreover*, $D_a(u) = \int_{-\infty}^{u} p_a(x)\,dx$ *for some probability density* $p_a(x)$; $p_a(x)$ *can be extended to the whole complex plane as an entire function of* x. *Furthermore*, *for real* x,

$$0 \le p_a(x) \ll \exp(-|x|^{4/(1+2|a|)-\varepsilon}).$$

Define

$$\text{tail of } D_a(u) = \begin{cases} D_a(u) & \text{if } u < 0, \\ 1 - D_a(u) & \text{if } u \ge 0. \end{cases}$$

In particular, Theorem 1 yields that tail of $D_a(u) \ll \exp(-|u|^{4/(1+2|a|)-\varepsilon})$. Our first result is to determine a more precise order of magnitude of $D_a(u)$.

THEOREM 2. *Let* $|u| \ge 2$. *Then*

$$\exp(-c_1(a)|u|^{4/(1+2|a|)}) \ll_a \text{tail of } D_a(u) \ll_a \exp(-c_2(a)|u|^{4/(1+2|a|)})$$

where $c_1(a)$ *and* $c_2(a)$ *are some constants depending on* a. *Also*, *the implied constants depend on* a.

The lower bound is derived by the method in [1, Theorem 5.1] while the upper bound is obtained from the study of its Laplace transform. Such an approach has appeared before, for example, in [2] and [5]. Our proof relies on their underlying principle.

The next result concerns the rate of convergence. The proof follows closely the argument in [7], so we shall give an outline only.

THEOREM 3. *Define*

$$D_{a,T}(u) = \frac{1}{T}\mu\{t \in [1,T] : t^{-(1/4+a/2)}\Delta_a(t) \le u\}$$

where μ *is the Lebesgue measure. Then*, *for* $-1/2 < a < 0$,

$$D_{a,T}(u) - D_a(u) \ll_a (\log\log T)^{-(1+2a)/8}$$

where the implied constant depends on a.

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2. Proof of the lower bound in Theorem 2. Write

$$A=\frac{1}{\pi\sqrt{2}}\sum_{r=1}^{\infty}\sigma_a(r^2)r^{-(3/2+a)}.$$

Then we have

$$\sup_{0\le t\le 1}|a_m(t)|\le A\sigma_a(m)m^{-(3/4+a/2)}, \tag{2.1}$$

$$\int_0^1 a_m(t)\,dt=0, \tag{2.2}$$

$$\int_0^1 a_m(t)^2\,dt=\frac{1}{4\pi^2}\cdot\frac{\mu(m)^2}{m^{3/2+a}}\sum_{r=1}^{\infty}\frac{\sigma_a(mr^2)^2}{r^{3+2a}}. \tag{2.3}$$

Define $a_m^{\pm}(t)=\max(0,\pm a_m(t))$. From (2.2) and (2.3), we have

$$\int_0^1 a_m^+(t)\,dt=\int_0^1 a_m^-(t)\,dt$$

and

$$\int_0^1 a_m^+(t)^2\,dt+\int_0^1 a_m^-(t)^2\,dt=\frac{1}{4\pi^2}\cdot\frac{\mu(m)^2}{m^{3/2+a}}\sum_{r=1}^{\infty}\frac{\sigma_a(mr^2)^2}{r^{3+2a}}.$$

Using (2.1), we obtain

$$\int_0^1 a_m^{\pm}(t)^2\,dt\le A\sigma_a(m)m^{-(3/4+a/2)}\int_0^1 a_m^+(t)\,dt$$

and hence for any squarefree m,

$$\int_0^1 a_m^+(t)\,dt\ge 2B^{-1}\sigma_a(m)m^{-(3/4+a/2)} \tag{2.4}$$

where $B=16\pi^2A(\sum_{r=1}^{\infty}r^{-(3+2a)})^{-1}>1$.

Let n be a large integer. For $1\le m\le n$, we define $A_m=[0,1]$ if m is non-squarefree, and

$$A_m=\{t\in[0,1]:a_m(t)>B^{-1}\sigma_a(m)m^{-(3/4+a/2)}\}$$

otherwise. For squarefree m, it is apparent that

$$\sup_{0\le t\le 1}|a_m(t)|\mu(A_m)+\frac{1}{B}\cdot\frac{\sigma_a(m)}{m^{3/4+a/2}}\mu(A_m^{\mathrm{c}})\ge\int_0^1 a_m^+(t)\,dt.$$

Hence from (2.1), (2.4) and $\mu(A_m^{\mathrm{c}}) \le 1$ we get

$$1/B' \le \mu(A_m) \le 1 \tag{2.5}$$

where $B' = AB$. By Markov's inequality, we have

$$\Pr\Big(\Big|\sum_{m=n+1}^{\infty} a_m(t_m)\Big| \le 2\sqrt{K}\Big) \ge 1 - \frac{1}{4K}\sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt \ge \frac{3}{4}$$

where $\Pr(\#)$ denotes the probability of the event $\#$ and

$$K = \sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt.$$

Define

$$E_n = \Big\{(t_1, t_2, \ldots) : t_m \in A_m \text{ for } 1 \le m \le n \text{ and } \Big|\sum_{m=n+1}^{\infty} a_m(t_m)\Big| \le 2\sqrt{K}\Big\}.$$

Then we get

$$\Pr(E_n) = \prod_{m=1}^{n} \Pr(A_m)\Pr\Big(\Big|\sum_{m=n+1}^{\infty} a_m(t_m)\Big| \le 2\sqrt{K}\Big) \ge \frac{3}{4}e^{-n\log B'}$$

due to $\Pr(A_m) = \mu(A_m)$ and (2.5). When $(t_1, t_2, \ldots) \in E_n$,

$$\sum_{m=1}^{\infty} a_m(t_m) \ge \frac{1}{B}\sum_{\substack{m\le n\\ m \text{ squarefree}}} \frac{\sigma_a(m)}{m^{3/4+a/2}} - 2\sqrt{K} \gg n^{1/4+|a|/2}.$$

Here and in what follows, the implied constants may depend on a. Replacing n by $[u^{4/(1+2|a|)}]$, we obtain $1 - D(u) \gg \exp(-c_1(a)u^{4/(1+2|a|)})$ for all sufficiently large u. The case of $D(-u)$ can be proved in a similar way.

3. Proof of the upper bound in Theorem 2. To prove it, we need the following result which is contained in [5]. For the sake of completeness, we give a proof as well.

A positive measurable function $\phi(x)$ defined for sufficiently large positive x is called a *regularly varying function* with index α if

$$\lim_{x\to\infty} \phi(\lambda x)/\phi(x) = \lambda^{\alpha} \quad \text{for any } \lambda > 1.$$

$\psi(x)$ is called an *asymptotic inverse* of $\phi(x)$ if $\lim_{x\to\infty}\psi(\phi(x))/x = 1$.

LEMMA 3.1. *Let X be a real random variable with probability distribution $D(x)$, let $\phi(x)$ be a regularly varying function with index $0 < \alpha < 1$, and let $\psi(x)$ be an asymptotic inverse of $x/\phi(x)$. Suppose that $D(x) > 0$ for any $x > 0$ and $L \in (0, \infty)$. We have*

(a) *if* $\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(\lambda X)) \le L$, *then*

$$\limsup_{x\to\infty} \frac{1}{x} \log(1 - D(\phi(x))) \le -\alpha\left(\frac{1-\alpha}{L}\right)^{(1-\alpha)/\alpha},$$

(b) *if* $\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(-\lambda X)) \le L$, *then*

$$\limsup_{x\to\infty} \frac{1}{x} \log D(-\phi(x)) \le -\alpha\left(\frac{1-\alpha}{L}\right)^{(1-\alpha)/\alpha}.$$

Proof. The proofs of (a) and (b) are similar and we prove part (b) only. Write $A = \limsup_{x\to\infty} \log D(-\phi(x))/x \le 0$; the result is obviously true if $A = -\infty$. Let $\xi > 0$ be fixed and $\eta > 0$. Then

$$\begin{aligned} E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) &= \int_{-\infty}^{\infty} \exp\left(-\frac{\eta}{\phi(\eta)}x\right) dD(x) \\ &\ge \int_{-\infty}^{-\phi(\xi\eta)} \exp\left(-\frac{\eta}{\phi(\eta)}x\right) dD(x) \\ &\ge \exp\left(\eta\frac{\phi(\xi\eta)}{\phi(\eta)}\right) D(-\phi(\xi\eta)). \end{aligned}$$

Hence,

$$\frac{1}{\eta} \log E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) \ge \frac{\phi(\xi\eta)}{\phi(\eta)} + \frac{1}{\eta}\log D(-\phi(\xi\eta)).$$

Then, for any $\varepsilon \in (0,1)$, there exist infinitely many $\eta \ge \eta_0(\varepsilon)$ such that

$$\frac{1}{\eta} \log E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) \ge \frac{\phi(\xi\eta)}{\phi(\eta)} + (A-\varepsilon)\xi.$$

Therefore,

$$\limsup_{\eta\to\infty} \frac{1}{\eta} \log E\left(\exp\left(-\frac{\eta}{\phi(\eta)}X\right)\right) \ge \xi^\alpha + A\xi.$$

Taking $\lambda = \eta/\phi(\eta)$, we have $\lambda \to \infty$ as $\eta \to \infty$ since $0 < \alpha < 1$ (see [8, Section 1.1]), and so

$$\limsup_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log E(\exp(-\lambda X)) \ge \xi^\alpha + A\xi.$$

From the hypothesis in (b), we obtain $L \ge \xi^\alpha + A\xi$, which holds for all $\xi > 0$. Let us write $A = -H$ ($H > 0$). Then we get $L \ge \xi^\alpha - H\xi$ and by taking $\xi = (\alpha/H)^{1/(1-\alpha)}$, we have

$$H \ge \alpha\left(\frac{1-\alpha}{L}\right)^{(1-\alpha)/\alpha}.$$

Our assertion follows.

As $X = \sum_{n=1}^{\infty} a_n(t_n)$ where t_n's are independent random variables uniformly distributed on [0,1], we have

$$E(\exp(\pm\lambda X)) = \prod_{n=1}^{\infty} E(\exp(\pm\lambda a_n(t_n))) = \prod_{n=1}^{\infty} \int_0^1 \exp(\pm\lambda a_n(t))\, dt.$$

Now, we take $\phi(x) = x^{(1+2|a|)/4}$, $\psi(x) = x^{4/(3+2a)}$ and $N = [\lambda^{4/(3+2a)}]$. We want to give an upper bound for $\log E(\exp(\pm\lambda X))$ where $\lambda \geq 1$. Therefore, we consider the integrals (inside the product) according to the following three cases.

Case (i): $n \leq N$. Using $\sigma_a(nr^2) \leq \sigma_a(n)\sigma_a(r^2)$,

$$\int_0^1 \exp(\pm\lambda a_n(t))\, dt \leq \exp\left(\lambda A \frac{\sigma_a(n)\mu(n)^2}{n^{3/4+a/2}}\right).$$

Recall that $A = (\pi\sqrt{2})^{-1} \sum_{r=1}^{\infty} \sigma_a(r^2) r^{-(3/2+a)}$.

Case (ii): $n > N$ and $\lambda A \sigma_a(n) < n^{3/4+a/2}$. Using the inequality $e^x \leq 1 + x + x^2$ for $-\infty < x \leq 1$ and $a_n(t) \leq \lambda A \sigma_a(n)/n^{3/4+a/2}$, we obtain with (2.2),

$$\begin{aligned}\int_0^1 \exp(\pm\lambda a_n(t))\, dt &\leq \int_0^1 (1 + \lambda a_n(t) + \lambda^2 a_n(t)^2)\, dt \\ &\leq 1 + (\lambda A)^2 \frac{\sigma_a(n)^2\mu(n)^2}{n^{3/2+a}} \\ &\leq \exp\left((\lambda A)^2 \frac{\sigma_a(n)^2\mu(n)^2}{n^{3/2+a}}\right)\end{aligned}$$

since $e^x \geq 1 + x$ for all real x.

Case (iii): $n > N$ and $\lambda A \sigma_a(n) \geq n^{3/4+a/2}$. As $e^x \leq e^{x^2}$ for $|x| \geq 1$ or $x = 0$,

$$\int_0^1 \exp(\pm\lambda a_n(t))\, dt \leq \exp\left(\lambda A \frac{\sigma_a(n)\mu(n)^2}{n^{3/4+a/2}}\right) \leq \exp\left((\lambda A)^2 \frac{\sigma_a(n)^2\mu(n)^2}{n^{3/2+a}}\right).$$

Since

$$\sum_{n \leq x} \sigma_a(n) = \zeta(1-a)x + O(x^{1+a}),$$
$$\sum_{n \leq x} \sigma_a(n)^2 = \zeta(1-2a)\zeta(1-a)^2\zeta(1-2a)^{-1}x + O(x^{1+a}),$$

we have

$$\begin{aligned}\log E(\exp(\pm\lambda X)) &\le \lambda A \sum_{n\le N} \frac{\sigma_a(n)\mu(n)^2}{n^{3/4+a/2}} + (\lambda A)^2 \sum_{n>N} \frac{\sigma_a(n)^2\mu(n)^2}{n^{3/2+a}} \\ &\le c_1\lambda A N^{1/4+|a|/2} + c_2(\lambda A)^2 N^{-1/2+|a|} \\ &\le c_3\lambda^{4/(3+2a)}\end{aligned}$$

where c_1, c_2 and c_3 are some positive constants depending on a.

Thus,

$$\limsup_{\lambda\to\infty} \frac{1}{\psi(\lambda)} \log E(\exp(\pm\lambda X)) \le c_3.$$

Note that $D_a(u) > 0$ for all u from the lower bound. By Lemma 3.1 and replacing $\phi(x)$ by u, i.e. $x = u^{4/(1+2|a|)}$, our proof is then complete.

4. Proof of Theorem 3. Define $F_a(t) = t^{-(1/2+a)}\Delta_a(t^2)$ and let $a_n(t)$ be defined as in (1.2). By taking $M = T^2$ in (1.1), we have for $1 \le N \le T$,

$$\begin{aligned}&\int_T^{2T} \Big|F_a(t) - \sum_{n\le N} a_n(\gamma_n t)\Big|^2\, dt \\ &\ll \int_T^{2T} \Big| \sideset{}{'}\sum_{N<n\le T^2} \frac{\sigma_a(n)}{n^{3/4+a/2}} \cos(4\pi\sqrt{n}t - \pi/4)\Big|^2\, dt \\ &\quad + \int_T^{2T} \Big| \sum_{n\le N} \frac{\mu(n)^2}{n^{3/4+a/2}} \sum_{r>T/\sqrt{n}} \frac{\sigma_a(nr^2)}{r^{3/2+a}} \cos(4\pi r\sqrt{n}t - \pi/4)\Big|^2\, dt + T^{2|a|+\varepsilon}\end{aligned}$$

where $\sum'$ sums over integers of the form $n = mr^2$ where $m > N$ is square-free. Then the first integral on the right hand side is evaluated as in Ivić [4, Theorem 13.5] while the second one is bounded trivially. We obtain for $1 \le N \le T$,

$$\int_T^{2T} \Big|F_a(t) - \sum_{n\le N} a_n(\gamma_n t)\Big|^2\, dt \ll TN^{|a|-1/2} + T^{2|a|+\varepsilon}N^{1+\varepsilon}. \tag{4.1}$$

Let us write $D_{F,T}(u) = T^{-1}\mu\{t \in [1,T] : F_a(t) \le u\}$. Then, applying the argument in [7, (5.1)], we have for any $r > 2$,

$$D_{a,T}(u) - D_a(u) \ll \sup_{T^{1/r}\le v\le T^{1/2}} |D_{F,v}(u) - D_a(u)| + T^{2/r-1}. \tag{4.2}$$

Thus, we consider $D_a(u) - D_{F,T}(u)$ and we have

$$D_a(u) - D_{F,T}(u) \ll \frac{1}{R} + \int_{-R}^{R} \Big|\frac{\chi_{a,T}(\alpha) - \chi_a(\alpha)}{\alpha}\Big|\, d\alpha$$

where $\chi_{a,T}(\alpha)$ and $\chi_a(\alpha)$ are the characteristic functions of $D_{F,T}$ and D_a respectively. Define

$$\chi_{N,T}(\alpha) = \frac{1}{T}\int_1^T \prod_{n=1}^N e(\alpha a_n(\gamma_n t))\,dt \quad \text{and} \quad \chi_N(\alpha) = \prod_{n=1}^N \int_0^1 e(\alpha a_n(t))\,dt.$$

Taking $N = 2[(\log\log T)/4]$, $R = N^{(1-2|a|)/8}$ and following [7, (5.3)–(5.4)], we obtain by (4.1),

$$D_a(u) - D_{F,T}(u) \ll \frac{1}{R} + RN^{|a|/2-1/4} + \int_{-R}^{R} \left|\frac{\chi_{N,T}(\alpha) - \chi_N(\alpha)}{\alpha}\right| d\alpha. \tag{4.3}$$

We follow [7, (5.5)–(5.7)] (with the same choices of M and δ) to evaluate $|\chi_{N,T}(\alpha) - \chi_N(\alpha)|$. Then we can get

$$|\chi_{N,T}(\alpha) - \chi_N(\alpha)| \ll (|\alpha| + 1)(\log T)^{-1/4+|a|/2+\varepsilon},$$

and $|\chi_{N,T}(\alpha) - \chi_N(\alpha)| \ll |\alpha| N^{1/4+|a|/2+\varepsilon}$ if $|\alpha| \le (\log T)^{-1}$. This yields

$$\int_{-R}^{R} \left|\frac{\chi_{N,T}(\alpha) - \chi_N(\alpha)}{\alpha}\right| d\alpha \ll (\log T)^{-1/4+|a|/2+\varepsilon}.$$

Together with (4.2) and (4.3), our result follows.

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